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LEŚNIEWSKI'S ONTOLOGY EXTENDED WITH THE AXIOM OF CHOICE

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Introduction This dissertation* deals with the Axiom of Choice in the field of Leśniewski's Ontology. Ontology, a theory of pure logic structured along the lines of a logical type theory, was developed by Stanisław Leśniewski (1886-1939) as a result of his own intensive analysis of the logical paradoxes and his dissatisfaction with the work of Russell and Whitehead in *Principia Mathematica* [34] and was intended to provide a secure and intuitively acceptable logical foundation for the formal development of mathematics. Now the importance of the Axiom of Choice in the development of classical mathematics is well known. In the field of Set Theory it has been intensively studied (*cf.*, e.g., [2], [3], [11], [12], and [24]). Less attention has been paid to the behavior of the Axiom of Choice in a type theory (*cf.*, e.g., [34] and [4]), and only minimal attention has been given to it in the field of Ontology (*cf.* [9] or [10]). However, it is well known that within the field of a logical type theory, the Axiom of Choice cannot be added to a logical basis by the mere addition of a single new axiom, but must be added as a spectrum of formulas each expressing the Axiom of Choice for some fixed logical type. It is also known that certain formulas known to be equivalent to the Axiom of Choice in the field of Set Theory are not directly equivalent in the field of a type theory.

In this dissertation we will show, first, that certain principles known to be equivalent to the Axiom of Choice in the field of Set Theory are also equivalent in Ontology. In particular we show the equivalence of the Axiom of Choice, the Kuratowski-Zorn Lemma, and the Well Ordering Principle though it will be noted that the sense of this equivalence in Ontology is analogous to, but not identical with, the sense of their equivalence in Set Theory. Second, since Ontology's type theoretical structure prevents the addition of the Axiom of Choice as a single formula, but requires the addition of a spectrum of formulas, we give a precise syntactical description of the conditions these formulas must meet. More specifically we provide a modification to the Rule of Ontology which will insure that the Axiom of Choice is available for each semantic category (logical type) expressible in Ontology.

The body of this dissertation is divided into three chapters. Chapter One is entirely introductory in nature and contains a brief description of Ontology as well as a large collection of basic theorems and definitions necessary for showing the equivalence of the three principles mentioned

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above. The actual equivalence of the Axiom of Choice, Zorn's Lemma, and the Well Ordering Principle in the field of Ontology is demonstrated in Chapter Two. Finally, Chapter Three contains a precise syntactical description of the modification to the Rule of Ontology required in order that the Axiom of Choice be available for each semantic category.

CHAPTER I: LEŚNIEWSKI'S ONTOLOGY

1.0 *General Description* In this chapter we develop a system of Ontology to a degree sufficient for the work of Chapter Two. The theorems and definitions of this development are contained in subsequent sections while in this section we give a general description of Ontology. Since rather detailed discussions of Ontology can be found elsewhere (*cf.* especially [5], [9], [13], and [29]), the material given here is intended only to highlight basic features and those details especially pertinent to our work.

Ontology can be characterized as a general theory dealing with the logical relationships between and among names. It can be developed from a single proper axiom containing a single primitive term. Ontology's single primitive term is the 'ε' which appears in sentences of the form

$$A \,\varepsilon\, a$$

read: A is a. The intended interpretation of the 'ε' is the copula (is, jest, est) in sentences such as "Elizabeth is female", "Margaret jest Margaret" or "Ann est uxor". Technically the 'ε' is a functor which forms a sentence when applied to two arguments. In accord with Leśniewski's view that names are capable of having no referent, exactly one referent, or many referents, both of the arguments connected by the 'ε' are regarded as names. Thus the 'ε' can be described as a sentence forming functor for two name arguments.[1]

There are several different formulas which can serve as Ontology's sole proper axiom,[2] but the one adopted here is:

$$A0^3 \quad [Aa] \therefore A \,\varepsilon\, a \;.\!\equiv\!:\, [\exists B] \,.\, B \,\varepsilon\, A : [BC] : B \,\varepsilon\, A \,.\, C \,\varepsilon\, A \;.\!\supset\!.\; B \,\varepsilon\, C :$$
$$[B] : B \,\varepsilon\, A \;.\!\supset\!.\; B \,\varepsilon\, a$$

Intuitively $A0$ can be regarded as stating the conditions for the truth of a sentence of the form '$A \,\varepsilon\, a$', i.e., A is a if and only if: 1) A is an unempty name, 2) A is a unique name, and 3) anything which is A is also a.

Though $A0$ can serve as Ontology's sole proper axiom, Ontology presupposes as its foundation Leśniewski's theory of the logic of propositions called Protothetic. Therefore, Ontology can be considered as an extension of Protothetic. The relationship between Protothetic and Ontology is somewhat similar to that between the propositional calculus and the ω-order functional calculus. Details regarding Protothetic can be found in [5], [15], [23], and [28]. In the present work we will assume that an adequately developed Protothetical base has been provided and no explicit reference to purely Protothetical reasoning is made.

The formal structure of Ontology is determined by Leśniewski's theory of semantic categories. With the exception of quantifiers, every well formed expression of Ontology can be assigned to exactly one semantic category: either to one of two primitive categories—statements or names—or to one of a potentially infinite number of functor categories which are generated from the two primitive categories. For example, Ontology's single axiom, $A0$, as well as its component '$[\exists B].B\,\varepsilon A$' are of the semantic category of sentences. Any individual name variable in $A0$ is, naturally, in the semantic category of names. The semantic category of a functor is determined by what it forms, e.g., something of the category of statements or the category of names, together with the number, order, and semantic categories of its arguments. This information is easily presented using a system of notation developed by Ajdukiewicz in [1]. Letting 'S' stand for the semantic category of statements and 'n' stand for the semantic category of names we can represent the semantic category of the 'ε', for example, as S/nn (i.e., sentence forming functor for two nominal arguments) or of the '≡' as S/SS (i.e., sentence forming functor for two sentential arguments). However, though the Ajdukiewicz notation provides a convenient extra-systemic way of referring to the semantic category of a functor, within the system the semantic category of a functor is indicated by the use of variously shaped parentheses. Canonically, Ontology requires the use of a Łukasiewicz style notation in which every functor precedes its arguments which are themselves enclosed in parentheses.[4] Given this, both the shape of the enclosing parentheses and the number of enclosed arguments can play a role in identifying the semantic category of a functor. Hence, in Ontology functors of different semantic categories must use different style parentheses and once a particular style has been chosen for use with a particular semantic category it must always be used for other functors of that same category. Parentheses thus serve to unambiguously identify—syntactically—the semantic category of a functor.

Besides the semantic categories of its Protothetical base,[5] a system of Ontology at its inception contains only two other semantic categories—n and S/nn—both of which are introduced by the axiom of Ontology. New categories are introduced by defining a constant functor of the new category. When and only when a category has been introduced can variables for that semantic category be used. There is, however, no pre-axiomatic prescription as to which semantic categories are actually introduced. It should be emphasized that the Rule of Ontology provides for the definition of constant and variable functors of a potentially infinite number of semantic categories and for the use variables ranging over the functors of those semantic categories. The following inductive definition indicates all possible semantic categories which are capable of being introduced into Ontology.

1. S and n are semantic categories.
2. If α and $\beta_1, \beta_2, \ldots \beta_n$ are semantic categories, the $\alpha/\beta_1, \beta_2, \ldots, \beta_n$ is a semantic category.
3. Only categories generated by steps 1 and 2 are semantic categories.

Included in the collection of possible functors are, besides the strictly sentential or nominal functors, the so-called many-link functors. Many-link functors are functor forming functors, that is, they form functors rather than sentences or names when they are applied to their arguments. The Ajdukiewicz symbol for a many-link functor is of the general form $\alpha/\beta_1, \beta_2, \ldots \beta_n$, but α is anything other than S or n.

The sense in which Leśniewski's theory of semantic categories determines Ontology's formal structure can be understood from the fact that Ontology's Rule permits, e.g., substitution for a variable only if what is substituted is of the same semantic category as the variable for which the substitution is made. Thus the theory of semantic categories characterizes Ontology as a kind of logical type theory.[6]

Development of Ontology from its single axiom is governed by a single multi-part rule composed of seven interrelated directives. The directives state the conditions under which a new formula can be added as a thesis. Ontology's Rule is formulated so as to prohibit certain additions where what is prohibited depends on what theses have already been added. However, the Rule dictates no specific additions and that is why we speak of *a given development* of Ontology or *a system of* Ontology rather than *the* development or *the* system. A system of Ontology is distinguished by which theses have actually been added and not by which theses are possible. A formal presentation of the Rule of Ontology requires a long sequence of definitions called terminological explanations which clarify the syntactical terms used in its statement.[7] (Many of the terminological explanations used in a formal statement of Ontology's Rule as well as illustrations of how they are used can be found in Chapter Three.) Given below is an informal description of the Rule.[8]

Assuming B is the last thesis added to a given development of Ontology, a new thesis, A, can be added if and only if the conditions included in one of the following directives are fulfilled:

1) *A is the result of detachment from a biconditional.* That is, among the theses added prior to B or B itself there must be theses of the forms: 'C' and '$C \equiv A$'.

2) *A is the result of distributing one or more quantifiers of a previously added thesis of biconditional form.* This directive stipulates that if a thesis added prior to B or B itself is a universal generalization of the form:

$$[a_1 \ldots a_n] : \alpha .\equiv. \beta$$

where $a_1 \ldots a_n$ are variables free in α or β or both, then some or all of the variables can be distributed between α and β. For example, we could obtain:

$$[a_2 \ldots a_n] : [a_1] . \alpha .\equiv. [a_1] . \beta$$

or $2^n - 2$ other distributions. Included in this directive is the proviso that in any distribution which results in a quantifier containing a variable which binds nothing, then that variable is dropped from the quantifier. This directive is the only one concerning quantifiers. All of the familiar quantification rules can be obtained as derived rules.

3) *A is the result of a uniform substitution into a previously added thesis.* This directive permits replacement of a variable bound by a main general quantifier by an expression of the same semantic category provided that no variables in the replacing expression are "captured" by interior quantifiers and that all variables free in the replacing expression are bound by main general quantifiers of the resultant.

4) *A is a well formed definition of the Protothetical type.* Definitions of this type have the general form:

$$[a_1 \dots a_n] : \alpha \ .\equiv.\ \beta$$

where α stands for the definiens, β stands for the definiendum and $a_1 \dots a_n$ represent the variables free in α and β. The main restrictions imposed by this directive are: that all variables free in β belong to semantic categories already introduced into the system; that β contain only one occurrence of each free variable; that no free variables occur in the definition; and that if the definition introduces a new semantic category, appropriate new style parentheses are used.

5) *A is a well formed definition of the Ontological type.* Ontological definitions have the general form:

$$[A, a_1 \dots a_n] : \beta \ .\equiv.\ A \,\varepsilon\, \alpha$$

where A is a variable in the category 'n', α is the definiendum, and $a_1 \dots a_n$ are variables free in α. Ontological type definitions allow definition of names and name forming functors. The main restrictions imposed here are: that the definition as a whole contains no free variables or vacuous quantifiers; that $a_1 \dots a_n$ be variables for semantic categories already introduced; that if α is many-link and introduces a new semantic category, then appropriate new style parentheses must be used; and, most important, that the definiens, β, either be of the form '$A \,\varepsilon\, \gamma$' or be a conjunction with a conjunct of this form.

6) *A is an extensionality thesis of the Protothetical type.* Extensionality theses of the Protothetical type are of the general form:

$$[fg] \therefore [a_1 \dots a_n] : f(a_1 \dots a_n) \ .\equiv.\ g(a_1 \dots a_n) \ :\equiv:\ [\theta] : \theta\langle f\rangle \ .\equiv.\ \theta\langle g\rangle$$

Here f and g are both of some given semantic category, say α, and θ is of category S/α. The main restriction here is that the semantic categories of all functors and variables used in expressing the thesis must have already been introduced into the system.

7) *A is an extensionality thesis of the Ontological type.* Extensionality theses of this type are of the general form:

$$[\alpha\beta] \therefore [A, a_1 \dots a_n] : A \,\varepsilon\, \alpha\langle a_1 \dots a_n\rangle \ .\equiv.\ A \,\varepsilon\, \beta\langle a_1 \dots a_n\rangle \ :\equiv:\cdot$$
$$[\mu] : \mu\{\alpha\} \ .\equiv.\ \mu\{\beta\}$$

Here too it is assumed that the semantic categories of all components of the extensionality thesis have already been introduced.

The preceding informal description of Ontology's Rule nearly completes our review. Only one other feature of Ontology need be mentioned. The great flexibility allowed by the Rule of Ontology regarding the definitions of constant functors and the introduction of new semantic categories makes possible the definition of analogous functors. For example, among expressions which it is possible to introduce into Ontology are those which correspond to the set theoretical notions of sets, families of sets, families of families of sets, etc. It is possible to define inclusion for sets, inclusion for families of sets, inclusion for families of families of sets, etc. Each of these inclusions are different in that they are defined for "objects" of different categories. However, the analogies between them are so strong that it is possible to show that for any thesis containing a functor of a lower semantic category, one can prove a completely analogous thesis containing an analogous functor of any higher semantic category.[9] Thus any proof containing functors of some given semantic categories can be "repeated" so that an analogous proof for functors of a higher semantic category is obtainable. This meta-result will play a role in Chapter Two where we argue that because two formulas containing functors of certain semantic categories are related by implication, all analogous formulas containing analogous functors for higher semantic categories are also related by implication.

Finally, before turning to the development and presentation of those definitions and theorems necessary for the work of Chapter Two, two comments on the style of exposition adopted here are warranted. First, the Rule of Ontology is so stated as to direct the development of a completely axiomatic system. In what follows we shall use familiar natural deduction techniques in presenting the demonstrations of various theses. Thus the demonstrations given are not strict proofs but proof outlines. Conversion of these outlines into strict proofs would be tedious but theoretically and practically possible. Second, since Ontology uses the shape of parentheses to indicate the semantic category of a functor and since our demonstrations make use of functors from a great number of different semantic categories, we adopt as an informal convention the use of subscripted parentheses. Strictly speaking Ontology requires the use of single continuously connected symbols in order that part of one symbol not be misidentified as part of another, but the procedure adopted here keeps the stock of symbols necessary for our presentation within reasonable bounds and presents no untoward difficulties.

1.1 *Elementary Theorems and Definitions* This section begins with the single axiom of Ontology and contains elementary theorems and definitions basic to subsequent work. Many of the definitions presented here give the Ontological analogues of fundamental set theoretical notions. For the most part, the theorems and definitions presented are well known in the literature of Ontology though in many instances the shapes of the symbols or parentheses used in defining particular functors differ from those used by other authors. Since the proofs of theorems listed in this section are trivial or can be found elsewhere (*cf.*, e.g., [13], [5], [26]) no demonstrations are given.

A0 $[Aa] \therefore \varepsilon\{Aa\} .\!\equiv\!: [\exists B] . \varepsilon\{BA\} : [CD] : \varepsilon\{CA\} . \varepsilon\{DA\} .\!\supset\!. \varepsilon\{CD\} : [C] : \varepsilon\{CA\} .\!\supset\!. \varepsilon\{Ca\}$

D1.1.01 $[ab] \therefore [A] : A \,\varepsilon\, a .\!\equiv\!. A \,\varepsilon\, b :\!\equiv\!. \circ\{ab\}$

a equals b. This defines equality for general names.

As we remarked earlier, canonical form in Ontology prescribes that functors always precede their arguments so that the shape of the enclosing parentheses can aid in fixing the semantic category of the functor. Nonetheless, it is often easier to read expressions involving binary functors if the functor is written between its arguments. We will always adhere to a canonical form when writing the definiendum of definitions. Thus the definition of a functor will always unambiguously identify the semantic category of the functor being defined. However, in instances where readability is improved, we will often write the functor between its arguments. Any ambiguity can be resolved by referring back to the definition. This relaxation of prescribed form is illustrated in the following theses.

T1.1.1 $[ab] : a \circ b .\!\equiv\!. b \circ a$ [*D1.1.01*]
T1.1.2 $[a] . a \circ a$ [*D1.1.01*]
T1.1.3 $[abc] : a \circ b . b \circ c .\!\supset\!. a \circ c$ [*D1.1.01*]

D1.1.02 $[a] : [\exists A] . A \,\varepsilon\, a .\!\equiv\!. \, !\{a\}$

a is unempty. This definition introduces the semantic category S/n, often called predicates.

