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A REMARK ON GENTZEN'S CALCULUS OF SEQUENTS

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In this short note we call attention to a simple but perhaps interesting property of Gentzen's calculus of sequents (*cf.* [1]): the restriction to sequents whose antecedent contains at most one formula does not affect the derivability of classically valid formulas without existential quantifier and implication sign (in contrast to the corresponding restriction concerning the succedent; as is well-known, in this case we get the intuitionistic calculus; see [1], p. 192). Let us call the system obtained from Gentzen's calculus by this restriction the "dual-intuitionistic calculus **DJ**". In [2] we prove by embeddings of propositional logics in S4: Each classically valid N-K-A-formula is derivable in **DJ**. Now we give a direct proof of this theorem, extending it to formulas containing the universal quantifier. The axioms of **DJ** are all the sequents of the form $\alpha \rightarrow \alpha$. The rules of inference are:

(V) $\dfrac{\Gamma \rightarrow \Delta, \alpha, \beta, \Theta}{\Gamma \rightarrow \Delta, \beta, \alpha, \Theta}$ (Kü) $\dfrac{\Gamma \rightarrow \Delta, \alpha, \alpha}{\Gamma \rightarrow \Delta, \alpha}$

(W1) $\dfrac{\rightarrow \Delta}{\alpha \rightarrow \Delta}$ (W2) $\dfrac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \alpha}$

(N1) $\dfrac{\rightarrow \Delta, \alpha}{N\alpha \rightarrow \Delta}$ (N2) $\dfrac{\alpha \rightarrow \Delta}{\rightarrow \Delta, N\alpha}$

(A1) $\dfrac{\alpha \rightarrow \Delta \qquad \beta \rightarrow \Delta}{A\alpha\beta \rightarrow \Delta}$ (A2a) $\dfrac{\Gamma \rightarrow \Delta, \alpha}{\Gamma \rightarrow \Delta, A\alpha\beta}$

(K1a) $\dfrac{\alpha \rightarrow \Delta}{K\alpha\beta \rightarrow \Delta}$ (A2b) $\dfrac{\Gamma \rightarrow \Delta, \beta}{\Gamma \rightarrow \Delta, A\alpha\beta}$

(K1b) $\dfrac{\beta \rightarrow \Delta}{K\alpha\beta \rightarrow \Delta}$ (K2) $\dfrac{\Gamma \rightarrow \Delta, \alpha \qquad \Gamma \rightarrow \Delta, \beta}{\Gamma \rightarrow \Delta, K\alpha\beta}$

(G1) $\dfrac{\alpha(a) \rightarrow \Delta}{\Pi x\alpha(x) \rightarrow \Delta}$ (G2) $\dfrac{\Gamma \rightarrow \Delta, \alpha(a)}{\Gamma \rightarrow \Delta, \Pi x\alpha(x)}$

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The free variable a may not occur in the conclusion of (G2). The sequence Γ is empty or contains exactly one formula. The rules (A2a) and (A2b) can be replaced by

$$(\mathrm{A2})\ \frac{\Gamma \to \Delta, \alpha, \beta}{\Gamma \to \Delta, A\alpha\beta}$$

without changing the set of derivable sequents. In view of (A2) we see the admissibility of the inverse rule

$$(\mathrm{IA2})\ \frac{\Gamma \to \Delta, A\alpha\beta}{\Gamma \to \Delta, \alpha, \beta}$$

Gentzen's Hauptsatz—that cuts

$$(\mathrm{Cut})\ \frac{\Gamma \to \Delta, \alpha \qquad \alpha \to \Theta}{\Gamma \to \Delta, \Theta}$$

are permitted—can be proved similarly as for **LK** and **LJ** in [1] or by reduction to the intuitionistic case, using the following natural mapping of **DJ** onto **LJ**:

π^* is π for each atomic formula π
$(N\alpha)^*$ is $N\alpha^*$
$(A\alpha\beta)^*$ is $K\alpha^*\beta^*$
$(K\alpha\beta)^*$ is $A\alpha^*\beta^*$
$(\Pi x\alpha(x))^*$ is $\Sigma x(\alpha(x))^*$
$(\Gamma \to \Delta)^*$ is $\Delta^* \to \Gamma^*$ where Δ^* rep. Γ^* is the result of replacing each formula α in Δ rep. Γ by α^*.

Then it follows: $\Gamma \to \Delta$ is derivable in **DJ** iff $(\Gamma \to \Delta)^*$ is derivable in **LJ**. We see that modus ponens in the form of

$$(\mathrm{MP})\ \frac{\to \alpha \to AN\alpha\beta}{\to \beta}$$

is an admissible rule (apply (N1), (IA2), and (Cut)).

