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RIGHT-DIVISIVE GROUPS

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1 *Introduction* In analogy with the recent developments of the Abelian groups with subtraction as primary operation (Güting [2] and Sobociński [3]) we pose ourselves the question, whether it is possible to develop group theory with right-division as primary operation (a development with left-division as primary operation will be completely analogous).

Right-division will be denoted by /.

$$a/b =_{Df} a \circ b^{-1}.$$

Then the following axioms are necessary (in all axioms the suppositions $a \in \mathfrak{G}$, $b \in \mathfrak{G}$, $c \in \mathfrak{G}$, and $a/b \in \mathfrak{G}$ are silently supposed).

D1 $a/b = a/c \rightarrow b = c$
D2 $(a/((d/d)/b))/c = a/(c/b)$

A set of elements $\mathfrak{D}$ satisfying these axioms is called a *right-divisive group*. It should be proved, that this system of axioms is consistent and independent.

The system is consistent, since it is fulfilled by $D = \{1, -1\}$ with the multiplication as groupoid operation. That *D1* is valid, is trivial. Since $dd = 1$, axiom *D2* takes the form $(a/b)/c = a/(c/b)$, what immediately follows from the commutative and associative laws for the multiplication of integers.

If D is the set of positive integers and the groupoid operation is the addition, axiom *D1* is fulfilled, but axiom *D2* is not, since for $a = b = c = d = 1$

$$(a + ((d + d) + b)) + c = 5 \text{ and } a + (c + b) = 3$$

If D is any set of minimally two elements and $a/b = a$ for all a and b, axiom *D2* is fulfilled, but axiom *D1* is not.

So the axioms are independent.

Theorem 1 *In a right-divisive group* $a/c = b/c \rightarrow a = b$.

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Proof: Suppose:

1. $a/c = b/c$

Then:

2. $a/(a/c) = (a/((d/d)/c))/a$ [$D2$]
3. $a/(a/c) = a/(b/c) = (a/((d/d)/c))/b$ [1; $D2$]
4. $(a/((d/d)/c))/a = (a/((d/d)/c))/b$ [2; 3]

Hence:

$a = b$ [4; $D1$]

Theorem 2 *A right-divisive group has an unique element* $\mathbf{e}$ *such that* $a/a = \mathbf{e}$ *and* $a/\mathbf{e} = a$ *for every* a.

Proof:

1. $(a/((d/d)/b))/c = a/(c/b) = (a/((f/f)/b))/c$ [$D2$]
2. $a/((d/d)/b) = a/((f/f)/b)$ [Theorem 1; 1]
3. $(d/d)/b = (f/f)/b$ [$D1$; 2]
4. $d/d = f/f$ [Theorem 1; 3]

I call $d/d = \mathbf{e}$, for every $d \in \mathfrak{D}$ [4]

5. $a/\mathbf{e} = a/(\mathbf{e}/\mathbf{e})$ [Definition of $\mathbf{e}$]
   $= (a/((\mathbf{e}/\mathbf{e})/\mathbf{e}))/\mathbf{e}$ [$D2$]
   $= (a/(\mathbf{e}/\mathbf{e}))/\mathbf{e} = (a/\mathbf{e})/\mathbf{e}$ [Definition of $\mathbf{e}$]
   $a = a/\mathbf{e}$ [Theorem 1; 5]

Suppose there were a second element $\mathbf{e}'$ such that also $a/\mathbf{e}' = a$. Then $a/\mathbf{e} = a = a/\mathbf{e}'$. Therefore, by $D1$: $\mathbf{e} = \mathbf{e}'$.

Theorem 3 *In a right-divisive group* $\mathfrak{D}$ $\mathbf{e}/(\mathbf{e}/a) = a$.

Proof:

1. $(\mathbf{e}/(\mathbf{e}/a))/a = (\mathbf{e}/((\mathbf{e}/\mathbf{e})/a))/a = \mathbf{e}/(a/a)$ [Theorem 2; $D2$]
   $= \mathbf{e}/\mathbf{e} = \mathbf{e} = a/a$ [Definition of $\mathbf{e}$]
2. $\mathbf{e}/(\mathbf{e}/a) = a$ [Theorem 1; 1]

Theorem 4 *In a right-divisive group* $\mathfrak{D}$ $(a/b)/(\mathbf{e}/b) = a$.

Proof: $(a/b)/(\mathbf{e}/b) = (a/(\mathbf{e}/(\mathbf{e}/b)))/(\mathbf{e}/b)$ [Theorem 3]
$= (a/((\mathbf{e}/\mathbf{e})/(\mathbf{e}/b)))/(\mathbf{e}/b)$ [Definition of $\mathbf{e}$]
$= a/((\mathbf{e}/b)/(\mathbf{e}/b))$ [$D2$]
$= a/\mathbf{e} = a$ [Definition of $\mathbf{e}$; Theorem 2]

Theorem 5 *In a right-divisive group* $\mathfrak{D}$ $\mathbf{e}/(a/b) = b/a$.