E1.1.01 $[ab] \therefore [A] : A \,\varepsilon\, a .\!\equiv\!. A \,\varepsilon\, b :\!\equiv\!: [\theta] : \theta\{a\} .\!\equiv\!. \theta\{b\}$

Given above is the Ontological thesis of extensionality for names. Notice that it follows *D1.1.02*, for according to the directive for extensionality, this thesis could not be introduced before the semantic category S/n had been introduced.

T1.1.4 $[ab] \therefore a \circ b . \theta\{a\} .\!\supset\!. \theta\{b\}$ [*D1.1.01*; *E1.1.01*]
T1.1.5 $[ab] : a \circ b . \sim(\theta\{a\}) .\!\supset\!. \sim(\theta\{b\})$ [*T1.1.4*]

D1.1.03 $[\rho\theta] \therefore [a] : \rho\{a\} .\!\equiv\!. \theta\{a\} :\!\equiv\!. \circ \,\text{⟨}\rho\theta\text{⟩}$

ρ equals θ. This defines equality for predicates. The analogy between equality for names and quality for predicates is indicated by employing an equiform symbol, '$\circ$', for both. Such use of equiform symbols for analogous constants will be frequent, but three points ought to be emphasized. First, '$\circ\{ab\}$' and '$\circ \,\text{⟨}\rho\theta\text{⟩}$', when used, express equality for *different* semantic categories and thus, when used, assert different propositions. Second, the difference in the shapes of the parentheses used to enclose the arguments serve to indicate the different semantic categories. Third, the shapes of the variables used in the definition play no formal role in determining the semantic category of any functor. It is only for convenience that Latin letters are used in some contexts and Greek letters in others.

T1.1.6 $[\rho\theta]:\rho\circ\theta\ .\equiv.\ \theta\circ\rho$ [*D1.1.03*]
T1.1.7 $[\theta].\theta\circ\theta$ [*D1.1.03*]
T1.1.8 $[\mu\rho\theta]:\mu\circ\rho\,.\,\rho\circ\theta\ .\supset.\ \mu\circ\theta$ [*D1.1.03*]
T1.1.9 $[\mu\rho\theta]:\mu\circ\rho\,.\sim\!(\rho\circ\theta)\ .\supset.\ \sim(\mu\circ\theta)$ [*T1.1.8*]
T1.1.10 $[a\,\mu\rho]:\mu\circ\rho\,.\,\mu\{a\}\ .\supset.\ \rho\{a\}$ [*D1.1.03*]
T1.1.11 $[a\,\mu\rho]:\mu\circ\rho\,.\sim(\mu\{a\})\ .\supset.\ \sim(\rho\{a\})$ [*T1.1.10*]

D1.1.04 $[\rho\theta]\therefore[a]:\rho\{a\}\ .\supset.\ \theta\{a\}\ :\equiv.\ \subset\langle\rho\theta\rangle$

ρ is included in θ. This defines inclusion for predicates.

T1.1.12 $[\theta].\theta\subset\theta$ [*D1.1.04*]
T1.1.13 $[\mu\rho\theta]:\mu\subset\rho\,.\,\rho\subset\theta\ .\supset.\ \mu\subset\theta$ [*D1.1.04*]
T1.1.14 $[a\rho\theta]:\rho\subset\theta\,.\,\rho\{a\}\ .\supset.\ \theta\{a\}$ [*D1.1.04*]
T1.1.15 $[\rho\theta]:\rho\subset\theta\,.\,\theta\subset\rho\ .\equiv.\ \rho\circ\theta$ [*D1.1.04*; *D1.1.03*]
T1.1.16 $[\mu\rho\theta]:\mu\circ\rho\,.\,\rho\subset\theta\ .\supset.\ \mu\subset\theta$ [*T1.1.13*; *T1.1.15*]

D1.1.05 $[a\rho\theta]\therefore\rho\{a\}.\mathrm{v}.\,\theta\{a\}\ :\equiv.\ \cup\langle\rho\theta\rangle\{a\}$

This defines the union of two predicates and introduces the semantic category (S/n)/((S/n)(S/n)).

T1.1.17 $[\rho\theta].\rho\subset\cup\langle\rho\theta\rangle$ [*D1.1.04*; *D1.1.05*]
T1.1.18 $[a\,\mu\rho\theta]\therefore\mu\subset\cup\langle\rho\theta\rangle\,.\,\mu\{a\}\ .\supset:\ \rho\{a\}\,.\mathrm{v}.\,\theta\{a\}$ [*T1.1.14*; *D1.1.05*]
T1.1.19 $[a\,\mu\rho\theta]:\cup\langle\mu\rho\rangle\subset\theta\,.\,\rho\{a\}\ .\supset.\ \theta\{a\}$ [*T1.1.14*; *D1.1.05*]
T1.1.20 $[\mu\rho\theta].\cup\langle\mu\rho\rangle\subset\theta\ .\supset.\ \rho\subset\theta$ [*T1.1.19*; *D1.1.04*]

D1.1.06 $[a\rho\theta]:\rho\{a\}.\,\theta\{a\}\ .\equiv.\ \cap\langle\rho\theta\rangle\{a\}$

This defines the intersections of two predicates.

T1.1.21 $[\mu\rho].\cap\langle\rho\mu\rangle\subset\rho$ [*D1.1.06*; *D1.1.04*]

D1.1.07 $[a\rho\theta]:\rho\{a\}.\sim(\theta\{a\})\ .\equiv.\ -\langle\rho\theta\rangle\{a\}$

This defines the difference of two predicates.

T1.1.22 $[a\,\mu\rho\theta]:\mu\subset\cup\langle\rho\theta\rangle\,.-\langle\mu\theta\rangle\{a\}\ .\supset.\ \rho\{a\}$ [*D1.1.07*; *T1.1.18*]
T1.1.23 $[\mu\rho\theta]:\mu\subset\cup\langle\rho\theta\rangle\ .\supset.\ -\langle\mu\theta\rangle\subset\rho$ [*T1.1.22*; *D1.1.04*]

D1.1.08 $[\theta]:[\exists a].\theta\{a\}\ .\equiv.\ !\langle\theta\rangle$

θ is unempty. This definition introduces the semantic category S/(S/n).

E1.1.02 $[\rho\mu]\therefore[a].\rho\{a\}\ .\equiv.\ \mu\{a\}\ :\equiv:\ [\xi]:\xi\langle\rho\rangle\ .\equiv.\ \xi\langle\mu\rangle$

Given above is the Protothetical extensionality thesis for predicates. As required by the directive for extensionality, it is presented after the semantic category S/(S/n) has been introduced.

D1.1.09 $[a\xi]:[\exists\theta].\xi\langle\theta\rangle.\theta\{a\}\ .\equiv.\ \cup\langle\!\langle\xi\rangle\!\rangle\{a\}$

a is in the generalized union of the family of sets determined by ξ. $\cup$ is in the semantic category (S/n)/(S/(S/n)).

T1.1.24	$[\theta\xi]:\xi\langle\theta\rangle\ .\supset.\ \theta\subset\cup\langle\!\langle\xi\rangle\!\rangle$	[*D1.1.09*; *D1.1.04*]

D1.1.010 $[a\xi]\therefore[\theta]:\xi\langle\theta\rangle\ .\supset.\ \theta\{a\}\ :\equiv.\ \cap\langle\!\langle\xi\rangle\!\rangle\{a\}$

a is in the generalized intersection of the family of sets determined by ξ.

T1.1.25	$[a\theta\xi]:\xi\langle\theta\rangle\,.\,\cap\langle\!\langle\xi\rangle\!\rangle\{a\}\ .\supset.\ \theta\{a\}$	[*D1.1.010*; *D1.1.04*]
T1.1.26	$[a\theta\xi]:\xi\langle\theta\rangle\ .\supset.\ \cap\langle\!\langle\xi\rangle\!\rangle\subset\theta$	[*T1.1.25*; *D1.1.04*]
T1.1.27	$[\theta\xi]:\theta\subset\cap\langle\!\langle\xi\rangle\!\rangle\,.\,\xi\langle\theta\rangle\ .\supset.\ \theta\circ\cap\langle\!\langle\xi\rangle\!\rangle$	[*T1.1.26*; *T1.1.15*]

D1.1.011 $[\psi\phi]\therefore[ab]\,.\,\psi\{ab\}\ .\equiv.\ \phi\{ab\}\ :\equiv.\ \circ\langle\!\langle\psi\phi\rangle\!\rangle$

ψ equals ϕ. This defines equality for binary connections.

T1.1.28	$[\psi\phi]:\psi\circ\phi\ .\equiv.\ \phi\circ\psi$	[*D1.1.011*]
T1.1.29	$[\psi]\,.\,\psi\circ\psi$	[*D1.1.011*]
T1.1.30	$[\psi\phi\Phi]:\psi\circ\phi\,.\,\phi\circ\Phi\ .\supset.\ \psi\circ\Phi$	[*D1.1.011*]
T1.1.31	$[ab\psi\phi]:\psi\circ\phi\,.\,\psi\{ab\}\ .\supset.\ \phi\{ab\}$	[*D1.1.011*]

D1.1.012 $[\psi\phi]\therefore[ab]:\psi\{ab\}\ .\supset.\ \phi\{ab\}\ :\equiv.\ \subset\langle\!\langle\psi\phi\rangle\!\rangle$

Connection ψ is included in connection ϕ.

T1.1.32	$[\phi]\,.\,\phi\subset\phi$	[*D1.1.012*]
T1.1.33	$[\phi\psi\Phi]:\phi\subset\psi\,.\,\psi\subset\Phi\ .\supset.\ \phi\subset\Phi$	[*D1.1.012*]
T1.1.34	$[\phi\psi]:\phi\subset\psi\,.\,\psi\subset\phi\ .\supset.\ \psi\circ\phi$	[*D1.1.012*; *D1.1.011*]
T1.1.35	$[ab\psi\phi]:\psi\subset\phi\,.\,\psi\{ab\}\ .\supset.\ \phi\{ab\}$	[*D1.1.012*]
T1.1.36	$[abc\phi]:\phi\{ab\}\,.\,b\circ c\ .\supset.\ \phi\{ac\}$	[*E1.1.01*]
T1.1.37	$[abc\phi]:\phi\{ab\}\,.\,a\circ c\ .\supset.\ \phi\{cb\}$	[*E1.1.01*]

D1.1.013 $[a\phi]:[\exists b]\,.\,\phi\{ab\}\ .\equiv.\ \mathbf{D}\langle\!\langle\phi\rangle\!\rangle\{a\}$

a is in the domain of connection ϕ. **D** is in semantic category (S/n)/(S/nn).

D1.1.014 $[b\phi]:[\exists a]\,.\,\phi\{ab\}\ .\equiv.\ \mathbf{\Omega}\langle\!\langle\phi\rangle\!\rangle\{b\}$

b is in the counter-domain or range of connection ϕ.

T1.1.38	$[ab\phi]:\phi\{ab\}\ .\supset.\ \mathbf{D}\langle\!\langle\phi\rangle\!\rangle\{a\}\,.\,\mathbf{\Omega}\langle\!\langle\phi\rangle\!\rangle\{b\}$	[*D1.1.013*; *D1.1.014*]

D1.1.015 $[\phi]\therefore[ab]:\phi\{ab\}\,.\,\phi\{ac\}\ .\supset.\ b\circ c\ :\equiv.\ \rightrightarrows\langle\!\langle\phi\rangle\!\rangle$

ϕ is a many-one or functional binary connection. $\rightrightarrows$ is in the semantic category S/(S/nn).

E1.1.03 $[\phi\psi]\therefore[ab]:\phi\{ab\}\ .\equiv.\ \psi\{ab\}\ :\equiv:\ [\eta]:\eta\langle\!\langle\phi\rangle\!\rangle\ .\equiv.\ \eta\langle\!\langle\psi\rangle\!\rangle$

E1.1.03 is the Protothetical extensionality thesis for connections.

D1.1.016 $[ab\mu\rho\phi]:\phi\{ab\}\,.\,\mu\{a\}\,.\,\rho\{b\}\ .\equiv.\ \mathbf{T}\langle\!\langle\!\langle\phi\mu\rho\rangle\!\rangle\!\rangle\{ab\}$

$\mathbf{T}\langle\!\langle\!\langle\phi\mu\rho\rangle\!\rangle\!\rangle$ denotes the restriction of a connection such that its first argument is an element of μ and its second argument is an element of ρ. **T** is in semantic category (S/nn)/((S/nn)(S/n)(S/n)).

T1.1.39	$[\rho\mu\phi]:\rho\circ\mu\ .\supset.\ \mathbf{T}\langle\!\langle\!\langle\phi\rho\rho\rangle\!\rangle\!\rangle\circ\mathbf{T}\langle\!\langle\!\langle\phi\mu\mu\rangle\!\rangle\!\rangle$	[*D1.1.016*; *E1.1.02*]

D1.1.017 $[ab\eta]:[\exists\phi]\,.\,\eta\langle\!\langle\phi\rangle\!\rangle\,.\,\phi\{ab\}\ .\equiv.\ \cup\langle\!\langle\!\langle\eta\rangle\!\rangle\!\rangle\{ab\}$

$\cup \{\!\!\{\eta\}\!\!\}$ denotes the generalized union of a family of connections determined by η. $\cup$ is in semantic category $(\mathsf{S}/\mathsf{nn})/\mathsf{S}/(\mathsf{S}/\mathsf{nn})$.

1.2 *Order Connections* In this section we present a number of definitions relating to the ordering properties of connections as well as a number of selected theorems involving these definitions.

D1.2.01 $[\theta\phi] : [a] \,.\, \theta\{a\} \,.\!\supset\!.\; \phi\{aa\} :\equiv.\; \mathsf{R}\langle\theta\rangle \{\!\!\{\phi\}\!\!\}$

Connection ϕ is reflexive in θ. R is of semantic category $(\mathsf{S}/(\mathsf{S}/\mathsf{nn}))/(\mathsf{S}/\mathsf{n})$.

T1.2.1 $[\theta] \,.\, \mathsf{R}\langle\theta\rangle \{\!\!\{\circ\}\!\!\}$ [*T1.1.2*; *D1.2.01*]
T1.2.2 $[\theta\psi\phi] : \mathsf{R}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \psi \subset \theta \,.\!\supset\!.\; \mathsf{R}\langle\psi\rangle \{\!\!\{\phi\}\!\!\}$ [*T1.1.14*; *D1.2.01*]

D1.2.02 $[\theta\phi] \therefore [abc] : \theta\{a\} \,.\, \theta\{b\} \,.\, \theta\{c\} \,.\, \phi\{ab\} \,.\, \phi\{bc\} \,.\!\supset\!.\; \phi\{ac\} :\equiv.\; \mathbf{T}\langle\theta\rangle \{\!\!\{\phi\}\!\!\}$

Connection ϕ is transitive in θ.

T1.2.3 $[\theta] \,.\, \mathbf{T}\langle\theta\rangle \{\!\!\{\circ\}\!\!\}$ [*T1.1.3*; *D1.2.02*]
T1.2.4 $[\theta\psi\phi] : \mathbf{T}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \psi \subset \theta \,.\!\supset\!.\; \mathbf{T}\langle\psi\rangle \{\!\!\{\phi\}\!\!\}$ [*T1.1.14*; *D1.2.01*]

D1.2.03 $[\theta\phi] \therefore [ab] : \theta\{a\} \,.\, \theta\{b\} \,.\, \phi\{ab\} \,.\, \phi\{ba\} \,.\!\supset\!.\, a \circ b :\equiv.\; \mathsf{A}\langle\theta\rangle \{\!\!\{\phi\}\!\!\}$

Connection ϕ is antisymmetric in θ.

T1.2.5 $[\theta] \,.\, \mathsf{A}\langle\theta\rangle \{\!\!\{\circ\}\!\!\}$ [*T1.1.1*; *D1.2.03*]
T1.2.6 $[\theta\mu\phi] : \mathsf{A}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \mu \subset \phi \,.\!\supset\!.\; \mathsf{A}\langle\mu\rangle \{\!\!\{\phi\}\!\!\}$ [*T1.1.14*; *D1.2.03*]

D1.2.04 $[\theta\phi] : \mathsf{R}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \mathbf{T}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \mathsf{A}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\!\equiv\!.\; \mathsf{PO}\langle\theta\rangle \{\!\!\{\phi\}\!\!\}$

Connection ϕ is a partial ordering connection in θ.

