510
Notre Dame Journal of Formal Logic
Volume XVIII, Number 3, July 1977
NDJFAM

A DEDUCTION RULE FOR VBTO $(\ )_{i=1}^{n}$

RICHARD BUTRICK

Under the provided deduction rule, the variable binding term operator $(\ )_{i=1}^{n}$ enables conversion of meta-theorems to theorems and provides a means of avoiding appeal to set theory. The following is a statement of the deduction rule along with semantic rules and adequacy proof.

$$\begin{aligned}&\text{From } (\Theta T_1)_{i=1}^{k} \text{ and } (\Theta T_2)_{i=1}^{j}\\ &\text{and } y = \Pi((\Pi(T_1))_{i=1}^{k}, (\Pi(T_2))_{i=1}^{j})\\ &\text{to infer } (\mathrm{E}x)(y = (\Pi x)_{i=1}^{n} \ \&\ (\Theta x)_{i=1}^{n})\end{aligned}$$

where $n = k + j$, Θ is a predicate letter, the T_i are terms, and Π is a dyadic associative function letter. Under the VBTO the degree of the function letter is not indicated by the number of terms following it.

By way of example, the deduction rule enables the inference from (k) & $(k + 1)$ to $(k + 2)$.

$$\begin{aligned}&(k)(Px + i)_{i=1}^{3} \ \&\ (P2i)_{i=1}^{4}\\ &(k+1) y = f((f(x+1))_{i=1}^{3}, (f(2i)_{i=1}^{4}))\\ &(k+2)(\mathrm{E}x)(y = (f(x))_{i=1}^{n} \ \&\ (Px)_{i=1}^{n})\end{aligned}$$

Semantic Rules: The VBTO $(\ )_{i=1}^{n}$ is contextually defined for predicate and functor contexts:

$$(\Theta T)_{i=1}^{n} =_{df} \Theta T(1) \ \&\ \Theta T(2) \ldots \ \&\ \Theta T(n)$$

where $T(k)$ is the result of substituting k for all occurrences (if any) of i in T.

$$(\Pi(T))_{i=1}^{n} = \underbrace{\Pi(\ldots\Pi(\Pi}_{N-1+\text{times}}(T(1), T(2)), T(3))\ldots T(n))$$

Adequacy Proof: Given the terms T_1 and T_2, we introduce the functions $f_{T_1T_2k}$ by the schema

$$f_{T_1T_2k}(x) = \begin{cases} T_1(i/x), \text{ if } x \leqslant k\\ T_2(i/x - k), \text{ if } x > k\end{cases}$$

Received November 5, 1971

Where $T(i/x)$ is the result of substituting x for i in T (strictly speaking, the result of substituting the numeral for the given value of 'x' for all occurrences of 'i' in T). Thus, if $x = 2$ and $T_1 =$ '$i + 3$' then $T(i/x)$ would be '$2 + 3$', i.e., the result of substituting '2' for all occurrences of 'i' in '$i + 3$'. It is perhaps helpful to run through a complete example. Suppose that $T_1 =$ '$i + 3$' and that $T_2 =$ 'i' then

$$f_{T_1T_24}(i) = \begin{cases} i + 3(i/i), \text{ i.e., } i + 3, \text{ if } i \leqslant 4 \\ i - 4, \text{ if } i > 4 \end{cases}$$

in which case, say, $(\Theta f_{T_1T_24}(i))_{i=1}^{6}$ would be Θ_4 & Θ_5 & Θ_6 & Θ_7 & Θ_1 & Θ_2.

The functions $f_{T_1T_2k}$ provide the needed basis for the existence inference of the deduction rule. To prove:

$$(\Theta T_1)_{i=1}^{k} \;\&\; (\Theta T_2)_{i=1}^{j} \equiv (\Theta T_3)_{i=1}^{n}, \text{ where } T_3 = \text{`}f_{T_1T_2k}(i)\text{'}.$$

Since $T_3 = T_1$ for $1 \leqslant i \leqslant k$ and T_3 for $k < i \leqslant n = T_2$ for $1 \leqslant i \leqslant j$, the expansion $(\Theta T_3)_{i=1}^{n}$ is $\Theta T_1(i/1)$ & $\Theta T_1(i/2)$ . . . & $\Theta T_1(i/k)$ & $\Theta T_2(i/1)$ & $\Theta T_2(i/2)$ . . . & $\Theta T_2(i/j)$.

