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ADDITIONAL EXTENSIONS OF S4

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1 The purpose of this paper is to explore the modal systems resulting from appending

I1 ALCLpqCLMLqp

to either the basis of S4 or any of its known extensions. All of the matrices employed are taken from Sobociński in [4]. In order to make it possible for our proofs to proceed with greater facility in the subsequent discussion, we shall make use of a Fitch-style natural deduction system for S4. Such systems are familiar enough for it to be recognized that the list of derivation rules given below constitute a natural deduction system for S4:

Negative Necessity Elimination (NLE)

· · · · NLX · · p MNX n, NLE

Negative Necessity Introduction (NLI)

· · · · . n MNX · . p NLX n, NLI

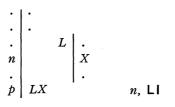
Negative Possibility Elimination (NME)

· · · · . . n NMX · . p LNX n, NME Negative Possibility Introduction (NMI)

· · · · . n LNX · . p NMX n, NMI

Necessity Elimination (LE)

Necessity Introduction (LI)



Strict Implication Elimination (**CE**)

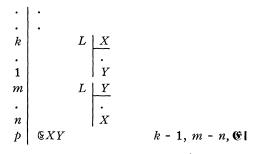
Strict Implication Introduction (©I)

$$\begin{array}{c|c} \cdot & \cdot \\ \cdot & \\ m & L & X \\ \cdot & & \\ \cdot & & Y \\ p & \mathbf{C}XY & m - n, \mathbf{C}\mathbf{I} \end{array}$$

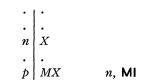
Strict Equivalence Elimination ($\ensuremath{\mathfrak{G}}\ensuremath{\mathsf{E}}\xspace$)

•	•		•	•	
• • m	&XY		m	G YX	
n	X		n	X	
Þ	• Y	m, n, E	р	Y	m, n, E

Strict Equivalence Introduction (©1)



Possibility Introduction (MI)



It is customary to formulate the rule of Possibility Elimination in the following fashion:

$$\begin{array}{c|c} \cdot & \cdot \\ \cdot & \cdot \\ 1 & MX \\ \cdot & \cdot \\ m & L & X \\ \cdot & \cdot \\ n & Y \\ \cdot & \cdot \\ p & MY \end{array} \quad 1, m - n, \mathsf{ME}$$

However, we shall adopt a formulation due to William A. Wisdom in [6]. It appears to me that more facility in constructing proofs is afforded by his formulation. It goes like this:

Possibility Elimination (ME)

Strict Reiteration (R')

LX may occur in a strict subordinate proof if LX occurs earlier in the proof to which it is subordinate. Schematically, we represent R' thus:

2 It is an easy matter to show that

I1 ALCLpqCLMLqp

is not a thesis of S4 since matrix $\mathfrak{M}4$ verifies S4 but falsifies 11 for p/5 and q/2: CNLCL52CLML25 = CNLC52CLM65 = CNL2CLM65 = CNL2CL15 = CN6C15 = C3C15 = C35 = 5. In fact, $\mathfrak{M}4$ also verifies S4.3.1; thus it is clear that 11 is not a thesis of any of the following systems: S4.3.1, S4.3, S4.2.1, S4.2, S4.1, S4.02, and S4.01. Now if we append 11 to the basis of S4, we obtain a new modal system which I call S4.03. Obviously, this system is not contained in any of the above systems; however, it is contained in the remaining systems up to and including S4.4. First we show that it is contained in S4.3.2 and hence S4.4 by demonstrating that F1, the proper axiom of S4.3.2, inferentially entails 11 in the field of S1:

(1)	ALCLpqCMLqp	F1
(2)	CNLCLpqCMLqp	1, Implication
(3)	CMLqCNLCLpqp	2, Permutation
(4)	СЬрр	S1
(5)	CLMLqMLq	4, p/MLq
(6)	CLMLqCNLCLpqp	3, 5, Syllogism
(7)	CNLCLpqCLMLqp	6, Permutation
(8)	ALCLÞqCLMLqÞ	7, Implication

Before showing that 11 is a thesis of S4.04 (and hence S4.1.2), we first demonstrate that

L4 CpLCMLpp

may also serve as the proper axiom of S4.04. Assume L4 and the field of S1:

(1)	СрЕСМЕрр	L4
(2)	CLCpqCLpLq	S1
(3)	CLCMLppCLMLpLp	2, p/MLp, q/p
(4)	CpCLMLpLp	1, 3, Syllogism
(5)	CLMLpCpLp	4, Permutation

Clearly L4 inferentially entails L1 in the field of S1. Now we prove that L1 inferentially entails L4 in the field of S4. In order to accomplish

this, we shall make use of our natural deduction system for S4. We shall also utilize the following formula:

CLMLCMLppCCMLppLCMLpp

This formula is merely a substitution instance (p/CMLpp) of L1.