T1.2.7 $[\theta] \,.\, \mathsf{PO}\langle\theta\rangle \{\!\!\{\circ\}\!\!\}$ [*T1.2.1*; *T1.2.3*; *T1.2.5*; *D1.2.04*]
T1.2.8 $[\theta\mu\phi] : \mathsf{PO}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \mu \subset \theta \,.\!\supset\!.\; \mathsf{PO}\langle\mu\rangle \{\!\!\{\phi\}\!\!\}$
[*T1.2.2*; *T1.2.4*; *T1.2.6*; *D1.2.04*]
T1.2.9 $[ab\theta\phi] \therefore \mathsf{PO}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \theta\{a\} \,.\, \theta\{b\} \,.\, \phi\{ab\} \,.\!\supset\!:\; \sim(\phi\{ba\}) \,.\!\vee\!.\, a \circ b$
[*D1.2.04*; *D1.2.03*]
T1.2.10 $[ab\theta\phi] : \mathsf{PO}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \theta\{a\} \,.\, \theta\{b\} \,.\, \phi\{ab\} \,.\, \sim(a \circ b) \,.\!\supset\!.\; \sim(\phi\{ba\})$ [*T1.2.9*]

D1.2.05 $[a\theta\phi] \therefore \mathsf{PO}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \theta\{a\} : [b] : \theta\{b\} \,.\, \phi\{ab\} \,.\!\supset\!.\, a \circ b :\equiv.\; \mathsf{MAXL}\, \{\!|\theta\phi|\!\}\, \{a\}$

a is a ϕ-maximal element of θ.

D1.2.06 $[a\theta\phi] \therefore \mathsf{PO}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \theta\{a\} : [b] : \theta\{b\} \,.\!\supset\!.\; \phi\{ab\} :\equiv.\; \mathsf{P}\, \{\!|\theta\phi|\!\}\, \{a\}$

a is the first element of θ with respect to connection ϕ.

T1.2.11 $[ab\theta\phi] : \mathsf{P}\, \{\!|\theta\phi|\!\}\, \{a\} \,.\, \mathsf{P}\, \{\!|\theta\phi|\!\}\, \{b\} \,.\!\supset\!.\, a \circ b$
PR $[ab\theta\phi] \therefore \mathrm{Hp}(2) \,.\!\supset\!:$
3. $\mathsf{PO}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.$ [1; *D1.2.06*]
4. $\mathsf{A}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.$ [3; *D1.2.04*]
5. $\theta\{a\} \,.$ [1; *D1.2.06*]
6. $\theta\{b\} :$ [2; *D1.2.06*]
7. $[d] : \theta\{d\} \,.\!\supset\!.\; \phi\{ad\} :$ [1; *D1.2.06*]
8. $[c] : \theta\{c\} \,.\!\supset\!.\; \phi\{bc\} :$ [2; *D1.2.06*]
9. $\phi\{ab\} \,.$ [6; 7]

10.	$\phi\{ba\}$.	[5; 8]
	$a \circ b$	[5; 6; 9; 10; 4; *D1.2.03*]

T1.2.11 establishes uniqueness of first elements.

D1.2.07 $[\theta\phi] \therefore \mathbf{PO}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} : [\rho] : \rho \subset \theta \,.\, !\{\!\!\{\rho\}\!\!\} \,.\!\supset\!.\, [\exists a] \,.\, \mathbf{P}\,\{\!|\rho\phi|\!\}\{a\} :\equiv.$
$\mathbf{WO}\langle\theta\rangle \{\!\!\{\phi\}\!\!\}$

ϕ well orders θ.

T1.2.12 $[\mu\rho\phi] : \mathbf{WO}\langle\mu\rangle \{\!\!\{\phi\}\!\!\} \,.\, \mu \circ \rho \,.\!\supset\!.\, \mathbf{WO}\langle\rho\rangle \{\!\!\{\phi\}\!\!\}$ [*E1.1.02*]

D1.2.08 $[a\rho\theta\phi] \therefore \mathbf{PO}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \theta\{a\} \,.\, \rho \subset \theta : [b] : \rho\{b\} \,.\!\supset\!.\, \phi\{ba\} :\equiv.$
$\mathbf{UB}\,\{\!|\rho\theta\phi|\!\}\{a\}$

a is a ϕ-upper bound of ρ in θ.

D1.2.09 $[a\rho\theta\phi] \therefore \mathbf{UB}\,\{\!|\rho\theta\phi|\!\}\{a\} : [b] : \mathbf{UB}\,\{\!|\rho\theta\phi|\!\}\{b\} \,.\!\supset\!.\, \phi\{ab\} :\equiv.\, \mathbf{LUB}\,\{\!|\rho\theta\phi|\!\}\{a\}$

a is the ϕ-least upper bound of ρ in θ.

T1.2.13 $[ab\rho\theta\phi] : \mathbf{LUB}\,\{\!|\rho\theta\phi|\!\}\{a\} \,.\, \mathbf{UB}\,\{\!|\rho\theta\phi|\!\}\{b\} \,.\!\supset\!.\, \phi\{ab\}$ [*D1.2.08*; *D1.2.09*]

D1.2.010 $[\theta\phi] :: [ab] \therefore \theta\{a\} \,.\, \theta\{b\} \,.\!\supset\!: \phi\{ab\} \,.\!\vee\!.\, \phi\{ba\} \therefore\equiv.\, \mathbf{C}\langle\theta\rangle \{\!\!\{\phi\}\!\!\}$

ϕ connects θ.

D1.2.011 $[\rho\theta\phi] : \mathbf{PO}\langle\theta\rangle \{\!\!\{\phi\}\!\!\} \,.\, \rho \subset \theta \,.\, \mathbf{C}\langle\rho\rangle \{\!\!\{\phi\}\!\!\} \,.\!\equiv\!.\, \mathbf{CH}\{\!\ast\theta\phi\ast\!\}\{\!\!\{\rho\}\!\!\}$

ρ is a ϕ-chain in θ.

T1.2.14 $[\rho\theta\phi] : \mathbf{CH}\{\!\ast\theta\phi\ast\!\}\{\!\!\{\mu\}\!\!\} \,.\, \theta \subset \rho \,.\, \mathbf{PO}\langle\rho\rangle \{\!\!\{\phi\}\!\!\} \,.\!\supset\!.\, \mathbf{CH}\{\!\ast\rho\phi\ast\!\}\{\!\!\{\mu\}\!\!\}$
[*D1.2.011*; *D1.2.010*; *T1.1.13*]

T1.2.15 $[\mu\theta\phi] : \mathbf{CH}\{\!\ast\theta\phi\ast\!\}\{\!\!\{\mu\}\!\!\} \,.\!\supset\!.\, \mathbf{CH}\{\!\ast\mu\phi\ast\!\}\{\!\!\{\mu\}\!\!\}$ [*D1.2.011*; *T1.2.8*; *T1.1.12*]

D1.2.012 $[\theta\phi] \therefore [\rho] : \mathbf{CH}\{\!\ast\theta\rho\ast\!\}\{\!\!\{\rho\}\!\!\} \,.\!\supset\!.\, [\exists a] \,.\, \theta\{a\} \,.\, \mathbf{LUB}\,\{\!|\rho\theta\phi|\!\}\{a\} :\equiv.\, \mathbf{W}\langle\theta\rangle \{\!\!\{\phi\}\!\!\}$

ϕ is such that every ϕ-chain in θ has a least upper bound in θ.

1.3 *Unit Sets* At several times in subsequent proofs we shall have occasion to use theorems containing a many-link functor which corresponds to the set theoretical notion of a unit set. In this section we define this functor and present a number of the needed theorems employing it.

D1.3.01 $[ab] : a \circ b \,.\!\equiv\!.\, \downarrow\underset{1\ 1}{\langle a\rangle}\{b\}$

T1.3.1 $[a] \,.\, \downarrow\underset{1\ 1}{\langle a\rangle}\{a\}$ [*D1.3.01*; *T1.1.2*]

T1.3.2 $[ab\rho] : \downarrow\underset{1\ 1}{\langle a\rangle}\{b\} \,.\, \rho\{a\} \,.\!\supset\!.\, \rho\{b\}$ [*D1.3.01*; *T1.1.4*]

T1.3.3 $[ab\rho] : \downarrow\underset{1\ 1}{\langle a\rangle}\{b\} \,.\, \sim(\rho\{a\}) \,.\!\supset\!.\, \sim(\rho\{b\})$ [*D1.3.01*; *T1.1.5*]

T1.3.4 $[ab\rho] : \sim(\rho\{a\}) \,.\, \rho\{b\} \,.\!\supset\!.\, \sim(\downarrow\underset{1\ 1}{\langle a\rangle}\{b\})$ [*T1.3.3*]

T1.3.5 $[a\rho] : \downarrow\underset{1\ 1}{\langle a\rangle} \subset \rho \,.\!\supset\!.\, \rho\{a\}$ [*T1.3.1*; *T1.1.14*]

T1.3.6 $[a\rho] : \rho\{a\} \,.\!\supset\!.\, \downarrow\underset{1\ 1}{\langle a\rangle} \subset \rho$ [*T1.3.1*; *D1.1.04*]

T1.3.7	$[ab\rho]:\cup \langle\rho\downarrow \underset{1\,1}{\langle a\rangle}\rangle\{b\}.\sim(\rho\{b\}).\supset.\ a\circ b$	$[D1.1.05;\ D1.3.01]$
T1.3.8	$[ab\rho\theta]:\rho\subset\theta.\theta\{a\}.\cup\langle\rho\downarrow\underset{1\,1}{\langle a\rangle}\rangle\{b\}.\supset.\ \theta\{b\}$	
PR	$[ab\rho\theta]\therefore \mathrm{Hp}(3).\supset:$	
4.	$\rho\{b\}.\mathrm{v}.\downarrow\underset{1\,1}{\langle a\rangle}\{b\}:$	$[3;\ D1.1.05]$
	$\theta\{b\}$	$[4;\ 1;\ T1.1.14;\ 4;\ 2;\ D1.3.01;\ T1.1.4]$
T1.3.9	$[a\rho\theta]:\rho\subset\theta.\theta\{a\}.\supset.\cup\langle\rho\downarrow\underset{1\,1}{\langle a\rangle}\rangle\subset\theta$	$[T1.3.8;\ D1.1.04]$
T1.3.10	$[a].\mathbf{P}\ddagger\downarrow\underset{1\,1}{\langle a\rangle}\circ\ddagger\{a\}$	$[T1.2.7;\ T1.3.1;\ D1.3.01;\ D1.2.06]$
T1.3.11	$[a].\mathbf{WO}\langle\downarrow\underset{1\,1}{\langle a\rangle}\rangle\not{+}\circ\not{+}$	$[T1.2.7;\ T1.3.10;\ D1.2.07]$

1.4 *A Lemma on Well Order Connections* In this section we prove a lemma which ultimately will be used in Chapter Two. The lemma is given here rather than in Chapter Two so as not to interrupt the main thrust of the proof in which it is used. It is also convenient to place it here because many of the definitions and theorems of the previous sections are employed in a non-trivial way. We wish to prove that if we have a connection which well orders a given set, we can always find a type of extension of that connection which well orders the original set enlarged by the addition of a single element. We begin by defining a connection which extends a previously given connection.

D1.4.01	$[abm\rho\phi]\therefore\mathbf{T}\ddagger\phi\rho\rho\ddagger\{ab\}.\mathrm{v}.\rho\{a\}.b\circ m.\mathrm{v}.a\circ b:\equiv.\ \mathbf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ab\}$	
T1.4.1	$[m\rho\theta\phi]:\mathbf{R}\langle\theta\rangle\not{+}\mathbf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\not{+}$	$[D1.4.01;\ T1.1.2;\ D1.2.01]$
T1.4.2	$[abc\rho\phi]:\mathbf{WO}\langle\rho\rangle\not{+}\phi\not{+}.\mathbf{T}\underset{1}{\ddagger}\phi\rho\rho\underset{1}{\ddagger}\{ab\}.\mathbf{T}\ddagger\phi\rho\rho\ddagger\{bc\}.\supset.\ \mathbf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	
PR	$[abc\rho\phi]:\mathrm{Hp}(3).\supset.$	
4.	$\rho\{a\}.$	$[2;\ D1.1.016]$
5.	$\rho\{b\}.$	$[2;\ D1.1.016]$
6.	$\rho\{c\}.$	$[3;\ D1.1.016]$
7.	$\mathbf{T}\langle\rho\rangle\not{+}\phi\not{+}.$	$[1;\ D1.2.07;\ D1.2.04]$
8.	$\phi\{ab\}.$	$[2;\ D1.1.016]$
9.	$\phi\{bc\}.$	$[3;\ D1.1.016]$
10.	$\phi\{ac\}.$	$[4;\ 5;\ 6;\ 8;\ 9;\ 7;\ D1.2.02]$
11.	$\mathbf{T}\ddagger\phi\rho\rho\ddagger\{ac\}.$	$[4;\ 6;\ 10;\ D1.1.016]$
	$\mathbf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	$[11;\ D1.4.01]$
T1.4.3	$[abcm\rho\phi]:\mathbf{T}\ddagger\phi\rho\rho\ddagger\{ab\}.c\circ m.\supset.\ \mathbf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	
PR	$[abcm\rho\phi]:\mathrm{Hp}(2).\supset:$	
3.	$\rho\{a\}.$	$[1;\ D1.1.016]$
	$\mathbf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	$[3;\ 2;\ D1.4.01]$
T1.4.4	$[abcm\rho\phi]:\mathbf{T}\ddagger\phi\rho\rho\ddagger\{ab\}.b\circ c.\supset.\ \mathbf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}.$	
PR	$[abcm\rho\phi]:\mathrm{Hp}(2).\supset.$	
3.	$\mathbf{T}\ddagger\phi\rho\rho\ddagger\{ac\}.$	$[1;\ 2;\ T1.1.36]$
	$\mathbf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	$[3;\ D1.4.01]$