To prove:

$$\Pi((\Pi(T_1))_{i=1}^{k}, (\Pi(T_2))_{i=1}^{j}) = (\Pi(T_3))_{i=1}^{n}, \text{ where } T_3 = \text{`}f_{T_1T_2k}(i)\text{'}.$$

By associativity, $\Pi(\Pi(\ldots\Pi(\Pi(T(1), T_1(2))\ldots T_1(j)), \Pi(\ldots\Pi(\Pi(T_2(1), T_2(2))\, T_2(3))\ldots T_2(k))) = \Pi(\Pi(\ldots\Pi(\Pi(\Pi\ldots\Pi(\Pi(T_1(1), T_1(2))\ldots T_1(j))\, T_2(1))\, T_2(2))\, T_2(3)\ldots T_2(k)))$ which equals $(\Pi T_3)_{i=1}^{n}$ by definition.

By way of example, the VBTO $(\ )_{i=1}^{n}$ can be used to convert the prime factor theorem ('Every integer >1 can be expressed as a product of primes') from a meta-theorem to a theorem. The meta-theorem,

$$(x)(x > 1 \rightarrow (\mathrm{E}x_1)(\mathrm{E}x_2)\ldots(\mathrm{E}x_n)(x = x_1 \cdot x_2 \ldots \cdot x_n \;\&\; Px_1 \;\&\; Px_2 \;\&\; \ldots \;\&\; Px_n))$$

does not yield any specific theorms but in effect guarantees their existence. On the other hand, use of set-theoretic concepts,

$$(y)\left(y > 1 \rightarrow (\mathrm{E}n)(z)\left(z \leqslant n \rightarrow (\mathrm{E}f)\left(Pf(z) \;\&\; Ff(z)y \;\&\; y = \prod_{i \leqslant n} f(i)\right)\right)\right),$$

involves appeal to a whole theory as opposed to the introduction of the VBTO which constitutes a mere addition to quantification theory. The statement of the prime factor theorem:

$$(x)(x > 1 \rightarrow (\mathrm{E}z)(x = (\bullet z)_{i=1}^{n} \;\&\; (Pz)_{i=1}^{n})).$$

We give the following informal proof. (In the statement of the theorem and in the proof '$\bullet$' is the product sign.) By hypothesis of induction $(y)(y < x \rightarrow (\mathrm{E}z)(y = (\bullet z)_{i=1}^{n} \;\&\; (Pz)_{i=1}^{n}))$. Assume Px. Since $x = (\bullet x)_{i=1}^{1}$ and $Px \equiv (Px)_{i=1}^{1}$, $(\mathrm{E}z)(x = (\bullet z)_{i=1}^{n} \;\&\; (Pz)_{i=1}^{n})$. On the other hand assume $-Px$. This would only be the case if $x = \alpha_1 \bullet \alpha_2$ and $1 < \alpha_1 < x$ and $1 < \alpha_2 < x$. Using the hypothesis of induction we obtain, $\alpha_1 < x \rightarrow (\alpha_1 > 1 \rightarrow (\mathrm{E}z)(\alpha_1 = (\bullet z)_{i=1}^{j} \;\&\; (Pz)_{i=1}^{j}))$ and,

$\alpha_2 < x \to (\alpha_2 > 1 \to (\mathrm{E}z)(\alpha_2 = (\bullet z)_{i=1}^{k}$ & $(Pz)_{i=1}^{k}))$. In which case $\alpha_1 = (\bullet\beta_1)_{i=1}^{j}$ & $(P\beta_1)_{i=1}^{j}$ and $\alpha_2 = (\bullet\beta_2)_{i=1}^{k}$ & $(P\beta_2)_{i=1}^{k}$, making $x = \bullet((\bullet(\beta_1))_{i=1}^{j}, (\bullet(\beta_2))_{i=1}^{k})$. By the deduction rule for VBTO's, $(\mathrm{E}z)(x = (\bullet z)_{i=1}^{n}$ & $(Pz)_{i=1}^{n})$. Since one or the other assumptions hold, $x > 1 \to (\mathrm{E}z)(x = (\bullet z)_{i=1}^{n}$ & $(Pz)_{i=1}^{n})$, and by induction $(x)(x > 1 \to (\mathrm{E}z)(x = (\bullet z)_{i=1}^{n}$ & $(Pz)_{i=1}^{n}))$.

It is worth noting that the price of converting meta-theorems of the form $(\mathrm{E}x_1)(\mathrm{E}x_2) \ldots (\mathrm{E}x_n)$ and their kith has, in this case at least, been the formulation of a deduction rule and definitional schemata in the meta-language.

Ohio University
Athens, Ohio