Now take the following list of axioms:

(*Ax1*) $C\alpha C\beta\alpha$
(*Ax2*) $CC\alpha C\beta\gamma CC\alpha\beta C\alpha\gamma$
(*Ax3a*) $CK\alpha\beta\alpha$
(*Ax3b*) $CK\alpha\beta\beta$
(*Ax4*) $CC\alpha\beta CC\alpha\gamma C\alpha K\beta\gamma$
(*Ax5a*) $C\alpha A\alpha\beta$
(*Ax5b*) $C\beta A\alpha\beta$
(*Ax6*) $CC\alpha\gamma CC\beta\gamma CA\alpha\beta\gamma$
(*Ax7*) $CCN\alpha N\beta C\beta\alpha$
(*Ax8*) $C\Pi x\alpha(x)\alpha(a)$

Together with (MP) and the quantification rule

$$\text{(G)}\quad \frac{C\alpha\beta(a)}{C\alpha\Pi x\beta(x)} \text{ (with the usual condition on } a\text{)}$$

these axioms give a complete system **L** of classical first-order logic without existential quantifier and implication, if we think of C as an abbreviation for AN.

For a proof of our theorem we use induction on derivations in **L**. The derivability of the sequents

$$\to (Axi) \qquad (i = 1, \ldots, 8)$$

can be shown without difficulties; for example, consider the case $(Ax6)$ (we do not mention applications of structural rules):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \to \gamma, \alpha, \beta \qquad \beta \to \gamma, \alpha, \beta}{A\alpha\beta \to \gamma, \alpha, \beta}}{\to NA\alpha\beta, \gamma, \alpha, \beta}}{N\beta \to NA\alpha\beta, \gamma, \alpha} \qquad \gamma \to NA\alpha\beta, \gamma, \alpha}{AN\beta\gamma \to NA\alpha\beta, \gamma, \alpha}}{\to NAN\beta\gamma, NA\alpha\beta, \gamma, \alpha}}{N\alpha \to NAN\beta\gamma, NA\alpha\beta, \gamma} \qquad \gamma \to NAN\beta\gamma, NA\alpha\beta, \gamma}{AN\alpha\gamma \to NAN\beta\gamma, NA\alpha\beta, \gamma}}{\to NAN\alpha\gamma, NAN\beta\gamma, NA\alpha\beta, \gamma}}{\to NAN\alpha\gamma, NAN\beta\gamma, ANA\alpha\beta\gamma}}{\to NAN\alpha\gamma, ANAN\beta\gamma ANA\alpha\beta\gamma}}{\to ANAN\alpha\gamma ANAN\beta\gamma ANA\alpha\beta\gamma}$$

Since (MP) is permitted in **DJ** and (G) can be checked as follows:

$$\dfrac{\dfrac{\dfrac{\to AN\alpha\beta(a)}{\to N\alpha, \beta(a)}\,\text{(IA2)}}{\to N\alpha, \Pi x\beta(x)}\,\text{(G2)}}{\to AN\alpha\Pi x\beta(x)}$$

we have proved our

Theorem *Each classically valid formula without existential quantifier and implication is derivable in* **DJ**.

That the restriction to formulas without implication is essential can be seen by the fact that $C\pi C\rho\pi$ with distinct atomic formulas π, ρ is not derivable in **DJ**, if we add the rules

$$\text{(C1)}\ \frac{\to \Delta, \alpha \qquad \beta \to \Theta}{C\alpha\beta \to \Delta, \Theta} \qquad \text{(C2)}\ \frac{\alpha \to \beta, \Delta}{\to C\alpha\beta, \Delta}$$

and consider C as a primitive sign. Let $[\alpha]^n$ for $n \geq 0$ denote a sequence of n occurrences of the formula α. The only possible sequents in a derivation of $C\pi C\rho\pi$ are of the form

$$[\pi]^i \to [C\rho\pi]^m, [C\pi C\rho\pi]^n$$

or

$$[\rho]^i \to [\pi]^k, [C\rho\pi]^m, [C\pi C\rho\pi]^n \qquad (i = 0 \text{ or } i = 1;\ k \geqslant 0,\ m \geqslant 0,\ n \geqslant 0)$$

But no one of them is derivable in **DJ**. Therefore, in contrast to the situation for formulas without implication, the system **DJ** is weak concerning the derivability of implicational formulas (especially formulas with iterated implicational parts). We were not able to prove the theorem also for formulas containing existential quantifiers.

REFERENCES

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