1 $ CLMLCMLppCCMLppLCMLpp$ L1, $p/CMLpp$					
2	Бигсигр	рссмьрр	LCMLpp		L1, $p/CMLpp$
$\begin{vmatrix} 2 \\ 3 \end{vmatrix}$	P	LCMLÞÞ			Нур
4		LCMLpp			Нур 3, NL E
5		NMLCMI	(
6		LNLCMI			Нур
7		NLCML			5, NLE
8		MNCML	-		6, LE
о 9		-	-		7, NL E
- 1			CMLpp		Нур
10			VLCMLpp		6, R'
11			<u>M</u> Lp		Нур
12			LNLCMLpp)	10, R'
13			$L \underline{L}p$		Нур
14			1 1	CMLpp	12, R'
15			NLCN		14, LE
16			MNCI		15, NLE
17				<u>N</u> CMLpp	Нур
18				Lp	13, R'
19				MLp	Нур
20				Lp	18, R
21				Þ	20, LE
22				CMLpp	19-21, Cl
23	4		NLp		16, 17-22, ME
24			Þ		11, 13-23, ME
25		1 .	ИLpp		11-24, Cl
26		MLCML	р		8, 9-25, ME
27		CMLpp			4, 5-26, ME
28	NNLMLCMLpp 3-27, NI				
29	LMLCM				28, NE
30	CLMLCMLppCCMLppLCMLpp 1, R				
31	CCMLpp	LCMLpp			30, 29, CE
32	MLp				Нур
33	Þ				2, R
34	CMLpp				32-33, CI
35	LCMLpp	•			31, 34, CE
36 <i>CpLCMLpp</i> 2-35, CI					
	, , , , , , , , , , , , , , , , , , , ,				

Having proved that L4 may also serve as the proper axiom of modal system S4.04, we now show that L4 inferentially entails 11 in the field of S4. This time we shall employ

CCLpqLCMLCLpqCLpq

in our proof. Note that this is merely a substitution instance of L4.

1 $CCLpqLCMLCLpqCLMLqp$ Hyp 3 $CCLpqLCMLCLpqCLpq$ 1, R 4 $NLCLpq$ Hyp 5 $CCLpqLCMLCLpqCLpq$ 3, R 6 $\lfloor LMLq$ Hyp 7 $NLCLpq$ Hyp 8 $CCLpqLCMLCLpqCLpq$ 5, R 9 $\lfloor Dp$ Hyp 10 $CCLpqLCMLCLpqCLpq$ 8, R 11 $\lfloor Dp$ Hyp 12 Np 9, R 13 $\lfloor Dp$ 11, R 14 $\lfloor Dp$ 14, LE 15 $\lfloor Dp$ Hyp 14 $\lfloor Dp$ 14, LE 15 $\lfloor Dp$ 14, LE 16 $\lfloor Np$ 12, R 17 NNq 13-16, NI 18 q 17, NE 19 $CLpq$ 10, 19, CE 21 $LMLq$ 6, R 22 $NLCLpq$ 7, R 23 $MNCLpq$ 21, R' 24 L $NCLpq$ 22, NLE 24	4	COT hat OMI OI haOI ha	11 b/OTha
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45 NNp 9-44, NI			
46 <i>p</i> 45, NE			-
	46	<i>p</i>	45, NE

47	CLMLqp	6-46, CI
48	ALCLpqCLMLqp	47, A I
49	NALCLpqCLMLqp	2, R
50	NNLCLpq	4-49, NI
51	LCLpq	50, NE
52	ALCLpqCLMLqp	51, Al
53	NNALCLpqCLMLqp	2-52, NI
54	ALCLpqCLMLqp	53, NE

Using the matrices of Sobociński ([4], p. 350), we find

1. $\mathfrak{M}7$ verifies 11, but rejects S4.02 ([3], p. 381) and hence also S4.04, S4.1.2, S4.1, S4.2.1, S4.3.1, and S4.4.

2. $\mathfrak{M5}$ verifies 11, but rejects S4.2 ([4], p. 354) and hence also S4.3 and and S4.3.2.

3. m11 verifies 11, but rejects S4.01 ([1], p. 569).

These considerations demonstrate that S4.03 is a proper extension of S4, properly contained in S4.04, S4.1.2, S4.3.2, and S4.4, and independent of S4.01, S4.02, S4.1, S4.2, S4.3, S4.2.1, and S4.3.1.