T1.4.5	$[abcm\rho\phi]:\sim(\rho\{m\}).\,b\circ m.\,\mathsf{T}\ddagger\phi\rho\rho\ddagger\{bc\}\,.\supset.\,\mathsf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	
PR	$[abcm\rho\phi]:\mathrm{Hp}(3)\,.\supset.$	
4.	$\sim(\rho\{b\}).$	[1; 2; *T1.1.5*]
5.	$\rho\{b\}.$	[3; *D1.1.016*]
	$\mathsf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	[4; 5]
T1.4.6	$[abcm\rho\phi]:\rho\{a\}.\,b\circ m.\,b\circ c\,.\supset.\,\mathsf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	
PR	$[abcm\rho\phi]:\mathrm{Hp}(3)\,.\supset.$	
4.	$c\circ m.$	[1; 2; *T1.1.3*; *T1.1.1*]
	$\mathsf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	[1; 4; *D1.4.01*]
T1.4.7	$[abcm\rho\phi]:a\circ b.\,\mathsf{T}\ddagger\phi\rho\rho\ddagger\{bc\}\,.\supset.\,\mathsf{E1}\underset{1}{\ddagger}m\phi\rho\underset{1}{\ddagger}\{ac\}$	[*T1.1.37*; *D1.4.01*]
T1.4.8	$[abcm\rho\phi]:a\circ b.\,\rho\{b\}.\,c\circ m\,.\supset.\,\mathsf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	[*T1.1.4*; *D1.4.01*]
T1.4.9	$[abcm\rho\phi]:a\circ b.\,b\circ c\,.\supset.\,\mathsf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	[*T1.1.3*; *D1.4.01*]
T1.4.10	$[abcm\rho\phi]:\mathsf{WO}\langle\rho\rangle\ddagger\phi\ddagger.\sim(\rho\{m\}).\,\mathsf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ab\}.\,\mathsf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{bc\}\,.\supset.$ $\mathsf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	
PR	$[abcm\rho\phi]\therefore\mathrm{Hp}(4)\,.\supset:$	
5.	$\mathsf{T}\ddagger\phi\rho\rho\ddagger\{ab\}.\vee.\,\rho\{a\}.\,b\circ m.\vee.\,a\circ b:$	[3; *D1.4.01*]
6.	$\mathsf{T}\ddagger\phi\rho\rho\ddagger\{bc\}.\vee.\,\rho\{b\}.\,c\circ m.\vee.\,b\circ c:$	[4; *D1.4.01*]
7.	$\mathsf{T}\ddagger\phi\rho\rho\ddagger\{ab\}.\,\mathsf{T}\ddagger\phi\rho\rho\ddagger\{bc\}.\vee.\,\mathsf{T}\ddagger\phi\rho\rho\ddagger\{ab\}.\,\rho\{b\}.\,c\circ m.\vee.$ $\mathsf{T}\ddagger\phi\rho\rho\ddagger\{ab\}.\,b\circ c.\vee.\,\rho\{a\}.\,b\circ m.\,\mathsf{T}\ddagger\phi\rho\rho\ddagger\{bc\}.\vee.$ $\rho\{a\}.\,b\circ m.\,\rho\{b\}.\,c\circ m.\vee.\,\rho\{a\}.\,b\circ m.\,b\circ c.\vee.$ $a\circ b.\,\mathsf{T}\ddagger\phi\rho\rho\ddagger\{bc\}.\vee.\,a\circ b.\,\rho\{b\}.\,c\circ m.\vee.\,a\circ b.\,b\circ c:$	[5; 6]
	$\mathsf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\{ac\}$	[7; 1; *T1.4.2*; 7; *T1.4.3*; 7; *T1.4.4*; 7; 2; *T1.4.5*; 7; *D1.4.01*; 7; *T1.4.7*; 7; *T1.4.8*; 7; *T1.4.9*]
T1.4.11	$[m\rho\theta\phi]:\mathsf{WO}\langle\rho\rangle\ddagger\phi\ddagger.\sim(\rho\{m\})\,.\supset.\,\mathbf{T}\langle\theta\rangle\ddagger\mathsf{E1}\underset{1}{\ddagger}m\rho\phi\underset{1}{\ddagger}\ddagger$	[*T1.4.10*; *D1.2.02*]
T1.4.12	$[ab\rho\phi]:\mathsf{WO}\langle\rho\rangle\ddagger\phi\ddagger.\,\mathsf{T}\ddagger\phi\rho\rho\ddagger\{ab\}.\,\mathsf{T}\ddagger\phi\rho\rho\ddagger\{ba\}\,.\supset.\,a\circ b$	
PR	$[ab\rho\phi]:\mathrm{Hp}(3)\,.\supset.$	
4.	$\mathsf{A}\langle\rho\rangle\ddagger\phi\ddagger.$	[1; *D1.2.07*; *D1.2.04*]
5.	$\rho\{a\}.$	[2; *D1.1.016*]
6.	$\rho\{b\}.$	[2; *D1.1.016*]
7.	$\phi\{ab\}.$	[2; *D1.1.016*]
8.	$\phi\{ba\}.$	[3; *D1.1.016*]
	$a\circ b$	[4; 5; 6; 7; 8; *D1.2.03*]
T1.4.13	$[abm\rho\phi]:\sim(\rho\{m\}).\,\mathsf{T}\ddagger\phi\rho\rho\ddagger\{ab\}.\,a\circ m\,.\supset.\,a\circ b$	
PR	$[abm\rho\phi]:\mathrm{Hp}(3)\,.\supset.$	
4.	$\sim(\rho\{a\}).$	[1; 3; *T1.1.5*]
5.	$\rho\{a\}.$	[2; *D1.1.016*]
	$a\circ b$	[4; 5]
T1.4.14	$[abm\rho\phi]:\sim(\rho\{m\}).\,b\circ m.\,\mathsf{T}\ddagger\phi\rho\rho\ddagger\{ba\}\,.\supset.\,a\circ b$	
PR	$[abm\rho\phi]:\mathrm{Hp}(3)\,.\supset.$	
4.	$\sim(\rho\{b\}).$	[1; 2; *T1.1.5*]

5. $\rho\{b\}$. [3; *D1.1.016*]

$a \circ b$ [4; 5]

T1.4.15 $[abm\rho\phi] : \mathbf{WO}\langle\rho\rangle \nleftarrow\phi\nrightarrow . \sim(\rho\{m\}) . \mathbf{E1}\,\underset{1}{\ddagger} m\rho\phi \underset{1}{\ddagger}\{ab\} . \mathbf{E1}\,\underset{1}{\ddagger} m\rho\phi\underset{1}{\ddagger}\{ba\} .\supset.$
$a \circ b$

PR $[abm\rho\phi] \therefore \mathrm{Hp}(4) .\supset:$

5. $\mathbf{T}\ddagger\phi\rho\rho\ddagger\{ab\} .\mathrm{v}. \rho\{a\} . b \circ m .\mathrm{v}. a \circ b :$ [3; *D1.4.01*]

6. $\mathbf{T}\ddagger\phi\rho\rho\ddagger\{ba\} .\mathrm{v}. \rho\{b\} . a \circ m .\mathrm{v}. b \circ a :$ [4; *D1.4.01*]

7. $\mathbf{T}\ddagger\phi\rho\rho\ddagger\{ab\} . \mathbf{T}\ddagger\phi\rho\rho\ddagger\{ba\} .\mathrm{v}. \mathbf{T}\ddagger\phi\rho\rho\ddagger\{ab\} . \rho\{b\} . a \circ m .\mathrm{v}.$
$\mathbf{T}\ddagger\phi\rho\rho\ddagger\{ab\} . b \circ a .\mathrm{v}. \rho\{a\} . b \circ m . \mathbf{T}\ddagger\phi\rho\rho\ddagger\{ba\} .\mathrm{v}. \rho\{a\} . b \circ m . \rho\{b\} .$
$a \circ m .\mathrm{v}. \rho\{a\} . b \circ m . b \circ a .\mathrm{v}. a \circ b . \mathbf{T}\ddagger\phi\rho\rho\ddagger\{ba\} .\mathrm{v}.$
$a \circ b . \rho\{b\} . a \circ m .\mathrm{v}. a \circ b . b \circ a :$ [5; 6]

$a \circ b$ [7; 1; *T1.4.12*; 7; 2; *T1.4.13*; 7; *T1.1.1*; 7; 2; *T1.4.14*; 7; *T1.1.1*; *T1.1.3*; 7; *T1.1.1*; 7; 7; 7]

T1.4.16 $[m\rho\theta\phi] : \mathbf{WO}\langle\rho\rangle \nleftarrow\phi\nrightarrow . \sim(\rho\{m\}) .\supset. \mathbf{A}\langle\theta\rangle \nleftarrow \mathbf{E1}\,\underset{1}{\ddagger} m\rho\phi\underset{1}{\ddagger}\nrightarrow$
[*T1.4.15*; *D1.2.03*]

T1.4.17 $[m\rho\theta] : \mathbf{WO}\langle\rho\rangle \nleftarrow\phi\nrightarrow . \sim(\rho\{m\}) .\supset. \mathbf{PO}\langle\theta\rangle \nleftarrow \mathbf{E1}\,\underset{1}{\ddagger} m\rho\phi\underset{1}{\ddagger}\nrightarrow$
[*T1.4.1*; *T1.4.11*; *T1.4.16*; *D1.2.04*]

T1.4.18 $[bm\rho\mu] : \sim(\mu\{m\}) . \mu\{b\} . \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\{b\} .\supset. \rho\{b\}$

PR $[bm\rho\mu] : \mathrm{Hp}(3) .\supset.$

4. $m \circ b .$ [3; *D1.3.01*]

5. $\sim(\mu\{b\}) .$ [1; 4; *T1.1.5*]

$\rho\{b\}$ [2; 5]

T1.4.19 $[bm\rho\mu] : \mu \subset \cup \langle\rho \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle . \sim(\mu\{m\}) . \mu\{b\} .\supset. \rho\{b\}$

PR $[bm\rho\mu] \therefore \mathrm{Hp}(3) .\supset:$

4. $\rho\{b\} .\mathrm{v}. \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\{b\} :$ [1; 3; *T1.1.14*]

$\rho\{b\}$ [4; 4; 2; 3; *T1.4.18*]

T1.4.20 $[m\rho\mu] : \mu \subset \cup \langle\rho \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle . \sim(\mu\{m\}) .\supset. \mu \subset \rho$ [*T1.4.19*; *D1.1.04*]

T1.4.21 $[abm\rho\mu\phi] : \mu \subset \cup \langle\rho \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle . \sim(\mu\{m\}) . \mathbf{P}\{\mu\phi\}\{a\} . \mu\{b\} .\supset.$
$\mathbf{E1}\,\underset{1}{\ddagger} m\rho\phi\underset{1}{\ddagger}\{ab\}$

PR $[abm\rho\mu\phi] \therefore \mathrm{Hp}(4) .\supset:$

5. $\mu \subset \rho .$ [1; 2; *T1.4.20*]

6. $\mu\{a\} :$ [3; *D1.2.06*]

7. $[b] : \mu\{b\} .\supset. \phi\{ab\} :$ [3; *D1.2.06*]

8. $\phi\{ab\} .$ [4; 7]

9. $\rho\{a\} .$ [5; 6; *T1.1.14*]

10. $\rho\{b\} .$ [4; 5; *T1.1.14*]

11. $\mathbf{T}\ddagger\phi\rho\rho\ddagger\{ab\} .$ [9; 10; 8; *D1.1.016*]

$\mathbf{E1}\,\underset{1}{\ddagger} m\rho\phi\underset{1}{\ddagger}\{ab\}$ [11; *D1.4.01*]

T1.4.22 $[am\rho\mu\phi] \therefore \mu \subset \cup \langle\rho \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle . \sim(\mu\{m\}) . \mathbf{P}\{\mu\phi\}\{a\} .\supset: [b] : \mu\{b\} .\supset.$
$\mathbf{E1}\,\underset{1}{\ddagger} m\rho\phi\underset{1}{\ddagger}\{ab\}$ [*T1.4.21*]

T1.4.23 $[am\rho\mu\phi]:\mathsf{WO}\langle\rho\rangle \nleftarrow\phi\nrightarrow . \sim(\rho\{m\}) . \mu \subset \cup \leftarrow\rho \downarrow\underset{1}{\langle} m\underset{1}{\rangle}\rightarrow . \sim(\mu\{m\}) .$
$\mathsf{P}\,\{\!|\mu\phi|\!\}\,\{a\} .\supset. \mathsf{P}\,\{\!|\mu\,\mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}|\!\}\,\{a\}$

PR $[am\rho\mu\phi] \therefore \mathrm{Hp}(5) .\supset:$

6. $\mathsf{PO}\langle\cup\leftarrow\rho\downarrow\underset{1}{\langle} m\underset{1}{\rangle}\rightarrow\rangle \nleftarrow\mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}\nrightarrow .$ [1; 2; *T1.4.17*]

7. $\mathsf{PO}\langle\mu\rangle \nleftarrow\mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}\nrightarrow .$ [3; 6; *T1.2.8*]

8. $\mu\{a\}:$ [5; *D1.2.06*]

9. $[b]:\mu\{b\} .\supset. \mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}\{ab\}:$ [3; 4; 5; *T1.4.22*]

$\mathsf{P}\,\{\!|\mu\,\mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}|\!\}\,\{a\}$ [7; 8; 9; *D1.2.06*]

T1.4.24 $[m\rho\mu\phi]:\mathsf{WO}\langle\rho\rangle \nleftarrow\phi\nrightarrow . \sim(\rho\{m\}) . \mu \subset \cup \leftarrow\rho \downarrow\underset{1}{\langle} m\underset{1}{\rangle}\rightarrow . !\leftarrow\mu\rightarrow . \sim(\mu\{m\}) .$
$\supset. [\exists a] . \mathsf{P}\,\{\!|\mu\,\mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}|\!\}\,\{a\}$

PR $[m\rho\mu\phi]:\mathrm{Hp}(5) .\supset.$

6. $\mu\subset\rho .$ [3; 5; *T1.4.20*]

$[\exists a] .$

7. $\quad\mathsf{P}\,\{\!|\mu\phi|\!\}\,\{a\} .$ [1; *D1.2.07*; 6; 4]

8. $\quad\mathsf{P}\,\{\!|\mu\,\mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}|\!\}\,\{a\} .$ [1; 2; 3; 5; 7; *T1.4.23*]

$[\exists a] . \mathsf{P}\,\{\!|\mu\,\mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}|\!\}\,\{a\}$ [8]

T1.4.25 $[m\rho\mu\phi]:\mu\circ\downarrow\underset{1}{\langle} m\underset{1}{\rangle} . \mu\{b\} .\supset. \mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}\{mb\}$

PR $[m\rho\mu\phi]:\mathrm{Hp}(2) .\supset.$

3. $\downarrow\underset{1}{\langle} m\underset{1}{\rangle}\{b\} .$ [1; 2; *T1.1.10*]

4. $m\circ b .$ [3; *D1.3.01*]

$\mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}\{mb\}$ [4; *D1.4.01*]

T1.4.26 $[m\rho\mu\phi]\therefore \mu\circ\downarrow\underset{1}{\langle} m\underset{1}{\rangle} .\supset: [b]:\mu\{b\} .\supset. \mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}\{mb\}$ [*T1.4.25*]

T1.4.27 $[m\rho\mu\phi]:\mathsf{WO}\langle\rho\rangle \nleftarrow\phi\nrightarrow . \sim(\rho\{m\}) . \mu \subset \cup \leftarrow\rho \downarrow\underset{1}{\langle} m\underset{1}{\rangle}\rightarrow . \mu\{m\} . \mu\circ\downarrow\underset{1}{\langle} m\underset{1}{\rangle} .$
$\supset. [\exists a] . \mathsf{P}\,\{\!|\mu\,\mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}|\!\}\,\{a\}$

PR $[m\rho\mu\phi]\therefore \mathrm{Hp}(5) .\supset:$

6. $\mathsf{PO}\langle\cup\leftarrow\rho\downarrow\underset{1}{\langle} m\underset{1}{\rangle}\rightarrow\rangle \nleftarrow\mathsf{E1}\,\{\!\|m\rho\phi\|\!\}\nrightarrow .$ [1; 2; *T1.4.17*]

7. $\mathsf{PO}\langle\mu\rangle \nleftarrow\mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}\nrightarrow :$ [3; 6; *T1.2.8*]

8. $[b]:\mu\{b\} .\supset. \mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}\{mb\}:$ [5; *T1.4.26*]

9. $\mathsf{P}\,\{\!|\mu\,\mathsf{E1}\,\underset{1}{\{\!\|}m\rho\phi\underset{1}{\|\!\}}|\!\}\,\{m\} .$ [7; 4; 8; *D1.2.06*]

$[\exists a] . \mathsf{P}\,\{\!|\mu\,\mathsf{E1}\,\{\!\|m\rho\phi\|\!\}|\!\}\,\{a\}$ [9]

T1.4.28 $[am\mu]:\mu\{m\} . \downarrow\underset{1}{\langle} m\underset{1}{\rangle}\{a\} . \sim(\mu\{a\}) .\supset. \mu\{a\} . \sim(\downarrow\underset{1}{\langle} m\underset{1}{\rangle}\{a\})$

PR $[am\mu]:\mathrm{Hp}(3) .\supset.$

4. $\mu\{a\} .$ [1; 2; *T1.3.2*]

$\mu\{a\} . \sim(\downarrow\underset{1}{\langle} m\underset{1}{\rangle}\{a\})$ [3; 4]

T1.4.29 $[m\mu]:\mu\{m\} . \sim(\mu\circ\downarrow\underset{1}{\langle} m\underset{1}{\rangle}) .\supset. !\leftarrow\!-\!\leftarrow\mu\downarrow\underset{1}{\langle} m\underset{1}{\rangle}\rightarrow\rightarrow$

PR $[m\mu]\therefore \mathrm{Hp}(2) .\supset:$

3. $\sim([a] : \mu\{a\} .\equiv. \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\{a\}) :$ [2; *D1.1.03*]

$[\exists a] :$

4. $\mu\{a\} . \sim(\downarrow\underset{1}{\langle} m \underset{1}{\rangle}\{a\}) .\vee. \sim(\mu\{a\}) . \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\{a\} :$ [3]

5. $\mu\{a\} . \sim(\downarrow\underset{1}{\langle} m \underset{1}{\rangle}\{a\}) .$ [4; 1; *T1.4.28*]

6. $-\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle\{a\} :$ [5; *D1.1.07*]

$! \nprec -\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle \nsucc$ [6; *D1.1.08*]

T1.4.30 $[abm\rho\mu\phi] : \sim(\rho\{m\}) . \mu \subset \cup \langle\!\langle \rho \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle . \mathbf{P}\{\!\{-\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle\phi\}\!\}\{a\} . \mu\{b\} .$
$\rho\{b\} .\supset. \mathbf{E1}\underset{1}{\ddagger} m\rho\phi \underset{1}{\ddagger}\{ab\}$

PR $[abm\rho\mu\phi] \therefore \mathrm{Hp}(5) .\supset:$

6. $-\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle \subset \rho .$ [2; *T1.1.23*]

