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A NOTE ON THE DECOMPOSITION OF THEORIES WITH RESPECT TO AMALGAMATION, CONVEXITY, AND RELATED PROPERTIES

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1 *Introduction* If T is any theory, it is well known that $T_\forall$, the universal part of T, can be uniquely represented as the intersection of irreducible components S_i; and, corresponding to this representation, there is a decomposition of the class $\mathcal{M}(T)$ of models of T into subclasses $\mathcal{M}(T \cup S_i)$. [This decomposition generalizes, for example, the classification of fields according to their characteristic.] In this note it is made clear, first, under what conditions the classes $\mathcal{M}(T \cup S_i)$ are mutually disjoint. This result is then used to show that any theory T having the amalgamation property can be decomposed into theories T_i such that each T_i has the joint extension property as well as the amalgamation property, and the classes $\mathcal{M}(T_i)$ are mutually disjoint. Then, turning to convex theories, it is shown that there is a one-to-one correspondence between the core models of a convex theory T and the components of $T_\exists$, hence T can be decomposed (according to the components of $T_\exists$) into convex theories with a unique core model. Decomposition results with similar intent have been obtained by Fisher and Robinson in [1], and by Fisher, Simmons, and Wheeler in [2].

We assume, throughout, that $\mathcal{L}$ is a countable, finitary, first-order language. A *theory* T is a consistent set of sentences of $\mathcal{L}$; $\mathcal{M}(T)$ is the class of models of T and, if $\mathfrak{A}$ is a structure of $\mathcal{L}$, $\mathrm{Th}(\mathfrak{A})$ is the set of all the sentences which are true in $\mathfrak{A}$. $\forall_1$ will designate the set of universal formulas of $\mathcal{L}$, and $\exists_1$ the set of existential formulas. $T_\forall$ designates the universal part of a theory T, and $T_\exists$ the existential part of T. By an *irreducible ideal* of $\forall_1$ (respectively $\exists_1$), we mean a deductively closed set S of universal (respectively existential) sentences such that $\phi \vee \psi \in S$ implies $\phi \in S$ or $\psi \in S$. A *component* of $T_\forall$ (respectively $T_\exists$) is a minimal irreducible extension of $T_\forall$ (respectively $T_\exists$).

2 *Components and the conditional joint extension property* Let T be a theory, let P be a component of $T_\forall$, and let $*P = \{\varepsilon \in \exists_1 : \neg\varepsilon \notin P\}$. It is trivial to verify that $*P$ is an irreducible ideal of $\exists_1$, and that $T \cup *P$ is consistent. Furthermore, $*P$ is maximal (among the ideals of $\exists_1$) with respect to being

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consistent with T, because P is a *minimal* irreducible extension of $T_\forall$. It follows that $T_\exists \subseteq {}^*P$. Now, if $T \cup P$ were not consistent, we would have some $\alpha \in P$ such that $T \vdash \neg\alpha$, whence $\neg\alpha \in {}^*P$; this is impossible because, by the definition of ${}^*P$, $\neg\alpha \notin {}^*P$. Thus,

Theorem 2.1 *For each component P of $T_\forall$, $T \cup P$ and $T \cup {}^*P$ are both consistent. In particular, $T \cup P \cup {}^*P$ is consistent.*

(Note that every model of $T \cup {}^*P$ has to be a model of P).

A theory T has the *joint extension property* (**JEP**) iff any two models of T have a joint extension which is a model of T. T is said to have the *conditional joint extension property* (**CJEP**) iff any two models $\mathfrak{A}, \mathfrak{B} \models T$ have a common extension $\mathfrak{D} \models T$ provided that they have a common submodel $\mathfrak{C} \models T$ (that is, provided there are injections $\mathfrak{C} \to \mathfrak{A}$ and $\mathfrak{C} \to \mathfrak{B}$).

Theorem 2.2 *Let T be a theory, and $\{P_i : i \in I\}$ the family of all the components of $T_\forall$. Then the following are equivalent:*

(i) *T has the* **CJEP**
(ii) *Every model of T is a model of no more than one P_i.*
(iii) *$\{\mathcal{M}(T \cup P_i) : i \in I\}$ is a partition of $\mathcal{M}(T)$.*

If these conditions hold, then $(T \cup P_i)_\forall = P_i$.

Proof: The equivalence of (ii) and (iii) is obvious, so it remains to show (i) $\Leftrightarrow$ (ii). Suppose T has the **CJEP**, P_i and P_j are distinct components of $T_\forall$, and $\mathfrak{C} \models T \cup P_i \cup P_j$. Then $\mathfrak{C}$ has extensions $\mathfrak{A} \models T \cup P_i \cup {}^*P_i$ and $\mathfrak{B} \models T \cup P_j \cup {}^*P_j$, and by hypothesis, $\mathfrak{A}$ and $\mathfrak{B}$ have a common extension $\mathfrak{D} \models T$. But then $\mathfrak{D} \models T \cup {}^*P_i \cup {}^*P_j$, which is impossible because $T \cup {}^*P_i \cup {}^*P_j$ is inconsistent; (recall that ${}^*P_i$, as well as ${}^*P_j$, is maximal with respect to being consistent with T). Conversely, suppose (ii) holds, and $\mathfrak{C} \to \mathfrak{A}$, $\mathfrak{C} \to \mathfrak{B}$ are injections of models of T. It is obvious that $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ must all be models of the same component P_i of $T_\forall$; but then $\mathfrak{A}, \mathfrak{B}$ have a common extension $\mathfrak{D}' \models P_i$, and $\mathfrak{D}'$ has an extension $\mathfrak{D} \models T \cup P_i \cup {}^*P_i$. The last assertion of the theorem follows immediately from the fact that $T \cup P_i \cup {}^*P_i$ is consistent.