We may now wonder whether the addition of 11 to the basis of any of the other Lewis extensions of S4 independent of S4.03 will also yield additional modal systems. But before directing our attention to this task, we first show that

12 CKLMLpqLApMq

may also serve as the proper axiom of S4.03. Assume 11 and the field of S1:

(1)	ALCLpqCLMLqp	11
(2)	ACLMLqpLCLpq	1, Commutation
(3)	CNCLMLqpLCLpq	2, Implication
(4)	CKLMLqNpLCLpq	3, Implication
(5)	CKLMLqNpLANLpq	4, Implication
(6)	CKLMLqNpLAMNpq	5, Modal Exchange
(7)	CKLMLqNpLAqMNp	6, Commutation
(8)	CKLMLpNNqLApMNNq	7, q/p , p/Nq
(9)	CKLMLpqLApMq	8, Double Negation

Quite obviously, this proof may also be carried out in reverse; thus 12 also inferentially entails 11 in the field of S1.

Having proved that 11 and 12 are inferentially equivalent in the field of S1, we now show that the addition of 12 to the basis of S4.02 yields S4.04. We prove this by showing that 12 and ± 1 together inferentially entail $\perp 1$ in a field at least as weak as S4:

-	ORT MT ENOLT ET ALMNOLT E	
	CKLMLpNCpLpLApMNCpLp	12, $q/NCpLp$
	LCLCLCpLppCLMLpp	上1
3	LMLp	Нур
4	CKLMLpNCpLpLApMNCpL	<i>p</i> 1, R
5	LCLCLCpLppCLMLpp	2, R
6		Нур
7		3, R
8		
9		5, R
10		
		Нур
11		7, R
12	CKLMLpNCpLpLApM	
13	LCLCLCpLppCLMLp	•
14		6, R
15		Нур
16	p	14, R
17		15, 16, CE
18		10, R
19		15-18, NI
20	KLMLpNCpLp	11, 19, K I
20		
		12, 20, CE
22		Нур
23		•
24	ApMNCpLp	23, LE
25		Нур
26	<u>MN</u> CpLp	Нур
27		22, R
28	Np	Нур
29		• =
30	NLCpL	
31		-
32		,
		28-31, NI
33		32, NE
34		24, 25-25, 26-33, AE
35	CLCpLpp	22-34, CI
36		35, LI
37	L LCLCLCpLppC	LMLpp 9, R'
38	CLCLCpLppCL	<i>MLpp</i> 37, LE
39		7, R'
40		36, R'
41		38, 40, CE
42	p	41, 39, CE
43	1 1 1 1-	41, 35, CE 42, LI
43 44		-
		10-43, NI
45		44, NE
46	CpLp	6-45, CI
47	ϹĹϺĹϼϹϼĹϼ	3-46, CI

Now because ≥ 1 is a thesis of S4.1, it is obvious that appending either 11 or 12 to the basis of S4.1 would yield S4.1.2 since S4.1.2 = {S4.1; L1}.

Now we show that 11 inferentially entails F1, the proper axiom of S4.3.2, in the field of S4.2:

(1)	ALCLpqCLMLqp	11
(2)	CNLCLpqCLMLqp	1, Implication
(3)	CLMLqCNLCLpqp	2, Permutation
(4)	CMLqLMLq	G2
(5)	CMLqCNLCLþqþ	3, 4, Syllogism
(6)	CNLCLpqCMLqp	5, Permutation
(7)	ALCLpqCMLqp	6, Implication

Clearly then, appending 11 to either S4.2 or S4.3 yields S4.3.2.

In [9], p. 297, Zeman demonstrates that appending F1 to S4.1 yields S4.4. Consequently, it follows that since S4.2.1 and S4.3.1 both contain S4.2, appending I1 to either S4.2.1 or S4.3.1 would also yield S4.4.

Adding formula 11 to the basis of S4.01 does, however, result in another new modal system. I call this new extension of S4 modal system S4.05. In [1], R. I. Goldblatt shows that S4.01 is properly contained in every known extension of S4 except S4.02 and S4.04. He also demonstrates that S4.01 is independent of both S4.02 and S4.04. Now again consider the following matrices:

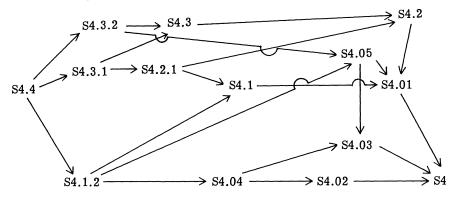
1. **W7** verifies both Γ 1, the proper axiom of S4.01, and I1, but rejects S4.02 ([3], p. 381) and hence also S4.04, S4.1.2, S4.1, S4.2.1, S4.3.1, and S4.4. 2. **W5** verifies both Γ 1 and I1 but rejects S4.2 ([4], p. 354) and hence also S4.3 and S4.2.

3. $\mathfrak{M}11$ verifies 11, but rejects $\Gamma1$ ([1], p. 569).

4. $\mathfrak{M}4$ verifies $\Gamma1$, but rejects 11.

