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AN AXIOMATIZATION OF HERZBERGER'S 2-DIMENSIONAL PRESUPPOSITIONAL SEMANTICS

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The purpose of this paper* is to axiomatize two 4-valued propositional logics suggested by Herzberger in [1], section VI. They are of philosophical interest because their interpretation makes use of two ideas inspired by Jean Buridan: (1) a proposition may correspond to the world and yet be untrue because it is semantically deviant, and (2) logically valid arguments preserve correspondence with reality, not truth. If the two non-classical truth-values of these systems are identified, the resulting tables for the classical connectives are the weak and strong systems of Kleene. Unlike Kleene's system, the 4-valued ones offer a choice of designated values that renders semantic entailment perfectly classical. Compare Herzberger [2] and Martin [5].

Let the set $\mathcal{F}$ of formulas be inductively defined over a denumerable set of atomic formulas such that $\neg A$, $A \,\&\, B$, $\mathbf{C}A$, $\mathbf{B}A$, $\mathbf{T}A$, $\mathbf{F}A$, $\mathbf{t}A$, and $\mathbf{f}A$ are formulas if A and B are. Let $\mathcal{W}$ be the set of all $\mathfrak{w}$ such that for some v and $\mathfrak{v}$,

(1) for any atomic formula A, $v(A)$, $\mathfrak{v}(A) \in \{0, 1\}$;
(2) $v(\neg A) = 1$ if $v(A) = 0$; $v(\neg A) = 0$ otherwise;
$v(A \,\&\, B) = 1$ if $v(A) = v(B) = 1$; $v(A \,\&\, B) = 0$ otherwise;
$v(\mathbf{C}A) = 1$ if $v(A) = 1$; $v(\mathbf{C}A) = 0$ otherwise;
$v(\mathbf{B}A) = 1$ if $\mathfrak{v}(A) = 1$; $v(\mathbf{B}A) = 0$ otherwise;
$v(\mathbf{T}A) = 1$ if $v(A) = \mathrm{v}(A) = 1$; $v(\mathbf{T}A) = 0$ otherwise;
$v(\mathbf{F}A) = 1$ if $v(A) = 0$ and $\mathfrak{v}(A) = 1$; $v(\mathbf{F}A) = 0$ otherwise;
$v(\mathbf{t}A) = 1$ if $v(A) = 1$ and $\mathfrak{v}(A) = 0$; $v(\mathbf{t}A) = 0$ otherwise;
$v(\mathbf{f}A) = 1$ if $v(A) = \mathrm{v}(A) = 0$; $v(\mathbf{f}A) = 0$ otherwise;
(3) $\mathfrak{v}(\neg A) = 1$ if $\mathfrak{v}(A) = 1$; $\mathfrak{v}(\neg A) = 0$ otherwise;
$\mathfrak{v}(A \,\&\, B) = 1$ if $\mathfrak{v}(A) = \mathfrak{v}(B) = 1$; $\mathfrak{v}(A \,\&\, B) = 0$ otherwise;
$\mathfrak{v}(\mathbf{C}A) = \mathfrak{v}(\mathbf{B}A) = \mathfrak{v}(\mathbf{T}A) = \mathfrak{v}(\mathbf{F}A) = \mathfrak{v}(\mathbf{t}A) = \mathfrak{v}(\mathbf{f}A) = 1$;

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(4) $\mathfrak{w}(A) = \langle \nu(A), \mathfrak{v}(A)\rangle$.

Let $\mathcal{L} = \langle \mathcal{I}, \mathcal{W}\rangle$, and abbreviate $\langle 11\rangle$ by T, $\langle 01\rangle$ by F, $\langle 10\rangle$ by t, and $\langle 00\rangle$ by f, and define $A \vee B$ as $\neg(\neg A \,\&\, \neg B)$, $A \rightarrow B$ as $\neg A \vee B$, and $A \leftrightarrow B$ as $(A \rightarrow B) \,\&\, (B \rightarrow A)$.

Intuitively, values on the first co-ordinate record whether a sentence corresponds to the world and values on the second whether it is semantically normal in the sense that all its presuppositions are satisfied. A sentence is assigned T for true iff it both corresponds and is normal and F for false iff though normal, it does not correspond. Hence '**C**' is read as 'corresponds' and '**B**' as 'is bivalent'. **C**A and **B**A could have been introduced by definition as **T**$A \vee$ **t**A and **T**$A \vee$ **F**A respectively.