7. $-\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle\{a\} .$ [3; *D1.2.06*]

8. $\rho\{a\} .$ [6; 7; *T1.1.14*]

9. $\sim(\downarrow\underset{1}{\langle} m \underset{1}{\rangle}\{b\}) .$ [1; 5; *T1.3.4*]

10. $-\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle\{b\} :$ [4; 9; *D1.1.07*]

11. $[b] : -\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle\{b\} .\supset. \phi\{ab\} :$ [3; *D1.2.06*]

12. $\phi\{ab\} .$ [10; 11]

13. $\mathbf{T}\ddagger\phi\rho\rho\ddagger\{ab\} .$ [8; 5; 12; *D1.6.016*]

$\mathbf{E1}\underset{1}{\ddagger} m\rho\phi \underset{1}{\ddagger}\{ab\}$ [13; *D1.4.01*]

T1.4.31 $[abm\rho\mu\phi] : \mu \subset \cup \langle\!\langle \rho \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle . \mathbf{P}\{\!\{-\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle\phi\}\!\}\{a\} . \mu\{b\} . \sim(\rho\{b\}) .\supset.$
$\mathbf{E1}\underset{1}{\ddagger} m\rho\phi \underset{1}{\ddagger}\{ab\}$

PR $[abm\rho\mu\phi] \therefore \mathrm{Hp}(4) .\supset:$

5. $-\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle \subset \rho .$ [1; *T1.1.23*]

6. $-\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle\{a\} .$ [2; *D1.2.06*]

7. $\rho\{a\} :$ [5; 6; *T1.1.14*]

8. $\rho\{b\} .\vee. \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\{b\} :$ [3; 1; *T1.1.18*]

9. $\downarrow\underset{1}{\langle} m \underset{1}{\rangle}\{b\} .$ [4; 8]

10. $m \circ b .$ [9; *D1.3.01*]

11. $\mathbf{E1}\underset{1}{\ddagger} m\rho\phi \underset{1}{\ddagger}\{ab\}$ [7; 10; *D1.4.01*]

T1.4.32 $[abm\rho\mu\phi] : \sim(\rho\{m\}) . \mu \subset \cup \langle\!\langle \rho \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle . \mathbf{P}\{\!\{-\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle\phi\}\!\}\{a\} . \mu\{b\} .\supset.$
$\mathbf{E1}\underset{1}{\ddagger} m\rho\phi \underset{1}{\ddagger}\{ab\}$ [*T1.4.30*; *T1.4.31*]

T1.4.33 $[am\rho\mu\phi] \therefore \sim(\rho\{m\}) . \mu \subset \cup \langle\!\langle \rho \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle . \mathbf{P}\{\!\{-\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle\phi\}\!\}\{a\} .\supset:$
$[b] : \mu\{b\} .\supset. \mathbf{E1}\underset{1}{\ddagger} m\rho\phi \underset{1}{\ddagger}\{ab\}$ [*T1.4.32*]

T1.4.34 $[am\rho\mu\phi] : \mathbf{WO}\langle\rho\rangle \nprec\phi\nsucc . \sim(\rho\{m\}) . \mu \subset \cup \langle\!\langle \rho \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle .$
$\mathbf{P}\{\!\{-\langle\!\langle \mu \downarrow\underset{1}{\langle} m \underset{1}{\rangle}\rangle\!\rangle\phi\}\!\}\{a\} .\supset. \mathbf{P}\{\!\{\mu\ \mathbf{E1}\underset{1}{\ddagger} m\rho\phi \underset{1}{\ddagger}\}\!\}\{a\}$

PR	$[am\rho\mu\phi] \therefore \mathrm{Hp}(4) \;.\supset:$	
5.	$\mathbf{PO}\langle\cup \prec\rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle}\succ\rangle \prec \mathbf{E1} \underset{1}{\llbracket} m\rho\phi \underset{1}{\rrbracket}\succ .$	[1; 2; *T1.4.17*]
6.	$\mathbf{PO}\langle\mu\rangle \prec \mathbf{E1} \underset{1}{\llbracket} m\rho\phi \underset{1}{\rrbracket}\succ .$	[3; 5; *T1.2.8*]
7.	$-\prec\mu \downarrow \underset{1}{\langle} m \underset{1}{\rangle}\succ \{a\} .$	[4; *D1.2.06*]
8.	$\mu\{a\}:$	[7; *D1.1.07*]
9.	$[b]: \mu\{b\} \;.\supset.\; \mathbf{E1} \underset{1}{\llbracket} m\rho\phi \underset{1}{\rrbracket}\{ab\}:$	[2; 3; 4; *T1.4.33*]
	$\mathbf{P}\{\mu\ \mathbf{E1} \underset{1}{\llbracket} m\rho\phi \underset{1}{\rrbracket}\}\{a\}$	[6; 8; 9; *D1.2.6*]
T1.4.35	$[m\rho\mu\phi]: \mathbf{WO}\langle\rho\rangle \prec\phi\succ . \sim(\rho\{m\}) . \mu \subset \cup \prec\rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle}\succ . \mu\{m\} . \sim(\mu \circ \downarrow \underset{1}{\langle} m \underset{1}{\rangle}) .$	
	$\supset. [\exists a] . \mathbf{P}\{\mu\ \mathbf{E1} \underset{1}{\llbracket} m\rho\phi \underset{1}{\rrbracket}\}\{a\}$	
PR	$[m\rho\mu\phi]: \mathrm{Hp}(5) \;.\supset.$	
6.	$-\prec\mu \downarrow \underset{1}{\langle} m \underset{1}{\rangle}\succ \subset \rho .$	[3; *T1.1.23*]
7.	$!\prec -\prec\mu \downarrow \underset{1}{\langle} m \underset{1}{\rangle}\succ\succ .$	[4; 5; *T1.4.29*]
	$[\exists a] .$	
8.	$\quad \mathbf{P}\{-\prec\mu \downarrow \underset{1}{\langle} m \underset{1}{\rangle}\succ\phi\}\{a\} .$	[1; 6; 7; *D1.2.07*]
9.	$\quad \mathbf{P}\{\mu\ \mathbf{E1} \underset{1}{\llbracket} m\rho\phi \underset{1}{\rrbracket}\}\{a\} .$	[12; 3; 8; *T1.1.34*]
	$[\exists a] . \mathbf{P}\{\mu\ \mathbf{E1} \underset{1}{\llbracket} m\rho\phi \underset{1}{\rrbracket}\}\{a\}$	[9]
T1.4.36	$[m\mu\rho\phi]: \mathbf{WO}\langle\rho\rangle \prec\phi\succ . \sim(\rho\{m\}) . \mu \subset \cup \prec\rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle}\succ . \mu\{m\} \;.\supset.$	
	$[\exists a]\, \mathbf{P}\{\mu\ \mathbf{E1} \underset{1}{\llbracket} m\rho\phi \underset{1}{\rrbracket}\}\{a\}$	[*T1.4.27*; *T1.4.35*]
T1.4.37	$[m\mu\rho\phi]: \mathbf{WO}\langle\rho\rangle \prec\phi\succ . \sim(\rho\{m\}) . \mu \subset \cup \prec\rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle}\succ . !\prec\mu\succ \;.\supset.$	
	$[\exists a] . \mathbf{P}\{\mu\ \mathbf{E1} \underset{1}{\llbracket} m\rho\phi \underset{1}{\rrbracket}\}\{a\}$	[*T1.4.24*; *T1.4.36*]
T1.4.38	$[m\rho\phi] \therefore \mathbf{WO}\langle\rho\rangle \prec\phi\succ . \sim(\rho\{m\}) \;.\supset: [\mu]: \mu \subset \cup \prec\rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle}\succ . !\prec\mu\succ \;.\supset.$	
	$[\exists a]\, \mathbf{P}\{\mu\ \mathbf{E1} \underset{1}{\llbracket} m\rho\phi \underset{1}{\rrbracket}\}\{a\}$	[*T1.4.37*]
T1.4.39	$[m\rho\phi]: \mathbf{WO}\langle\rho\rangle \prec\phi\succ . \sim(\rho\{m\}) \;.\supset.\; \mathbf{WO}\, \langle\cup\prec\rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle}\succ\rangle \prec \mathbf{E1} \underset{1}{\llbracket} m\rho\phi \underset{1}{\rrbracket}\succ$	
		[*T2.4.17*; *T1.4.38*; *D1.2.07*]

T1.4.39 is the thesis which will be used as a lemma in section **2.3**.

1.5 *Definitions for the Generalized Choice Principle* In this section we present a number of definitions necessary for stating the generalized choice principle together with a number of related theorems. For the most part the definitions and theorems presented here have analogues in section **1.1**.

D1.5.01 $\quad [\xi] \therefore [\exists\theta] . \xi\prec\theta\succ \;:\equiv.\; !\underset{1}{\prec}\xi\underset{1}{\succ}$

ξ is unempty. ! introduces semantic category $\mathsf{S}/(\mathsf{S}/(\mathsf{S}/\mathsf{n}))$. This can be compared with *D1.1.02* and *D1.1.08*. Note that in conformity with Ontology's directive for definition, the semantic category to which ξ belongs, i.e., $\mathsf{S}/(\mathsf{S}/\mathsf{n})$, has been introduced by *D1.1.08*.

D1.5.02 $[\xi]:[\theta].\,\xi\{\!\{\theta\}\!\}\ .\supset.\ !\{\!\{\theta\}\!\}\ :\equiv.\ \boldsymbol{\lambda}\underset{1}{\{\!\{}\xi\underset{1}{\}\!\}}$

ξ is a family of non-empty sets, i.e., each set satisfying ξ is non-empty.

D1.5.03 $[\xi\zeta]\therefore[\theta]:\xi\{\!\{\theta\}\!\}\ .\equiv.\ \zeta\{\!\{\theta\}\!\}\ :\equiv.\ \circ\underset{1}{\{\!\{}\xi\zeta\underset{1}{\}\!\}}$

ξ equals ζ. This defines equality for functors of semantic category S/(S/(S/n)) and can be compared with *D1.1.01*, *D1.1.03*, or *D1.1.11*.

T1.5.1 $[\xi\zeta]:\xi\circ\zeta\ .\equiv.\ \zeta\circ\xi$ [*D1.5.03*]

T1.5.2 $[\xi\zeta\theta]:\xi\circ\zeta\,.\,\xi\{\!\{\theta\}\!\}\ .\supset.\ \zeta\{\!\{\theta\}\!\}$ [*D1.5.03*]

D1.5.04 $[\xi\zeta]\therefore[\theta]:\xi\{\!\{\theta\}\!\}\ .\supset.\ \zeta\{\!\{\theta\}\!\}\ :\equiv.\ \subset\{\!\{\xi\zeta\}\!\}$

ξ is included in ζ. This defines inclusion for functors of semantic category S/(S/(S/n)) and can be compared with *D1.1.04*.

T1.5.3 $[\xi\zeta]:\xi\subset\zeta\,.\,\zeta\subset\xi\ .\supset.\ \xi\circ\zeta$ [*D1.5.04*; *D1.5.03*]

D1.5.06 $[a\theta].\circ\{\!\{\theta\theta\}\!\}.\circ\{aa\}\ .\equiv.\ \vee\underset{1}{\{}\theta a\underset{1}{\}}$

V is a universal connection between objects of categories S/n and n. This definition is given solely to introduce the semantic category S/(S/n)(n) to which V belongs.

The next two definitions define the domain and range (counter-domain) respectively of connections of semantic category S/(S/n)(n) and are analogues of *D1.1.013* and *D1.1.014*.

D1.5.07 $[\theta\eta]:[\exists a].\,\eta\underset{1}{\{}\theta a\underset{1}{\}}\ :\equiv.\ \mathsf{D}\underset{1}{\{\!\{\!\{}\eta\underset{1}{\}\!\}\!\}}\{\!\{\theta\}\!\}$

D1.5.08 $[a\eta]:[\exists\theta].\,\eta\underset{1}{\{}\theta a\underset{1}{\}}\ .\equiv.\ \mathsf{G}\underset{1}{\{\!\{\!\{}\eta\underset{1}{\}\!\}\!\}}\{a\}$

D1.5.09 $[\eta]\therefore[ab\theta]:\eta\underset{1}{\{}\theta a\underset{1}{\}}.\,\eta\underset{1}{\{}\theta b\underset{1}{\}}\ .\supset.\ a\circ b\ :\equiv.\ \rightrightarrows\underset{1}{\{\!\{\!\{}\eta\underset{1}{\}\!\}\!\}}$

η is many-one or function-like. $\rightrightarrows$ is in semantic category S/(S/(S/n)(n)).

T1.5.4 $[a\mu\rho\eta]:\eta\underset{1}{\{}\mu a\underset{1}{\}}.\,\mu\circ\rho\ .\supset.\ \eta\underset{1}{\{}\rho a\underset{1}{\}}$ [*E1.1.02*]

1.6 *More Analogous Functors* The theses presented in Chapter Two involve the statement of the maximal principle known as Zorn's Lemma for two different semantic categories. The definitions necessary for stating it for the lower of the two categories are found in section **1.2**. The definitions required for stating the principle for the higher category are completely analogous to those found in section **1.2**, and are presented below together with some related theses.

D1.6.01 $[\theta\phi]:\circ\{\!\{\theta\theta\}\!\}.\circ\{\!\{\!\{\phi\phi\}\!\}\!\}\ .\equiv.\ \vee\underset{4}{\{\!\{}\theta\phi\underset{4}{\}\!\}}$

This defines the universal connection of semantic category S/(S/n)(S/nn) and is given here solely to introduce this particular semantic category.

D1.6.02 $[\Gamma\Delta]\therefore[\theta\phi]:\Gamma\underset{4}{\{\!\{}\theta\phi\underset{4}{\}\!\}}\ .\equiv.\ \Delta\underset{4}{\{\!\{}\theta\phi\underset{4}{\}\!\}}\ :\equiv.\ \circ\underset{2}{\{}\Gamma\Delta\underset{2}{\}}$

Γ equals Δ. This defines equality for objects of semantic category S/(S/n)(S/nn). $\circ$ is in semantic category S/((S/n)(S/nn))((S/n)(S/nn)).

T1.6.1	$[\Gamma\Delta] : \Gamma \circ \Delta \;.\!\equiv\!.\; \Delta \circ \Gamma$	[*D1.6.02*]
T1.6.2	$[\Gamma] . \Gamma \circ \Gamma$	[*D1.6.02*]
T1.6.3	$[\Gamma\Delta\Upsilon] : \Gamma \circ \Delta . \Delta \circ \Upsilon \;.\!\supset\!.\; \Gamma \circ \Upsilon$	[*D1.6.02*]
T1.6.4	$[\theta\phi\Gamma\Delta] : \Gamma\underset{2}{\{}\theta\phi\underset{2}{\}} . \Gamma \circ \Delta \;.\!\supset\!.\; \Delta\underset{2}{\{}\theta\phi\underset{2}{\}}$	[*D1.6.02*]
T1.6.5	$[\Gamma\Delta\Upsilon\Omega] : \Gamma \circ \Upsilon . \Delta \circ \Omega . \Upsilon \circ \Omega \;.\!\supset\!.\; \Delta \circ \Gamma$	[*T1.6.1*; *T1.6.3*]

D1.6.03 $[\Gamma] : [\exists\theta\phi] . \Gamma\underset{4}{-\!\!(}\theta\phi\underset{4}{)\!\!-} \;.\!\equiv\!.\; !\underset{2}{\{}\Gamma\underset{2}{\}}$

Γ is unempty. This definition is given here to introduce the semantic category S/((S/n)(S/nn)).

D1.6.04 $[\Sigma\theta] \therefore [\Gamma] : \Sigma\underset{2}{\{}\Gamma\underset{2}{\}} \;.\!\supset\!.\; \theta\underset{2}{\{}\Gamma\underset{2}{\}} :\equiv.\; \subset \underset{2}{-\!\!(}\Sigma\theta\underset{2}{)\!\!-}$

Here we define inclusion for objects of semantic category S/((S/n)(S/nn)). An obvious thesis follows.

T1.6.6 $[\Gamma\Sigma\theta] : \Sigma \subset \theta . \Sigma\underset{2}{\{}\Gamma\underset{2}{\}} \;.\!\supset\!.\; \theta\underset{2}{\{}\Gamma\underset{2}{\}}$ [*D1.6.04*]

The following definition introduces a many-link functor which corresponds to the set theoretical notion of an ordered pair whose first element is a set and whose second element is a connection.