Proof: $\mathbf{e}/(a/b) = (\mathbf{e}/((\mathbf{e}/\mathbf{e})/b))/a$ [$D2$]
$= (\mathbf{e}/(\mathbf{e}/b))/a = b/a$ [Definition of $\mathbf{e}$; Theorem 3]

2 *Multiplicative and right-divisive groups* The axioms for a multiplicative group $\mathfrak{G}$ with group operation $\circ$ are assumed to be:
(a, $b \in \mathfrak{G}$ is always silently assumed)

G1 $a \circ b \in G$
G2 $a \circ (b \circ c) = (a \circ b) \circ c$
G3 *For every a, b there is an x such that* $a \circ x = b$
G4 *For every a, b there is an y such that* $y \circ a = b$.

We shall now prove, that there is a one-to-one correspondence between every multiplicative group $\langle S, \circ\rangle$ satisfying the axioms *G1, G2, G3,* and *G4* and the right-divisive group $\langle S, /\rangle$ satisfying the axioms *D1* and *D2*, in which $/$ is defined by $a/b =_{Df} a \circ (b^{-1})$ and if we start from $\langle S, /\rangle$, $\circ$ is defined by $a \circ b =_{Df} a/(\mathbf{e}/b)$.

Theorem 6 *If* $\langle S, \circ\rangle$ *is a multiplicative group and* $a/b =_{Df} a \circ (b^{-1})$, *then* $\langle S, /\rangle$ *is a right-divisive group satisfying the axioms D1 and D2.*

Proof: $a/b = a \circ (b^{-1}) \in S$ in virtue of well-known properties of the group theory, *cf.* e.g., [1].

D1 $a/c = b/c$. Then
$a \circ c^{-1} = b \circ c^{-1}$
$(a \circ c^{-1}) \circ c = a \circ (c^{-1} \circ c) = a \circ \mathbf{e} = a$
$(a \circ c^{-1}) \circ c = (b \circ c^{-1}) \circ c = b \circ (c^{-1} \circ c) = b \circ \mathbf{e} = b$
So $a = b$

D2 $(a/((d/d)/b))/c = (a \circ ((d \circ d^{-1}) \circ b^{-1})^{-1}) \circ c^{-1} =$
$(a \circ (\mathbf{e} \circ b^{-1})^{-1}) \circ c^{-1} = (a \circ (b^{-1})^{-1}) \circ c^{-1} = (a \circ b) \circ c^{-1} =$
$a \circ (b \circ c^{-1}) = a \circ (c \circ b^{-1})^{-1} = a/(c/b)$

Theorem 7 *If* $\langle S, /\rangle$ *is a right-divisive group satisfying the axioms D1 and D2 and* $a \circ b =_{Df} a/(\mathbf{e}/b)$, *then* $\langle S, \circ\rangle$ *is a multiplicative group satisfying the axioms G1, G2, G3 and G4.*

Proof: G1 is evident.

G2 $(a \circ b) \circ c = (a/(\mathbf{e}/b))/(\mathbf{e}/c) = a/((\mathbf{e}/c)/b)$ (*D2*) $=$
$a/((\mathbf{e}/(\mathbf{e}/(\mathbf{e}/c)))/b)$ (Theorem 3) $=$
$a/(\mathbf{e}/((b/(\mathbf{e}/c))))$ (*D2*) $= a \circ (b \circ c)$

G3 The solution of the equation $a \circ x = b$ is $x = a^{-1} \circ b = (\mathbf{e}/a)/(\mathbf{e}/b)$. Since
$a \circ x = a/(\mathbf{e}/((\mathbf{e}/a)/(\mathbf{e}/b))) = a/((\mathbf{e}/b)/(\mathbf{e}/a))$ (Theorem 5) $=$
$(a/(\mathbf{e}/(\mathbf{e}/a)))/(\mathbf{e}/b)$ (*D2*) $=$
$(a/a)/(\mathbf{e}/b)$ (Theorem 3) $=$
$\mathbf{e}/(\mathbf{e}/b)$ (Theorem 2) $=$
b (Theorem 3)

G4 The solution of the equation $y \circ a = b$ is $y = b \circ a^{-1} = b/a$
$y \circ a = (b/a) \circ a = (b/a)/(\mathbf{e}/a) = b$ (Theorem 4)

Theorem 8 *Let* $\langle S, +\rangle$ *be the multiplicative group associated to the right-divisive group* $\langle S, /\rangle$, *which in turn is associated to the multiplicative group* $\langle S, \circ\rangle$. *Then* $a + b = a \circ b$ *for all* $a, b \in S$.

Proof: $a + b = a/(\mathbf{e}/b) = a \circ (b^{-1})^{-1} = a \circ b$.

Theorem 9 *Let $\langle S, \sim\rangle$ be the right-divisive group associated to the multiplicative group $\langle S, \circ\rangle$, which in turn is associated to the right-divisive group $\langle S, /\rangle$. Then $a \sim b = a/b$ for all $a, b \in S$.*

Proof: $a \sim b = a \circ b^{-1} = a/(\mathrm{e}/b^{-1}) = a/(\mathrm{e}/(\mathrm{e}/b)) = a/b$. [Theorem 3]

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