The values on the first coordinate of members of $\mathcal{W}$, those on the second, and the compound values for members of $\mathcal{W}$ conform to tables under I, II, and I × II respectively:

I

	¬	&	10	**C**
1	0		10	1
0	1		00	0

	B	**T**	**F**	**t**	**f**
11	1	1	0	0	0
01	1	0	1	0	0
10	0	0	0	1	0
00	0	0	0	0	1

II

	¬	&	10	**C**	**B**	**T**	**F**	**t**	**f**
1	1		10	1	1	1	1	1	1
0	0		00	1	1	1	1	1	1

I × II

	¬	&	T F t f	∨	T F t f	→	T F t f	**C**	**B**	**T**	**F**	**t**	**f**
T	F		T F t f		T T t t		T F t f	T	T	T	F	F	F
F	T		F F f f		T F t f		T T t t	F	T	F	T	F	F
t	f		t f t f		t t t t		t f t f	T	F	F	F	T	F
f	t		f f f f		t f t f		t t t t	F	F	F	F	F	T

The operations of I × II are functionally incomplete as is seen from the fact that T and F are never taken into t or f. Further, substitution of truth-functional equivalents fails among the non-classical formulas, e.g., if $\mathfrak{w}(A)$ = T and $\mathfrak{w}(B)$ = t, then $\mathfrak{w}(A \leftrightarrow B)$ = t but $\mathfrak{w}(\mathbf{T}A \leftrightarrow \mathbf{T}B)$ = F.

If t and f are identified, ¬, &, and ∨ become Kleene's weak connectives (*cf.* Kleene [3]). Let $D = \{\mathrm{T}, \mathrm{t}\}$ be the set of designated values, and let a set Γ of formulas *semantically entail* A, briefly $\Gamma \Vdash A$, iff $\forall \mathfrak{w} \in \mathcal{W}$, and $\forall B \in \Gamma$, if $\mathfrak{w}(B) \in D$, then $\mathfrak{w}(A) \in D$. Observe also that $\mathcal{L}$ is a conservative extension of classical logic. That is, for all formulas shared by both $\mathcal{L}$ and classical logic, $\Gamma \Vdash A$ iff the argument from Γ to A is classically valid. For, given any formula A made up from just $\neg$ and &, $v(A)$ conforms to the classical matrix for $\neg$ and &, and $\mathfrak{w}(A)$ is designated iff $v(A) = 1$.

The set of *axioms* for $\mathcal{L}$ is defined as the least set both containing all classical tautologies and all instances of the following axiom schemata, and closed under *modus ponens*:

1. $(A \,\&\, \mathsf{B}A) \rightarrow \mathsf{T}A$
2. $\neg A \,\&\, \mathsf{B}A \rightarrow \mathsf{F}A$
3. $(A \,\&\, \neg\mathsf{B}A) \rightarrow \mathsf{t}A$
4. $(\neg A \,\&\, \neg\mathsf{B}A) \rightarrow \mathsf{f}A$
5. $\mathsf{B}A \rightarrow (\neg\mathsf{t}A \,\&\, \neg\mathsf{f}A)$
6.* $(\mathsf{T}A \vee \mathsf{F}A) \rightarrow \mathsf{B}A$
7.* $\mathsf{C}A \leftrightarrow A$
8. $\mathsf{B}A \leftrightarrow \mathsf{B}\neg A$
9.** $(\mathsf{B}A \,\&\, \mathsf{B}B) \leftrightarrow \mathsf{B}(A \,\&\, B)$
10. $\neg(\mathsf{T}A \,\&\, \mathsf{F}A)$
11. $\neg(\mathsf{t}A \,\&\, \mathsf{f}A)$
12. $\mathsf{BT}A$
13. $\mathsf{BF}A$
14. $\mathsf{Bt}A$
15. $\mathsf{Bf}A$
16. $\mathsf{BB}A$
17.* $\mathsf{BC}A$

Let A be *deducible from* Γ, briefly $\Gamma \vdash A$, iff there is a finite sequence $A_1, \ldots, A_n$ such that $A_n = A$ and A_m, $m < n$, is either an axiom, a member of Γ, or a consequent of previous A_i by *modus ponens*. The *theorems* of $\mathcal{L}$ are all formulas deducible from the empty set. They include the following as well as all instances of 6*, 7*, and 17* if C and B are introduced by definition:

18. $\mathsf{T}A \vee \mathsf{F}A \vee \mathsf{t}A \vee \mathsf{f}A$
19. $\neg(\mathsf{T}A \,\&\, \mathsf{t}A)$
20. $\neg(\mathsf{T}A \,\&\, \mathsf{f}A)$
21. $\neg(\mathsf{F}A \,\&\, \mathsf{t}A)$
22. $\neg(\mathsf{F}A \,\&\, \mathsf{f}A)$
23.** $(\mathsf{B}A \,\&\, \mathsf{B}B) \leftrightarrow \mathsf{B}(A \rightarrow B)$
24. $\mathsf{T}A \rightarrow \mathsf{C}A$
25. $\mathsf{t}A \rightarrow \mathsf{C}A$
26. $\mathsf{F}A \rightarrow \neg\mathsf{C}A$
27. $\mathsf{f}A \rightarrow \neg\mathsf{C}A$
28. $\mathsf{C}A \rightarrow (\mathsf{T}A \vee \mathsf{t}A)$
29. $\mathsf{B}A \rightarrow (\mathsf{T}A \vee \mathsf{F}A)$
30. $\mathsf{B}\neg\mathsf{C}A$
31. $\mathsf{B}\neg\mathsf{B}A$
32. $\mathsf{B}\neg\mathsf{T}A$
33. $\mathsf{B}\neg\mathsf{F}A$
34. $\mathsf{B}\neg\mathsf{t}A$
35. $\mathsf{B}\neg\mathsf{f}A$

Let a set Γ of formulas be *consistent* iff for some A, $\Gamma \nvdash A$, and let Γ be *maximally consistent* iff Γ is consistent and for all A, $A \in \Gamma$ or $\neg A \in \Gamma$. The proof that every consistent set is contained in a maximally consistent set carries over unaltered from classical logic.

Lemma *Any maximally consistent* Γ *is the set of all designated formulas of some* $\mathfrak{w} \in \mathcal{W}$.

Proof: Let Γ be maximally consistent and define v, $\mathfrak{v}$, and $\mathfrak{w}$ as follows: $v(A) = 1$ if $A \in \Gamma$, $v(A) = 0$ otherwise, $\mathfrak{v}(A) = 1$ if $\mathsf{B}A \in \Gamma$, $\mathfrak{v}(A) = 0$ otherwise, and $\mathfrak{w}(A) = \langle v(A), \mathfrak{v}(A)\rangle$. Clearly, Γ is the set of formulas designated by $\mathfrak{w}$. To show $\mathfrak{w} \in \mathcal{W}$, it suffices to show v and $\mathfrak{v}$ satisfy (1)-(3) of the definition of $\mathcal{W}$. Since v and $\mathfrak{v}$ are both functions from $\mathcal{I}$ into $\{1,0\}$, (1) is satisfied. For (2) consider first $\neg A$. If $v(A) = 1$, then $A \in \Gamma$, and $v(\neg A) = 0$. If $v(A) = 0$, then $\neg A \in \Gamma$, and $v(\neg A) = 1$. Consider next $A \,\&\, B$. If $v(A) = v(B) = 1$, then $A, B \in \Gamma$, $A \,\&\, B \in \Gamma$, and $v(A \,\&\, B) = 1$. If $v(A)$ or $v(B)$ is 0, then $\neg A$ or $\neg B$ is in Γ, $\neg(A \,\&\, B) \in \Gamma$, and $v(A \,\&\, B) = 0$. Consider $\mathsf{C}A$. If $\mathfrak{w}(A) \in \{\mathrm{T}, \mathrm{t}\}$, then $A \in \Gamma$, $\mathsf{C}A \in \Gamma$, and $v(\mathsf{C}A) = 1$. If $\mathfrak{w}(A) \in \{\mathrm{F}, \mathrm{f}\}$, then $\neg A \in \Gamma$, $\neg\mathsf{C}A \in \Gamma$, and $v(\mathsf{C}A) = 0$. Consider $\mathsf{B}A$. If $\mathfrak{w}(A) \in \{\mathrm{T}, \mathrm{F}\}$, then $\mathsf{B}A \in \Gamma$, and $v(\mathsf{B}A) = 1$. If $\mathfrak{w}(A) \in \{\mathrm{t}, \mathrm{f}\}$, then $\neg\mathsf{B}A \in \Gamma$, and $v(\mathsf{B}A) = 0$. Consider $\mathsf{T}A$. If $\mathfrak{w}(A) = \mathrm{T}$, then