D1.6.05 $[\rho\mu\phi\psi] : \circ -\!\!(\rho\mu)\!\!- . \circ =\!\!(\phi\psi)\!\!= \;.\!\equiv\!.\; \downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}\underset{4}{-\!\!(}\mu\psi\underset{4}{)\!\!-}$

$\downarrow$ is in semantic category (S/(S/n)(S/nn))/(S/n)(S/nn). Note, however, that $\downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}$ is in semantic category S/(S/n)(S/nn).

T1.6.7	$[\rho\phi] : \downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}\underset{4}{-\!\!(}\rho\phi\underset{4}{)\!\!-}$	[*T1.1.7*; *T1.1.29*; *D1.6.05*]
T1.6.8	$[\rho\mu\phi\psi] : \downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle} \circ \downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle} \;.\!\supset\!.\; \mu \circ \rho . \phi \circ \psi$	
PR	$[\rho\mu\phi\psi] \therefore \mathrm{Hp}(1) \;.\!\supset\!:$	
2.	$[\sigma\Phi] : \downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}\underset{4}{-\!\!(}\sigma\Phi\underset{4}{)\!\!-} \;.\!\equiv\!.\; \downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}\underset{4}{-\!\!(}\sigma\Phi\underset{4}{)\!\!-} :$	[1; *D1.6.05*]
3.	$\downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}\underset{4}{-\!\!(}\rho\phi\underset{4}{)\!\!-} .$	[2; *T1.6.7*]
	$\mu \circ \rho . \phi \circ \psi$	[3; *D1.6.05*]
T1.6.9	$[\rho\mu\sigma\phi\psi\Phi] . \rho \circ \mu . \phi \circ \psi . \downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}\underset{4}{-\!\!(}\sigma\Phi\underset{4}{)\!\!-} \;.\!\supset\!.\; \downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}\underset{4}{-\!\!(}\sigma\Phi\underset{4}{)\!\!-}$	
PR	$[\rho\mu\sigma\phi\psi\Phi] . \mathrm{Hp}(3) \;.\!\supset\!.$	
4.	$\rho \circ \sigma .$	[3; *D1.6.05*]
5.	$\phi \circ \Phi .$	[3; *D1.6.05*]
6.	$\mu \circ \sigma .$	[1; 4; *T1.1.1*; *T1.1.3*]
7.	$\psi \circ \Phi .$	[2; 5; *T1.1.28*; *T1.1.30*]
	$\downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}\underset{4}{-\!\!(}\sigma\Phi\underset{4}{)\!\!-}$	[6; 7; *D1.1.06*]
T1.6.10	$[\rho\mu\sigma\phi\psi\Phi] . \rho \circ \mu . \phi \circ \psi . \downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}\underset{4}{-\!\!(}\sigma\Phi\underset{4}{)\!\!-} \;.\!\supset\!.\; \downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}\underset{4}{-\!\!(}\sigma\Phi\underset{4}{)\!\!-}$	[similar to *T1.6.9*]

T1.6.11 $[\rho\mu\phi\psi].\rho\circ\mu.\phi\circ\psi\ .\supset.\ \downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}\circ\downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}$ [*T1.6.10*; *T1.6.9*; *D1.6.02*]

T1.6.12 $[\rho\mu\phi\psi\Gamma]:\Gamma\circ\downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}.\Gamma\circ\downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}\ .\supset.\ \rho\circ\mu.\phi\circ\psi$ [*T1.6.1*; *T1.6.8*]

T1.6.13 $[a\rho\mu\phi\psi\Gamma]:\Gamma\circ\downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}.\Gamma\circ\downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}.\rho\{a\}\ .\supset.\ \mu\{a\}$ [*T1.6.12*; *T1.1.4*]

T1.6.14 $[ab\rho\mu\phi\psi\Gamma]:\Gamma\circ\downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}.\Gamma\circ\downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}.\phi\{ab\}\ .\supset.\ \psi\{ab\}$
[*T1.6.12*; *T1.1.31*]

T1.6.15 $[\mu\rho\sigma\phi\psi]:\rho\circ\mu.\downarrow\underset{4}{\langle}\mu\,\mathbf{T}\ddagger\psi\mu\mu\ddagger\underset{4}{\rangle}\underset{4}{\{\!\!\!-}\sigma\phi\underset{4}{-\!\!\!\}}\ .\supset.\ \downarrow\underset{4}{\langle}\rho\,\mathbf{T}\ddagger\psi\rho\rho\ddagger\underset{4}{\rangle}\underset{4}{\{\!\!\!-}\sigma\phi\underset{4}{-\!\!\!\}}$

PR $[\mu\rho\sigma\phi\psi]:\mathrm{Hp}(2)\ .\supset.$

3. $\mu\circ\sigma.$ [2; *D1.6.05*]
4. $\mathbf{T}\ddagger\psi\mu\mu\ddagger\circ\phi.$ [2; *D1.6.05*]
5. $\mathbf{T}\ddagger\psi\mu\mu\ddagger\circ\mathbf{T}\ddagger\psi\rho\rho\ddagger.$ [1; *T1.1.39*]
6. $\rho\circ\sigma.$ [1; 3; *T1.1.3*]
7. $\mathbf{T}\ddagger\psi\rho\rho\ddagger\circ\phi.$ [4; 5; *T1.1.28*; *T1.1.30*]

$\downarrow\underset{4}{\langle}\rho\,\mathbf{T}\ddagger\psi\rho\rho\ddagger\underset{4}{\rangle}\underset{4}{\{\!\!\!-}\sigma\phi\underset{4}{-\!\!\!\}}$ [6; 7; *D1.6.05*]

T1.6.16 $[\mu\rho\sigma\phi\psi]:\rho\circ\mu.\downarrow\underset{4}{\langle}\rho\,\mathbf{T}\ddagger\psi\rho\rho\ddagger\underset{4}{\rangle}\underset{4}{\{\!\!\!-}\sigma\phi\underset{4}{-\!\!\!\}}\ .\supset.\ \downarrow\underset{4}{\langle}\mu\,\mathbf{T}\ddagger\psi\mu\mu\ddagger\underset{4}{\rangle}\underset{4}{\{\!\!\!-}\sigma\phi\underset{4}{-\!\!\!\}}$
[similar to *T1.6.15*]

T1.6.17 $[\mu\rho\psi]:\rho\circ\mu\ .\supset.\ \downarrow\underset{4}{\langle}\mu\,\mathbf{T}\ddagger\psi\mu\mu\ddagger\underset{4}{\rangle}\circ\downarrow\underset{4}{\langle}\rho\,\mathbf{T}\ddagger\psi\rho\rho\ddagger\underset{4}{\rangle}$
[*T1.6.15*; *T1.6.16*; *D1.6.02*]

T1.6.18 $[\mu\rho\psi\Gamma]:\rho\circ\mu.\Gamma\circ\downarrow\underset{4}{\langle}\mu\,\mathbf{T}\ddagger\psi\mu\mu\ddagger\underset{4}{\rangle}\ .\supset.\ \Gamma\circ\downarrow\underset{4}{\langle}\rho\,\mathbf{T}\ddagger\psi\rho\rho\ddagger\underset{4}{\rangle}$
[*D1.6.02*; *T1.6.17*]

Definitions *D1.6.06* through *D1.6.016* which follow are completely analogous to *D1.1.08*, *D1.2.01* through *D1.2.05*, and *D1.2.08* through *D1.2.012*.

D1.6.06 $[\Sigma].[\exists\Gamma].\Sigma\underset{2}{\{}\Gamma\underset{2}{\}}\ .\equiv.\ !\underset{2}{\{\!\!\!-}\Sigma\underset{2}{-\!\!\!\}}$

Σ is unempty.

D1.6.07 $[\theta\Pi]\therefore[\Gamma]:\theta\underset{2}{\{}\Gamma\underset{2}{\}}\ .\supset.\ \Pi\underset{2}{\{}\Gamma\Gamma\underset{2}{\}}\ :\equiv.\ \mathbf{R}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\!=}\Pi\underset{2}{=\!\!\!\}}$

Π is a reflexive connection on θ.

D1.6.08 $[\theta\Pi]\therefore[\Gamma\Delta\Upsilon]:\theta\underset{2}{\{}\Gamma\underset{2}{\}}.\theta\underset{2}{\{}\Delta\underset{2}{\}}.\theta\underset{2}{\{}\Upsilon\underset{2}{\}}.\Pi\underset{2}{\{}\Gamma\Delta\underset{2}{\}}.\Pi\underset{2}{\{}\Delta\Upsilon\underset{2}{\}}\ .\supset.\ \Pi\underset{2}{\{}\Gamma\Upsilon\underset{2}{\}}\ :\equiv.$
$\mathbf{T}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\!=}\Pi\underset{2}{=\!\!\!\}}$

Π is a transitive connection on θ.

D1.6.09 $[\theta\Pi]\therefore[\Gamma\Delta]:\theta\underset{2}{\{}\Gamma\underset{2}{\}}.\theta\underset{2}{\{}\Delta\underset{2}{\}}.\Pi\underset{2}{\{}\Gamma\Delta\underset{2}{\}}.\Pi\underset{2}{\{}\Delta\Gamma\underset{2}{\}}\ .\supset.\ \Delta\circ\Gamma\ :\equiv.\ \mathbf{A}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\!=}\Pi\underset{2}{=\!\!\!\}}$

Π is an antisymmetric connection on θ.

D1.6.010 $[\theta\Pi]:\mathbf{R}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\!=}\Pi\underset{2}{=\!\!\!\}}.\mathbf{T}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\!=}\Pi\underset{2}{=\!\!\!\}}.\mathbf{A}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\!=}\Pi\underset{2}{=\!\!\!\}}\ .\equiv.\ \mathbf{PO}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\!=}\Pi\underset{2}{=\!\!\!\}}$

Π partially orders θ.

D1.6.011 $[\Gamma\theta\Pi]:\mathbf{PO}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\!=}\Pi\underset{2}{=\!\!\!\}}.\theta\underset{2}{\{}\Gamma\underset{2}{\}}:[\Delta]:\theta\underset{2}{\{}\Delta\underset{2}{\}}.\Pi\underset{2}{\{}\Gamma\Delta\underset{2}{\}}\ .\supset.\ \Gamma\circ\Delta\ :\equiv.$
$\mathbf{MAXL}\underset{2}{\{\!\!\!-}\theta\Pi\underset{2}{-\!\!\!\}}\underset{2}{\{}\Gamma\underset{2}{\}}$

Γ is a maximal elment of θ with respect to Π.

D1.6.012 $[\Gamma\Sigma\theta\Pi] \therefore \mathbf{PO}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\{}\Pi\underset{2}{\}\!\!\}} . \theta\underset{2}{\{}\Gamma\underset{2}{\}} . \Sigma \subset \theta : [\Delta] : \Sigma\underset{2}{\{}\Delta\underset{2}{\}} .\supset. \Pi\underset{2}{\{}\Delta\Gamma\underset{2}{\}} :\equiv. \mathbf{UB}\underset{2}{\{\!|}\Sigma\theta\Pi\underset{2}{|\!\}}\underset{2}{\{}\Gamma\underset{2}{\}}$

Γ is a Π-upper bound of Σ in θ.

D1.6.013 $[\Gamma\Sigma\theta\Pi] \therefore \mathbf{UB}\underset{2}{\{\!|}\Sigma\theta\Pi\underset{2}{|\!\}}\underset{2}{\{}\Gamma\underset{2}{\}} : [\Delta] : \mathbf{UB}\underset{2}{\{\!|}\Sigma\theta\Pi\underset{2}{|\!\}}\underset{2}{\{}\Delta\underset{2}{\}} .\supset. \Pi\underset{2}{\{}\Gamma\Delta\underset{2}{\}} :\equiv. \mathbf{LUB}\underset{2}{\{\!|}\Sigma\theta\Pi\underset{2}{|\!\}}\underset{2}{\{}\Gamma\underset{2}{\}}$

Γ is a Π-least upper bound of Σ in θ.

D1.6.014 $[\theta\Pi] :: [\Gamma\Delta] \therefore \theta\underset{2}{\{}\Gamma\underset{2}{\}} . \theta\underset{2}{\{}\Delta\underset{2}{\}} .\supset: \Pi\underset{2}{\{}\Gamma\Delta\underset{2}{\}} . \vee . \Pi\underset{2}{\{}\Delta\Gamma\underset{2}{\}} \therefore\equiv. \mathbf{C}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\{}\Pi\underset{2}{\}\!\!\}}$

Π connects θ.

D1.6.015 $[\Sigma\theta\Pi] : \mathbf{PO}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\{}\Pi\underset{2}{\}\!\!\}} . \Sigma \subset \theta . \mathbf{C}\underset{2}{\langle}\Sigma\underset{2}{\rangle}\underset{2}{\{\!\!\{}\Pi\underset{2}{\}\!\!\}} .\equiv. \mathbf{CH}\underset{2}{\{\!\!|}\theta\Pi\underset{2}{|\!\!\}}\underset{2}{\{\!|}\Sigma\underset{2}{|\!\}}$

Σ is a Π-chain in θ.

T1.6.19 $[\Gamma\Sigma\theta\Pi] : \mathbf{CH}\underset{2}{\{\!\!|}\theta\Pi\underset{2}{|\!\!\}}\underset{2}{\{\!|}\Sigma\underset{2}{|\!\}} . \Sigma\underset{2}{\{}\Gamma\underset{2}{\}} .\supset. \theta\underset{2}{\{}\Gamma\underset{2}{\}}$ [*D1.6.015*; *T1.6.6*]

D1.6.016 $[\theta\Pi] : [\Sigma] : \mathbf{CH}\underset{2}{\{\!\!|}\theta\Pi\underset{2}{|\!\!\}}\underset{2}{\{\!|}\Sigma\underset{2}{|\!\}} .\supset. [\exists\Gamma] . \mathbf{LUB}\underset{2}{\{\!|}\Sigma\theta\Pi\underset{2}{|\!\}}\underset{2}{\{}\Gamma\underset{2}{\}} .\equiv. \mathbf{W}\underset{2}{\langle}\theta\underset{2}{\rangle}\underset{2}{\{\!\!\{}\Pi\underset{2}{\}\!\!\}}$

Π is such that every Π-chain in θ has a least upper bound in θ.

CHAPTER II: THE AXIOM OF CHOICE AND SOME RELATED PRINCIPLES IN ONTOLOGY

2.0 *Introduction* In this chapter we investigate the inferability relationships among the Ontological analogues of the Axiom of Choice, the Kuratowski-Zorn Lemma, and the Well Ordering Principle. These three, stated for the semantic category S/n, are:

ACF $[\xi] :: !\underset{1}{\{\!|}\xi\underset{1}{|\!\}} . \curlywedge\underset{1}{\{\!|}\xi\underset{1}{|\!\}} .\supset\therefore [\exists\eta] . \rightrightarrows\underset{1}{\{\!\!\{}\eta\underset{1}{\}\!\!\}} . \mathbf{D}\underset{1}{\{\!\!\{}\eta\underset{1}{\}\!\!\}} \circ \xi : [a\theta] : \eta\underset{1}{\{}\theta a\underset{1}{\}} .\supset. \theta\{a\}$

KZL $[\theta\phi] : \mathbf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\} . !\{\!|\theta|\!\} . \mathbf{W}\langle\theta\rangle\{\!\!\{\phi\}\!\!\} .\supset. [\exists a] . \mathbf{MAXL}\{\!|\theta\phi|\!\}\{a\}$

WO $[\theta] : [\exists\phi] . \mathbf{WO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\}$

ACF, a functional form of the Axiom of Choice known as the Generalized Principle of Choice, states that for every non-empty family of non-empty sets, there exists a functional connection which associates with each element of the family a member of that element. **KZL** is a form of the maximal principle formulated by Kuratowski and Zorn. It states that every non-empty, partially ordered set has a maximal element provided the least upper bound of every chain contained in the set is an element of the set. **WO** formulates the Well Ordering Principle and says that for every set there is a connection which well orders it.