These considerations establish that S4.05 is a proper extension of S4, S4.01, and S4.03, properly contained in S4.1.2, S4.3.2, and S4.4, and independent of S4.2, S4.3, S4.1, S4.2.1, S4.3.1, S4.02, and S4.04.

The following diagram enables us to visualize the relationships existing among S4.03, S4.05, and the other Lewis extensions of S4 at the time of writing:



3 In this section we consider the relationship of formula 11 to modal family Z. We have already observed that $\mathfrak{M}4$ falsifies 11; but this matrix verifies system Z7. Consequently, 11 is not a thesis of any of the following systems: Z7, Z6, Z4, Z5, Z3, and Z1. Now we have also observed that 11 is a thesis of S4.04, S4.1.2, S4.3.2, and S4.4; thus it must be the case that 11 is a thesis of both Z8 and Z2.

Appending 11 to Z1 generates a new system to be called Z1.5. Now consider the following matrices:

1. $\mathfrak{M8}$ verifies both 11 and $\Gamma1$, but rejects Z1 ([7], p. 354).

2. $\mathfrak{M5}$ verifies both 11 and Z1 but rejects G1 ([5], pp. 310-311), and hence Z4, Z5, Z6, Z7, and Z8.

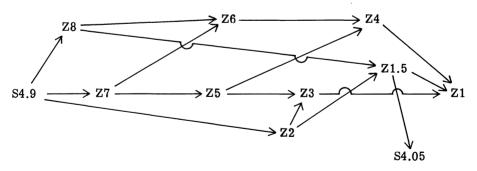
3. $\mathfrak{M}\mathfrak{b}$ verifies both 11 and Z1 but falsifies N1 ([9], pp. 296-298), and hence Z3 and Z2.

Now in view of Goldblatt's finding (cf. [1], p. 568) that Γ 1 is entailed by Z1 and the considerations given above, we may conclude that modal system Z1.5 is a proper extension of S4.05 and Z1, properly contained in Z8 and Z2, and independent of Z3, Z4, Z5, Z6, and Z7.

In the last section, we proved that appending 11 to either S4.2 or S4.3 yields S4.3.2. Thus appending 11 to either Z4 or Z6 must yield Z8. Again in the last section, we saw that appending 11 to S4.1 gives S4.1.2; hence it follows that adding it to Z3 yields Z2.

In [2], Sobociński constructs a system which he calls Z9 by appending Z1 to the basis of S4.4. However, in [8], Zeman proves that Z9 is S4.9. Hence appending Z1 to S4.4 yields S4.9. Now we observed, in the last section, that adding 11 to either S4.2.1 or S4.3.1 gives S4.4. Consequently, appending 11 to either Z5 or Z7 would yield S4.9.

The relationships holding between Z1.5 and the other systems of family Z are exhibited by the following diagram:



4 Let us now consider 11 with respect to family K. Now $\mathfrak{M}4$ verifies the entire basis of K3.1, but, as we have seen, falsifies 11. Thus it is clear that 11 is not a thesis of any of the following systems: K3.1, K3, K2.1, K2, K1.1, and K1. Now since 11 is a thesis of S4.04, S4.3.2, and S4.4, it follows that it is also a thesis of K1.2, K3.2, and K4.

The addition of 11 to K1 also generates a new modal system to be called K1.1.5. Now consider the following:

1. $\mathfrak{M}2$ verifies both 11 and Z1, but rejects, as is well-known, K1 and hence K1.1.5.

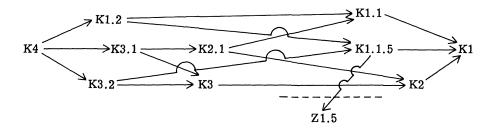
2. $\mathfrak{M5}$ verifies both 11 and K1, but rejects G1 ([5], pp. 310-311), and hence K2, K3, K3.2, K2.1, K3.1, and K4.

3. Mt verifies both 11 and K1, but rejects N1 ([9], pp. 296-298), and hence K1.1 and K1.2.

These considerations show that K1.15 is a proper extension of K1 and and Z1.5, properly contained in K1.2, K3.2, and K4, and independent of K1.1, K2, K3, K2.1, and K3.1.

Now, as we have already remarked, appending 11 to S4.1 yields S4.1.2, hence appending it to K1.1 would generate K1.2. Adding 11 to either S4.2 or S4.3 gives S4.3.2, hence appending it to either K2 or K3 would yield K3.2. It is well-known that $\{S4.4; K1\} = K4$, and, as we have already pointed out, $\{S4; N1; F1\} = S4.4$, hence appending 11 to either K2.1 or K3.1, since they both contain S4.2, would yield K4.

The diagram given below exhibits the relationships holding between K1.1.5 and the other systems of family K at the time of writing:



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