A, $\mathbf{B}A \in \Gamma$, $\mathbf{T}A \in \Gamma$, and $\upsilon(\mathbf{T}A) = 1$. If $\mathfrak{w}(A) = \mathsf{F}$, then $\neg A$, $\mathbf{B}A \in \Gamma$, $\mathbf{F}A \in \Gamma$, $\neg\mathbf{T}A \in \Gamma$, and $\upsilon(\mathbf{T}A) = 0$. If $\mathfrak{w}(A) = \mathsf{t}$, then $\mathbf{t}A \in \Gamma$, $\neg\mathbf{T}A \in \Gamma$, and $\upsilon(\mathbf{T}A) = 0$. If $\mathfrak{w}(A) = \mathsf{f}$, then $\mathbf{f}A \in \Gamma$, $\neg\mathbf{T}A \in \Gamma$, and $\upsilon(\mathbf{T}A) = 0$. Consider $\mathbf{F}A$. If $\mathfrak{w}(A) = \mathsf{T}$, then $\mathbf{T}A \in \Gamma$, $\neg\mathbf{F}A \in \Gamma$, and $\upsilon(\mathbf{F}A) = 0$. If $\mathfrak{w}(A) = \mathsf{F}$, then $\mathbf{F}A \in \Gamma$, $\upsilon(\mathbf{F}A) = 1$. If $\mathfrak{w}(A) \in \{\mathsf{t}, \mathsf{f}\}$, then $\neg\mathbf{B}A \in \Gamma$, $\neg\mathbf{F}A \in \Gamma$, $\upsilon(\mathbf{F}A) = 0$. Consider $\mathbf{t}A$. If $\mathfrak{w}(A) = \mathsf{T}$, then $\mathbf{T}A \in \Gamma$, $\neg\mathbf{t}A \in \Gamma$, and $\upsilon(\mathbf{t}A) = 0$. If $\mathfrak{w}(A) = \mathsf{F}$, then $\mathbf{F}A \in \Gamma$, $\neg\mathbf{t}A \in \Gamma$, and $\upsilon(\mathbf{t}A) = 0$. If $\mathfrak{w}(A) = \mathsf{t}$, then $\mathbf{t}A \in \Gamma$, and $\upsilon(\mathbf{t}A) = 1$. If $\mathfrak{w}(A) = \mathsf{f}$, then $\mathbf{f}A \in \Gamma$, $\neg\mathbf{t}A \in \Gamma$, and $\upsilon(\mathbf{t}A) = 0$. Consider $\mathbf{f}A$. If $\mathfrak{w}(A) = \mathsf{T}$, then $\mathbf{T}A \in \Gamma$, $\neg\mathbf{f}A \in \Gamma$, and $\upsilon(\mathbf{f}A) = 0$. If $\mathfrak{w}(A) = \mathsf{F}$, then $\mathbf{F}A \in \Gamma$, $\neg\mathbf{f}A \in \Gamma$, and $\upsilon(\mathbf{f}A) = 0$. If $\mathfrak{w}(A) = \mathsf{t}$, then $\mathbf{T}A \in \Gamma$, $\neg\mathbf{f}A \in \Gamma$, and $\upsilon(\mathbf{f}A) = 0$. If $\mathfrak{w}(A) = \mathsf{f}$, then $\mathbf{f}A \in \Gamma$, and $\upsilon(\mathbf{f}A) = 1$. For (3) consider first $\neg A$. If $\mathfrak{v}(A) = 1$, then $\mathbf{B}A \in \Gamma$, $\mathbf{B}\,\neg A \in \Gamma$, and $\mathfrak{v}(A) = 1$. If $\mathfrak{v}(A) = 0$, then $\neg\mathbf{B}A \in \Gamma$, $\neg\mathbf{B}\,\neg A \in \Gamma$, $\mathbf{B}\,\neg A \notin \Gamma$, and $\mathfrak{v}(\neg A) = 0$. Consider $A \,\&\, B$. If $\mathfrak{v}(A) = \mathfrak{v}(B) = 1$, then $\mathbf{B}A$, $\mathbf{B}B \in \Gamma$, $\mathbf{B}(A \,\&\, B) \in \Gamma$, and $\mathfrak{v}(A \,\&\, B) = 1$. If $\mathfrak{v}(A)$ or $\mathfrak{v}(B)$ is 0, then $\neg\mathbf{B}A$ or $\neg\mathbf{B}B$ is in Γ. In either case $\neg\mathbf{B}(A \,\&\, B) \in \Gamma$ and $\mathfrak{v}(A \,\&\, B) = 0$. For the other connectives observe that since $\mathbf{BC}A$, $\mathbf{BB}A$, $\mathbf{BT}A$, $\mathbf{BF}A$, $\mathbf{Bt}A$, $\mathbf{Bf}A \in \Gamma$, $\mathfrak{v}(\mathbf{C}A) = \mathfrak{v}(\mathbf{B}A) = \mathfrak{v}(\mathbf{T}A) = \mathfrak{v}(\mathbf{F}A) = \mathfrak{v}(\mathbf{t}A) = \mathfrak{v}(\mathbf{f}A) = 1$, no matter what $\mathfrak{v}(A)$ is.