The inferential equivalences of the Axiom of Choice, the Kuratowski-Zorn Lemma, and the Well Ordering Principle within the field of set theory are well known. It is one of the purposes of this chapter to show that similar equivalences can also be established within the field of Ontology. But, as the work of this chapter will show, there is a difference between the sense in which these three principles are equivalent in the field of set theory and the sense in which they are equivalent in Ontology. In set theory the equivalence of, e.g., the Axiom of Choice and the Kuratowski-Zorn Lemma is established by proving a biconditional one of whose arguments is equiform to a formula expressing the Axiom of Choice and the other to a formula expressing the Kuratowski-Zorn Lemma. In Ontology, however, though we can prove **ACF** implies **KZL**, the first step in establishing a biconditional, we cannot prove the converse. Nonetheless, we can prove that a formula completely analogous to **KZL** but which states the Lemma for a higher semantic category than **KZL** above does imply **ACF**. Hence, the two formulas **ACF** and **KZL** are not directly equivalent. Furthermore, even if we could establish the direct equivalence of **ACF** to **KZL**, this would not in itself be sufficient to establish that the Axiom of Choice and Zorn's Lemma are equivalent within the field of Ontology because **ACF** and **KZL** formulate their respective principles for a single semantic category, S/n. Given that there are a potentially infinite number of semantic categories, the establishment of a single biconditional with **ACF** as one of its arguments and **KZL** as the other would show only that the two are equivalent when stated for the semantic category S/n. The import of these remarks is that within a type theory such as Ontology, the question of the equivalence of the Axiom of Choice to some other principle is not resolved by showing two formulas equivalent, but rather by showing that the acceptance of a rule allowing the addition of the Axiom of Choice for any semantic category yields the same results as the acceptance of a rule allowing, e.g., the addition of the Kuratowski-Zorn Lemma for any semantic category. This sense of equivalence we call rule equivalence and the immediate goal of the work of this chapter is to establish the rule equivalence in Ontology of the three principles mentioned.

The first step in achieving this goal is the establishment of certain implication relations between the Axiom of Choice, the Kuratowski-Zorn Lemma, and the Well Ordering Principle for certain fixed semantic categories, and it is to this end that the bulk of the material of this chapter is devoted. Section **2.1** contains a proof of a "fixed point" theorem which, when used subsequently as a lemma in section **2.2**, serves to bridge the gap between **ACF** and **KZL**. Though this "fixed point" theorem is used merely as a lemma in proving one of the desired implications, it is interesting in its own right and its lengthy proof justifies devoting an entire section to it alone. In section **2.2** we prove that **ACF** implies **KZL**. In section **2.3**, we show that the Kuratowski-Zorn Lemma, stated for a higher semantic category than **KZL**, above, implies the Well Ordering Principle as stated above (**WO**). Section **2.4** contains a proof that **WO** implies **ACF**. Finally, section **2.5** indicates how the results of the preceding sections suffice to establish the rule equivalence within Ontology of the Axiom of Choice, the Kuratowski-Zorn Lemma, and the Well Ordering Principle.

2.1 *A Fixed Point Theorem* This section contains a proof of a "fixed point" theorem which, in brief, states that under certain conditions, a functional connection maps at least one element of a set onto itself.[11]

The formal statement of the theorem requires the following definition:

D2.1.01 $[\rho\theta f] : \rightrightarrows \langle\!\langle f \rangle\!\rangle \,.\, \mathsf{D}\langle\!\langle f \rangle\!\rangle \circ \theta \,.\, \mathsf{C}\langle\!\langle f \rangle\!\rangle \subset \rho \;.\!\equiv\!.\; \mathsf{IN}\langle\theta\rho\rangle \langle\!\langle f \rangle\!\rangle$

f is a many-one functional connection which maps θ *into* ρ.

Given this definition, the theorem we want to prove can be stated:

FP $[\theta\phi f] : ! \prec\theta\succ \,.\, \mathsf{PO}\langle\theta\rangle \langle\!\langle \phi \rangle\!\rangle \,.\, \mathsf{W}\langle\theta\rangle \langle\!\langle \phi \rangle\!\rangle \,.\, \mathsf{IN}\langle\theta\theta\rangle \langle\!\langle f \rangle\!\rangle \,.\, \subset \langle\!\langle f\phi \rangle\!\rangle \;.\!\supset\!.$
$[\exists b] \,.\, \theta\{b\} \,.\, f\{bb\}$

The proof itself will make use of the following auxiliary definitions.

D2.1.02 $[\rho f] \therefore [cd] : \rho\{c\} \,.\, f\{cd\} \;.\!\supset\!.\; \rho\{d\} :\equiv.\; \mathsf{C1}\langle\rho\rangle \langle\!\langle f \rangle\!\rangle$

f is closed on ρ.

D2.1.03 $[\rho\theta\phi] \therefore [a\sigma] : ! \prec\sigma\succ \,.\, \mathsf{CH}\{\!\{\rho\phi\}\!\} \prec\sigma\succ \,.\, \mathsf{LUB}\{\!\{\sigma\theta\phi\}\!\}\{a\} \;.\!\supset\!.\; \rho\{a\} :\equiv.$
$\mathsf{C2}\{\!\{\theta\phi\}\!\} \prec\rho\succ$

ρ is such that every unempty chain within it has its least upper bound in ρ as long as that least upper bound is in θ.

D2.1.04 $[a\theta\phi\rho f] : \rho \subset \theta \,.\, \rho\{a\} \,.\, \mathsf{C1}\langle\rho\rangle \langle\!\langle f \rangle\!\rangle \,.\, \mathsf{C2}\{\!\{\theta\phi\}\!\} \prec\rho\succ \;.\!\equiv\!.\; \mathsf{ADM}\{\!\{a\theta\phi f\}\!\} \prec\rho\succ$

ρ is an admissible subset of θ relative to a with respect to ϕ and f.

We begin the proof by showing that $\mathsf{ADM}\{\!\{a\theta\phi f\}\!\} \prec\theta\succ$. This is established in *T2.1.5*.

T2.1.1	$[cd\theta f] : \mathsf{IN}\langle\theta\theta\rangle \langle\!\langle f \rangle\!\rangle \,.\, f\{cd\} \;.\!\supset\!.\; \theta\{d\}$	
PR	$[cd\theta f] :$ Hp(2) $.\!\supset\!.$	
3.	$\mathsf{C}\langle\!\langle f \rangle\!\rangle \subset \theta \,.$	[1; *D2.1.01*]
4.	$\mathsf{C}\langle\!\langle f \rangle\!\rangle\{d\} \,.$	[2; *T1.1.38*]
	$\theta\{d\}$	[3; 4; *T1.1.14*]
T2.1.2	$[\theta f] : \mathsf{IN}\langle\theta\theta\rangle \langle\!\langle f \rangle\!\rangle \;.\!\supset\!.\; \mathsf{C1}\langle\theta\rangle \langle\!\langle f \rangle\!\rangle$	[*T2.1.1*; *D2.1.02*]
T2.1.3	$[b\sigma\theta\phi] : \mathsf{LUB}\{\!\{\sigma\theta\phi\}\!\}\{b\} \;.\!\supset\!.\; \theta\{b\}$	[*D1.2.09*; *D1.2.08*]
T2.1.4	$[\theta\phi] : \mathsf{C2}\{\!\{\theta\phi\}\!\} \prec\theta\succ$	[*T2.1.3*; *D2.1.03*]
T2.1.5	$[a\theta\phi f] : \mathsf{IN}\langle\theta\theta\rangle \langle\!\langle f \rangle\!\rangle \,.\, \theta\{a\} \;.\!\supset\!.\; \mathsf{ADM}\{\!\{a\theta\phi f\}\!\} \prec\theta\succ$	
PR	$[a\theta\phi f] :$ Hp(2) $.\!\supset\!.$	
3.	$\theta \subset \theta \,.$	[*T1.1.12*]
4.	$\mathsf{C1}\langle\theta\rangle \langle\!\langle f \rangle\!\rangle \,.$	[1; *T2.1.2*]
	$\mathsf{ADM}\{\!\{a\theta\phi f\}\!\} \prec\theta\succ$	[2; 3; 4; *T2.1.4*; *D2.1.40*]

We now proceed to show that the generalized intersection of all admissible sets contained in θ is itself admissible. This is established in *T2.1.19*.

T2.1.6	$[ab\theta\phi f] : \mathsf{IN}\langle\theta\theta\rangle \langle\!\langle f \rangle\!\rangle \,.\, \theta\{a\} \,.\, \cap \prec\mathsf{ADM}\{\!\{a\theta\rho f\}\!\}\succ \{b\} \;.\!\supset\!.\; \theta\{b\}$	
PR	$[ab\theta\phi f] \therefore$ Hp(3) $.\!\supset\!:$	
4.	$[\mu] : \mathsf{ADM}\{\!\{a\theta\phi f\}\!\} \prec\mu\succ \;.\!\supset\!.\; \mu\{b\} :$	[3; *D1.1.010*]

5. $\mathrm{ADM}\{\!|a\theta\phi f|\!\}\langle\!\langle\theta\rangle\!\rangle\ .\supset.\ \theta\{b\}:$ [4]
6. $\mathrm{ADM}\{\!|a\theta\phi f|\!\}\langle\!\langle\theta\rangle\!\rangle.$ [1; 2; *T2.1.5*]
$\theta\{b\}$ [5; 6]

T2.1.7 $[a\theta\phi f]:\mathrm{IN}\langle\theta\theta\rangle\langle\!\langle f\rangle\!\rangle.\theta\{a\}\ .\supset.\ \cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\subset\theta$ [*T2.1.6*; *D1.1.04*]

T2.1.8 $[a\sigma\theta\phi f]:\mathrm{ADM}\{\!|a\theta\phi f|\!\}\langle\!\langle\sigma\rangle\!\rangle\ .\supset.\ \sigma\{a\}$ [*D2.1.04*]

T2.1.9 $[a\theta\phi f].\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\{a\}$ [*T2.1.8*; *D1.1.010*]

T2.1.10 $[acd\sigma\theta\phi f]:\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\{c\}.f\{cd\}.\mathrm{ADM}\{\!|a\theta\phi f|\!\}\langle\!\langle\sigma\rangle\!\rangle\ .\supset.\ \sigma\{d\}$
PR $[acd\sigma\theta\phi f]:\mathrm{Hp}(3)\ .\supset.$
4. $\mathrm{C1}\langle\sigma\rangle\langle\!\langle f\rangle\!\rangle.$ [3; *D2.1.04*]
5. $\sigma\{c\}.$ [1; 3; *T1.1.25*]
$\sigma\{d\}$ [4; 5; 2; *D2.1.02*]

T2.1.11 $[acd\theta\phi f]:\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\{c\}.f\{cd\}\ .\supset.\ \cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\{d\}$
[*T2.1.10*; *D1.1.010*]

T2.1.12 $[a\theta\phi f]:\mathrm{C1}\langle\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\rangle\langle\!\langle f\rangle\!\rangle$ [*T2.1.11*; *D2.1.02*]

T2.1.13 $[ad\sigma\mu\theta\phi f]:\sigma\subset\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle.\mathrm{ADM}\{\!|a\theta\phi f|\!\}\langle\!\langle\mu\rangle\!\rangle.\sigma\{d\}\ .\supset.\ \mu\{d\}$
PR $[ad\sigma\mu\theta\phi f]\therefore\mathrm{Hp}(3)\ .\supset:$
4. $\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\{d\}:$ [1; 3; *T1.1.14*]
5. $[\mu]:\mathrm{ADM}\{\!|a\theta\phi f|\!\}\langle\!\langle\mu\rangle\!\rangle\ .\supset.\ \mu\{d\}:$ [4; *D1.1.010*]
$\mu\{d\}$ [2; 5]

T2.1.14 $[a\sigma\mu\theta\phi f]:\sigma\subset\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle.\mathrm{ADM}\{\!|a\theta\phi f|\!\}\langle\!\langle\mu\rangle\!\rangle\ .\supset.\ \sigma\subset\mu$
[*T2.1.13*; *D1.1.04*]

T2.1.15 $[a\sigma\mu\theta\phi f]:\mathrm{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.\mathrm{CH}\{\!\{\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\phi\}\!\}\langle\!\langle\sigma\rangle\!\rangle.$
$\mathrm{ADM}\{\!|a\theta\phi f|\!\}\langle\!\langle\mu\rangle\!\rangle\ .\supset.\ \mathrm{CH}\{\!\{\mu\phi\}\!\}\langle\!\langle\sigma\rangle\!\rangle$
PR $[a\sigma\mu\theta\phi f]:\mathrm{Hp}(3)\ .\supset.$
4. $\sigma\subset\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle.$ [2; *D1.2.011*]
5. $\mu\subset\theta.$ [3; *D2.1.04*]
6. $\mathrm{PO}\langle\mu\rangle\langle\!\langle\phi\rangle\!\rangle.$ [1; 5; *T1.2.8*]
7. $\sigma\subset\mu.$ [4; 3; *T2.1.14*]
8. $\mathrm{C}\langle\sigma\rangle\langle\!\langle\phi\rangle\!\rangle.$ [2; *D1.2.011*]
$\mathrm{CH}[\mu\phi]\langle\!\langle\sigma\rangle\!\rangle$ [6; 7; 8; *D1.2.011*]

T2.1.16 $[ab\sigma\mu\theta\phi f]:\mathrm{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.!\langle\!\langle\sigma\rangle\!\rangle.\mathrm{CH}\{\!\{\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\phi\}\!\}\langle\!\langle\sigma\rangle\!\rangle.$
$\mathrm{LUB}\{\!\{\sigma\theta\phi\}\!\}\{b\}.\mathrm{ADM}\{\!|a\theta\phi f|\!\}\langle\!\langle\mu\rangle\!\rangle\ .\supset.\ \mu\{b\}$
PR $[ab\sigma\xi\theta\phi f]:\mathrm{Hp}(5)\ .\supset.$
6. $\sigma\subset\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle.$ [3; *D1.2.011*]
7. $\mathrm{C2}\{\!\{\theta\phi\}\!\}\langle\!\langle\mu\rangle\!\rangle.$ [5; *D2.1.04*]
8. $\sigma\subset\mu.$ [5; 6; *T2.1.14*]
9. $\mathrm{CH}\{\!\{\mu\phi\}\!\}\langle\!\langle\sigma\rangle\!\rangle.$ [1; 3; 5; *T2.1.15*]
$\mu\{b\}$ [5; *D2.1.04*; 2; 3; 9; *D2.1.03*]

T2.1.17 $[ab\sigma\theta\phi f]:\mathrm{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.!\langle\!\langle\sigma\rangle\!\rangle.\mathrm{CH}\{\!\{\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\phi\}\!\}\langle\!\langle\sigma\rangle\!\rangle.$
$\mathrm{LUB}\{\!\{\sigma\theta\phi\}\!\}\{b\}\ .\supset.\ \cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\{b\}$ [*T2.1.16*; *D1.1.010*]

T2.1.18 $[a\theta\phi f]:\mathrm{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle\ .\supset.\ \mathrm{C2}\{\!\{\theta\phi\}\!\}\langle\!\langle\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\rangle\!\rangle$
[*T2.1.17*; *D2.1.03*]

T2.1.19 $[a\theta\phi f]:\mathrm{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.\mathrm{IN}\langle\theta\theta\rangle\langle\!\langle f\rangle\!\rangle.\theta\{a\}\ .\supset.$
$\mathrm{ADM}[a\theta\phi f]\langle\!\langle\cap\langle\!\langle\mathrm{ADM}\{\!|a\theta\phi f|\!\}\rangle\!\rangle\rangle\!\rangle$
[*T2.1.7*; *T2.1.9*; *T2.1.12*; *T2.1.18*; *D2.1.04*]

The above shows that under the given conditions, the generalized intersection of all admissible sets is itself admissible. We now give an auxiliary definition which will be used to show that a is a minimal element $\cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle$.

D2.1.05 $[ab\theta\phi f] : \cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle\{b\} . \phi\{ab\} \;.\!\equiv\!.\; \mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{b\}$

b is an element of $\cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle$ which follows a.