3 *The amalgamation property* A theory T has the *amalgamation property* (**AP**) iff each diagram

$$(3.1) \qquad \begin{tikzcd}[column sep=small] & \mathfrak{B} \arrow[dr, dashed] & \\ \mathfrak{A} \arrow[ur] \arrow[dr] & & \mathfrak{D} \\ & \mathfrak{C} \arrow[ur, dashed] & \end{tikzcd}, \quad \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \models T$$

can be completed. It is obvious that if T has the **AP** then T has the **CJEP**, hence T satisfies all the conditions of Theorem 2.2.

Theorem 3.2 *Let T have the* **AP**, *and let $\{P_i : i \in I\}$ be the family of all the components of $T_\forall$. Each $T \cup P_i$ has the* **AP** *as well as the* **JEP**. *Furthermore, the classes $\mathcal{M}(T \cup P_i)$, $i \in I$, are mutually disjoint, so $T = \bigcap_{i \in I} (T \cup P_i)$.*

Proof: By Theorem 2.2, $(T \cup P_i)_\forall = P_i$, hence each $T \cup P_i$ has the **JEP**. It remains only to show that each $T \cup P_i$ has the **AP**. Well, suppose $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C} \models T \cup P_i$, and $\mathfrak{A} \to \mathfrak{B}$, $\mathfrak{A} \to \mathfrak{C}$ are embeddings. Because T has the **AP**, there is a model $\mathfrak{D} \models T$ such that the diagram (3.1) can be completed. By 2.2 (ii), $\mathfrak{D}$ must be a model of P_i, that is, $\mathfrak{D} \models T \cup P_i$; thus, $T \cup P_i$ has the **AP**.

4 *Convexity* A theory T is called *convex* if, whenever $\mathfrak{A} \models T$, $\mathfrak{A}_j \models T$, and $\mathfrak{A}_j \subseteq \mathfrak{A}$ for each $j \in J$, then $\bigcap_{j \in J} \mathfrak{A}_j \models T$. If T is convex and $\mathfrak{A} \models T$, the *core submodel* of $\mathfrak{A}$ is $\bigcap \{\mathfrak{B}: \mathfrak{B} \subseteq \mathfrak{A} \text{ and } \mathfrak{B} \models T\}$. (We assume here that the language $\mathcal{L}$ has at least one constant symbol, hence the above intersection is always non-empty). It is easily seen that if $\mathfrak{A}$, $\mathfrak{B} \models T$ have a common extension $\mathfrak{D} \models T$, then they must contain the same core model.

In connection with convex theories T, we will be interested not in the components P of $T_\forall$, but in the components Q of $T_\exists$. If Q is any component of $T_\exists$, we let $*Q = \{\alpha \in \forall_1: \neg\alpha \notin Q\}$. By the same reasoning as in (2.1), we deduce that:

(4.1) *For each component Q of $T_\exists$, $T \cup Q$ and $T \cup *Q$ are consistent.*

Now, let T be a convex theory and Q a component of $T_\exists$. Because $T \cup *Q$ is consistent, there must be a core model $\mathfrak{C}$ of T such that $\mathfrak{C} \models T \cup *Q$. If $\mathfrak{A} \models T \cup Q$, then $\mathrm{Th}(\mathfrak{A})_\forall \subseteq *Q$, so $\mathfrak{C}$ can be extended to a model of $\mathrm{Th}(\mathfrak{A})$, and it easily follows (from the observation at the end of the first paragraph of this section) that $\mathfrak{C}$ is the core model contained in $\mathfrak{A}$. We have now shown that *all the models of $T \cup Q$ contain the same core model $\mathfrak{C}$*. On the other hand, if Q_1 and Q_2 are distinct components of $T_\exists$, then the core models $\mathfrak{C}_1 \models T \cup *Q_1$ and $\mathfrak{C}_2 \models T \cup *Q_2$ must be distinct, for $T \cup *Q_1 \cup *Q_2$ is inconsistent. Since each model of T contains *one and only one* core model, we conclude that the classes $\mathcal{M}(T \cup Q_1)$, $\mathcal{M}(T \cup Q_2)$ are disjoint. In particular, *each model of T is a model of only one component Q_i of $T_\exists$.* We conclude as follows:

Theorem 4.2 *Let T be a convex theory, and $\{Q_i: i \in I\}$ the family of all the components of $T_\exists$. Each $T \cup Q_i$ is a convex theory with a unique core model. Furthermore, the classes $\mathcal{M}(T \cup Q_i)$, $i \in I$, are mutually disjoint, and $\mathcal{M}(T) = \bigcup_{i \in I} \mathcal{M}(T \cup Q_i)$. In particular, $T = \bigcap_{i \in I} (T \cup Q_i)$.*

Corollary 4.3 *There is a one-to-one correspondence between the core models of T and the components of $T_\exists$.*

Corollary 4.4 *A convex theory T has a unique core model iff $T_\exists$ is irreducible.*

If T has the **CJEP**, it is clear that $\mathfrak{A}$, $\mathfrak{B} \models T$ contain the same core model iff they have a common extension, that is, iff they are models of the same component P of $T_\forall$. Thus,

Corollary 4.5 *If T is a convex theory with the* **CJEP**, *then there is a one-to-one correspondence between the core models of T and the components of $T_\forall$.*

5 *Conclusion* By means of the method used here, one can obtain similar decompositions of theories with respect to other properties, for example, different kinds of amalgamation property, the congruence extension property, and properties relating to the existence of existentially closed and algebraically closed models.

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