Theorem $\Gamma \vdash A$ *iff* $\Gamma \Vdash A$.

Proof: (1) Let $\Gamma \vdash A$. Then there exist a finite sequence $A_1, \ldots, A_n$ such that $A_n = A$ and for all A_m, $m < n$, A_n is either an axiom, a member of Γ, or a consequent by *modus ponens* of previous members. Assume that $\forall B \in \Gamma$, $\mathfrak{w}(B) \in D$. But then since all the axioms are designated by any $\mathfrak{w}$, and *modus ponens* preserves designation, $\mathfrak{w}(A) \in D$. (2) Assume $\Gamma \nvdash A$. Then $\Gamma \cup \{\neg A\}$ is consistent and contained in some maximally consistent Δ. Further there is a $\mathfrak{w}$ such that Δ is the set of designated formulas of $\mathfrak{w}$. Hence $\mathfrak{w}$ satisfies Γ, yet $\mathfrak{w}(A) \notin D$. Hence $\Gamma \nVdash A$. Q.E.D.

This axiom system is also adaptable to Herzberger's 2-dimensional rendering of Kleene's strong connectives. Let $*\mathcal{W}$ be defined like $\mathcal{W}$ except that clause (3) is altered as follows:

$$\mathfrak{v}(A \,\&\, B) = 1 \text{ if } \upsilon(A) = 0 \text{ and } \mathfrak{v}(A) = 1, \text{ or } \upsilon(B) = 0 \text{ and } \mathfrak{v}(B) = 1,$$
$$\text{or } \mathfrak{v}(A) = \mathfrak{v}(B) = 1;\ \mathfrak{v}(A \,\&\, B) = 0 \text{ otherwise.}$$

We retain the same abbreviations and defined connectives as before. The truth tables remain the same except for the following changes.

*II

&	T	F	t	f
T	1	1	0	0
F	1	1	1	1
t	0	1	0	0
f	0	1	0	0

I × *II

&	T	F	t	f	v	T	F	t	f	→	T	F	t	f
	T	F	t	f		T	T	T	T		T	F	t	f
	F	F	F	F		T	F	t	f		T	T	T	T
	t	F	t	f		T	t	t	t		T	f	t	f
	f	F	f	f		T	f	t	f		T	t	t	t

The tables for the strong connectives are obtained by identifying t and f with N. (*Cf.* Kleene [4], pp. 334-335.) Also, the new language $*\mathcal{L} = \langle \mathcal{I}, *\mathcal{W} \rangle$ remains a conservative extension of classical logic. For the axiomatization, all the previous schemata are retained except 9** which is replaced by

*9. $\mathbf{B}(A \mathbin{\&} B) \leftrightarrow (\mathbf{F}A \vee \mathbf{F}B \vee (\mathbf{B}A \mathbin{\&} \mathbf{B}B))$.

The list of previous theorems remains unchanged except for 23** which is replaced by:

*23. $\mathbf{B}(A \rightarrow B) \leftrightarrow (\mathbf{F}A \vee \mathbf{T}B \vee (\mathbf{B}A \mathbin{\&} \mathbf{B}B))$.

The proof of the soundness and completeness results remains the same except that the proof of the lemma for clause (3) of the definition of $*\mathcal{W}$ should be altered as follows: Consider $A \mathbin{\&} B$. If $\boldsymbol{v}(A) = \boldsymbol{v}(B) = \mathfrak{v}(A) = \mathfrak{v}(B) = 1$, then $\mathbf{B}A$, $\mathbf{B}B \in \Gamma$, $\mathbf{B}(A \mathbin{\&} B) \in \Gamma$, and $\mathfrak{v}(A \mathbin{\&} B) = 1$. If $\boldsymbol{v}(A) = 0$ and $\mathfrak{v}(A) = 1$, or $\boldsymbol{v}(B) = 0$ and $\mathfrak{v}(B) = 1$, then either $\neg A$, $\mathbf{B}A \in \Gamma$ or $\neg B$, $\mathbf{B}B \in \Gamma$, either $\mathbf{F}A \in \Gamma$ or $\mathbf{F}B \in \Gamma$, $\mathbf{B}(A \mathbin{\&} B) \in \Gamma$, and $\mathfrak{v}(A \mathbin{\&} B) = 1$. If $\mathfrak{v}(A) = \mathfrak{v}(B) = 0$, then $\neg\mathbf{B}A$, $\neg\mathbf{B}B \in \Gamma$, $\neg\mathbf{B}(A \mathbin{\&} B) \in \Gamma$, and $\mathfrak{v}(A \mathbin{\&} B) = 0$.

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