T2.1.20 $[a\theta\phi f] : \mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}} \subset \cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle$ [*D2.1.05*; *D1.1.04*]

T2.1.21 $[ab\theta\phi f] : \mathbf{IN}\langle\theta\theta\rangle \langle\!\langle f\rangle\!\rangle . \theta\{a\} . \mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{b\} \;.\!\supset\!.\; \theta\{b\}$

PR $[ab\theta\phi f] : \mathrm{Hp}(3) \;.\!\supset\!.$

4. $\cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle\{b\}$. [3; *D2.1.05*]

$\theta\{b\}$ [1; 2; 4; *T2.1.6*]

T2.1.22 $[a\theta\phi f] : \mathbf{IN}\langle\theta\theta\rangle \langle\!\langle f\rangle\!\rangle . \theta\{a\} \;.\!\supset\!.\; \mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}} \subset \theta$ [*T2.1.21*; *D1.1.04*]

T2.1.23 $[a\theta\phi f] : \mathbf{PO}\langle\theta\rangle \langle\!\langle \phi\rangle\!\rangle . \phi\{a\} \;.\!\supset\!.\; \mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{a\}$

PR $[a\theta\phi f] : \mathrm{Hp}(2) \;.\!\supset\!.$

3. $\mathbf{R}\langle\theta\rangle \langle\!\langle \phi\rangle\!\rangle$. [1; *D1.2.04*]

4. $\phi\{aa\}$. [2; 4; *D1.2.01*]

$\mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{a\}$ [*T2.1.9*; 3; 5; *D2.1.05*]

T2.1.24 $[acd\theta\phi f] : \mathbf{PO}\langle\theta\rangle \langle\!\langle \phi\rangle\!\rangle . \mathbf{IN}\langle\theta\theta\rangle \langle\!\langle f\rangle\!\rangle . f \subset \phi . \theta\{a\} . \mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{c\} . f\{cd\} .$
$\supset\!.\; \mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{d\}$

PR $[acd\theta\phi f] : \mathrm{Hp}(6) \;.\!\supset\!.$

7. $\theta\{c\}$. [2; 4; 5; *T2.1.21*]

8. $\theta\{ac\}$. [5; *D2.1.05*]

9. $\theta\{cd\}$. [3; 6; *T1.1.35*]

10. $\theta\{d\}$. [2; 6; *T2.1.1*]

11. $\mathbf{T}\langle\theta\rangle \langle\!\langle \phi\rangle\!\rangle$. [1; *D1.2.04*]

12. $\phi\{ad\}$. [4; 7; 10; 8; 9; *D1.2.02*]

13. $\cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle\{c\}$. [5; *D2.1.05*]

14. $\cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle\{d\}$. [*T2.1.12*; *D2.1.02*; 13; 6]

$\mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{d\}$ [8; 14; *D2.1.05*]

T2.1.25 $[a\theta\phi f] : \mathbf{PO}\langle\theta\rangle \langle\!\langle \phi\rangle\!\rangle . \mathbf{IN}\langle\theta\theta\rangle \langle\!\langle f\rangle\!\rangle . f \subset \phi . \theta\{a\} \;.\!\supset\!.\; \mathbf{Cl}\langle \mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\rangle \langle\!\langle f\rangle\!\rangle$
[*T2.1.24*; *D2.1.02*]

T2.1.26 $[a\theta\phi f] : \mathbf{PO}\langle\theta\rangle \langle\!\langle \sigma\rangle\!\rangle . \mathbf{IN}\langle\theta\theta\rangle \langle\!\langle f\rangle\!\rangle . \theta\{a\} \;.\!\supset\!.\; \mathbf{PO}\langle \cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle\rangle \langle\!\langle \phi\rangle\!\rangle$

PR $[a\theta\phi f] : \mathrm{Hp}(3) \;.\!\supset\!.$

4. $\cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle \subset \theta$. [2; 3; *T2.1.7*]

$\mathbf{PO}\langle \cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle\rangle \langle\!\langle \phi\rangle\!\rangle$ [1; 4; *T1.2.8*]

T2.1.27 $[a\sigma\mu\theta\phi f] : \mathbf{PO}\langle\theta\rangle \langle\!\langle \phi\rangle\!\rangle . \mathbf{IN}\langle\theta\theta\rangle \langle\!\langle f\rangle\!\rangle . \theta\{a\} . \mathbf{CH}\{\!\{\mu\phi\}\!\} \langle\sigma\rangle .$
$\mu \subset \cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle \;.\!\supset\!.\; \mathbf{CH}\{\!\{\cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle\phi\}\!\} \langle\sigma\rangle$

PR $[a\sigma\mu\theta\phi f] : \mathrm{Hp}(5) \;.\!\supset\!.$

6. $\mathbf{PO}\langle \cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle\rangle \langle\!\langle \phi\rangle\!\rangle$. [1; 2; 3; *T2.1.26*]

$\mathbf{CH}\{\!\{\cap \langle\!\langle \mathbf{ADM}\{a\theta\phi f\}\rangle\!\rangle\phi\}\!\} \langle\sigma\rangle$ [4; 5; 6; *T1.2.14*]

T2.1.28 $[ab\sigma\theta\phi f] : \mathbf{PO}\langle\theta\rangle \langle\!\langle \phi\rangle\!\rangle . \mathbf{IN}\langle\theta\theta\rangle \langle\!\langle f\rangle\!\rangle . \theta\{a\} . !\langle\sigma\rangle .$
$\mathbf{CH}\{\!\{\mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\phi\}\!\} \langle\sigma\rangle . \mathbf{LUB}\{\!\{\sigma\theta\phi\}\!\}\{b\} \;.\!\supset\!.\; \mathbf{F1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{b\}$

PR $[ab\sigma\theta\phi f] \therefore \mathrm{Hp}(6) \;.\supset:$

7. $\sigma \subset \mathbf{F1}\underset{1}{\{\!\!\{} a\theta\phi f \underset{1}{\}\!\!\}}.$ [5; *D1.2.011*]

8. $\mathbf{C2}\{\!|\theta\phi|\!\}\,(\!-\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle-\!)\,.$ [1; *T2.1.18*]

9. $\sigma \subset \cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle\,.$ [7; *T2.1.20*; *T1.1.13*]

10. $\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle \subset \theta\,.$ [2; 3; *T2.1.7*]

11. $\sigma \subset \theta\,.$ [9; 10; *T1.1.13*]

12. $\mathbf{CH}\{\!|\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle\,\phi|\!\}\,(\!-\sigma-\!)\,.$ [1; 2; 3; 5; 9; *T2.1.27*]

13. $\mathbf{CH}\{\!|\theta\phi|\!\}\,(\!-\sigma-\!):$ [1; 10; 12; *T1.2.14*]

14. $[b\sigma]:!(\!-\sigma-\!)\,.\,\mathbf{CH}\{\!|\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle\,\phi|\!\}\,(\!-\sigma-\!)\,.\,\mathbf{LUB}\{\!-\sigma\theta\phi-\!\}\{b\}\;.\supset.$
$\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle\{b\}:$ [8; *D2.1.03*]

15. $\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle\{b\}:$ [14; 4; 12; 6]

16. $[d]:\sigma\{d\}\;.\supset.\;\phi\{db\}:$ [6; *D1.2.09*; *D1.2.08*]
$[\exists m]\,.$

17. $\sigma\{m\}\,.$ [4; *D1.1.08*]

18. $\phi\{mb\}\,.$ [16; 17]

19. $\mathbf{F1}\underset{1}{\{\!\!\{} a\theta\phi f \underset{1}{\}\!\!\}}\{m\}\,.$ [7; 17; *T1.1.14*]

20. $\phi\{am\}\,.$ [19; *D2.1.05*]

21. $\mathbf{T}\langle\theta\rangle\,(\!=\phi=\!)\,.$ [1; *D1.2.04*]

22. $\theta\{m\}\,.$ [17; 11; *T1.1.14*]

23. $\theta\{b\}\,.$ [6; *D1.2.09*; *D1.2.08*]

24. $\phi\{ab\}:$ [3; 22; 23; 20; 18; 21; *D1.2.02*]

$\mathbf{F1}\underset{1}{\{\!\!\{} a\theta\phi f \underset{1}{\}\!\!\}}\{b\}$ [15; 24; *D2.1.05*]

T2.1.29 $[a\theta\phi f]:\mathbf{PO}\langle\theta\rangle\,(\!=\phi=\!)\,.\,\mathbf{IN}\langle\theta\theta\rangle\,(\!=f=\!)\,.\,\theta\{a\}\;.\supset.\;\mathbf{C2}\{\!|\theta\phi|\!\}\,(\!-\mathbf{F1}\underset{1}{\{\!\!\{} a\theta\phi f \underset{1}{\}\!\!\}}-\!)$
[*T2.1.28*; *D2.1.03*]

T2.1.30 $[a\theta\phi f]:\mathbf{PO}\langle\theta\rangle\,(\!=\phi=\!)\,.\,\mathbf{IN}\langle\theta\theta\rangle\,(\!=f=\!)\,.\,f \subset \phi\,.\,\theta\{a\}\;.\supset.$
$\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\,(\!-\mathbf{F1}\underset{1}{\{\!\!\{} a\theta\phi f \underset{1}{\}\!\!\}}-\!)$
[*T2.1.22*; *T2.1.23*; *T2.1.25*; *T2.1.29*; *D2.1.04*]

T2.1.31 $[a\theta\phi f]:\mathbf{PO}\langle\theta\rangle\,(\!=\phi=\!)\,.\,\mathbf{IN}\langle\theta\theta\rangle\,(\!=f=\!)\,.\,f \subset \phi\,.\,\theta\{a\}\;.\supset.$
$\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle \circ \mathbf{F1}\underset{1}{\{\!\!\{} a\theta\phi f \underset{1}{\}\!\!\}}$ [*T2.1.30*; *T2.1.20*; *T1.1.27*]

T2.1.32 $[ab\theta\phi f]:\mathbf{PO}\langle\theta\rangle\,(\!=\phi=\!)\,.\,\mathbf{IN}\langle\theta\theta\rangle\,(\!=f=\!)\,.\,f \subset \phi\,.\,\theta\{a\}\,.\,\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle\{b\}\,.$
$\supset.\;\phi\{ab\}$

PR $[ab\theta\phi f]:\mathrm{Hp}(5)\;.\supset.$

6. $\mathbf{F1}\underset{1}{\{\!\!\{} a\theta\phi f \underset{1}{\}\!\!\}}\{b\}\,.$ [1; 2; 3; 4; *T2.1.31*; *T1.1.10*]

$\phi\{ab\}$ [6; *D2.1.05*]

T2.1.32 establishes that every element of $\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle$ follows a. The next definitions given are also auxiliary. They will be used to show that a subset of $\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle$ equals $\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle$.

D2.1.06 $[ab\theta\phi f]\therefore\theta\{b\}:[cd]:\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle\{c\}\,.\,\phi\{cb\}\,.\sim(\phi\{cb\})\,.\,f\{cd\}\;.\supset.$
$\phi\{db\}\;:\equiv.\;\mathbf{G1}\underset{1}{\{\!\!\{} a\theta\phi f \underset{1}{\}\!\!\}}\{b\}$

D2.1.07 $[abwz\theta\phi f]\therefore\cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle\{z\}:\phi\{zb\}\,.\mathrm{v}.\,\phi\{wz\}\;:\equiv.\;\mathbf{B1}\underset{2}{\{\!\!\{} ba\theta\phi fw \underset{2}{\}\!\!\}}\{z\}$

T2.1.33 $[abw\theta\phi f]:\mathbf{B1}\underset{2}{\{\!\!\{} ba\theta\phi fw \underset{2}{\}\!\!\}} \subset \cap\,\langle\!\{\mathbf{ADM}\{\!\!\{a\theta\phi f\}\!\!\}\}\!\rangle$ [*D2.1.07*; *D1.1.04*]

T2.1.34	$[abcw\theta\phi f]:\mathbf{IN}\langle\theta\theta\rangle \nprec f \nsucc .\, \theta\{a\}.\, \mathbf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\{c\}\ .\supset.\ \theta\{c\}$	
PR	$[abcw\theta\phi f] \therefore \mathrm{Hp}(3)\ .\supset:$	
4.	$\cap \nprec \mathbf{ADM}\{a\theta\phi f\}\nsucc\{c\}.$	$[3;\ D2.1.07]$
5.	$\mathbf{ADM}\{a\theta\phi f\} \prec\theta\succ :$	$[1;\ 2;\ T2.1.5]$
6.	$[\mu]:\mathbf{ADM}\{a\theta\phi f\}\prec\mu\succ\ .\supset.\ \mu\{c\}:$	$[4;\ D1.1.010]$
	$\theta\{c\}$	$[5;\ 6]$
T2.1.35	$[abw\theta\phi f]:\mathbf{IN}\langle\theta\theta\rangle \nprec f \nsucc .\, \theta\{a\}\ .\supset.\ \mathbf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}} \subset \theta$	$[T2.1.34;\ D1.1.04]$
T2.1.36	$[abw\theta\phi f]:\mathbf{PO}\langle\theta\rangle \nprec\phi\nsucc .\, \mathbf{IN}\langle\theta\theta\rangle \nprec f \nsucc .\, f\subset\phi .\, \theta\{a\}.$ $\cap \nprec \mathbf{ADM}\{a\theta\phi f\}\nsucc\{b\}\ .\supset.\ \mathbf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\{a\}$	
PR	$[abw\theta\phi f] \therefore \mathrm{Hp}(5)\ .\supset:$	
6.	$\phi\{ab\}:$	$[1;\ 2;\ 3;\ 4;\ 5;\ T2.1.32]$
7.	$\phi\{ab\}.\mathrm{v}.\,\phi\{wa\}:$	$[6]$
	$\mathbf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\{a\}$	$[7;\ T2.1.9;\ D2.1.07]$
T2.1.37	$[abcdw\theta\phi f].\, \mathbf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\{c\}.\, f\{cd\}\ .\supset.\ \cap \nprec \mathbf{ADM}\{a\theta\phi f\}\nsucc\{d\}$	
PR	$[abcdw\theta\phi f]:\mathrm{Hp}(2)\ .\supset.$	
3.	$\cap \nprec \mathbf{ADM}\{a\theta\phi f\}\nsucc\{c\}.$	$[1;\ D2.1.07]$
	$\cap \nprec \mathbf{ADM}\{a\theta\phi f\}\nsucc\{d\}$	$[2;\ 3;\ T2.1.11]$
T2.1.38	$[abcdw\theta\phi f] \therefore \mathbf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{b\}.\, \mathbf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\{c\}.\, f\{cd\}.\, \phi\{cd\}.\, \sim(\phi\{bc\}).$ $\supset:\ \phi\{db\}.\mathrm{v}.\,\phi\{wd\}$	
PR	$[abcdw\theta\phi f] \therefore \mathrm{Hp}(5)\ .\supset:$	
6.	$[lm]:\cap \nprec \mathbf{ADM}\{a\theta\phi f\}\nsucc\{l\}.\, \phi\{lb\}.\sim(\phi\{bl\}).\, f\{lm\}\ .\supset.\ \phi\{mb\}:$	$[1;\ D2.1.06]$
7.	$\cap \nprec \mathbf{ADM}\{a\theta\phi f\}\nsucc\{c\}.$	$[2;\ D2.1.07]$
8.	$\phi\{db\}:$	$[6;\ 7;\ 3;\ 4;\ 5]$
	$\phi\{db\}.\mathrm{v}.\,\phi\{wd\}$	$[8]$
T2.1.39	$[abcdw\theta\phi f] \therefore \mathbf{PO}\langle\theta\rangle \nprec\phi\nsucc .\, \mathbf{IN}\langle\theta\theta\rangle \nprec f \nsucc .\, \theta\{a\}.\, \cap \nprec \mathbf{ADM}\{a\theta\phi f\}\nsucc\{b\}.$ $f\{bw\}.\, f\{cd\}.\, c\circ b\ .\supset:\ \phi\{db\}.\mathrm{v}.\,\phi\{wd\}$	
PR	$[abcdw\theta\phi f] \therefore \mathrm{Hp}(7)\ .\supset:$	
8.	$f\{cw\}.$	$[7;\ 5;\ T1.1.37]$
9.	$\rightrightarrows \nprec f \nsucc .$	$[2;\ D2.1.01]$
10.	$w\circ d.$	$[6;\ 8;\ 9;\ D1.1.015]$
11.	$\cap \nprec \mathbf{ADM}\{a\theta\phi f\}\nsucc\{w\}.$	$[4;\ 5;\ T2.1.11]$
12.	$\theta\{w\}.$	$[2;\ 3;\ 11;\ T2.1.6]$
13.	$\mathbf{R}\langle\theta\rangle \nprec\phi\nsucc .$	$[1;\ D1.2.04]$
14.	$\phi\{ww\}.$	$[13;\ 12;\ D1.2.01]$
15.	$\phi\{wd\}:$	$[14;\ 10;\ T1.1.36]$
	$\phi\{db\}.\mathrm{v}.\,\phi\{wd\}$	$[15]$
T2.1.40	$[abcdw\theta\phi f] \therefore \mathbf{PO}\langle\theta\rangle \nprec\phi\nsucc .\, \mathbf{IN}\langle\theta\theta\rangle \nprec f \nsucc .\, f\subset\phi .\, \theta\{a\}.$ $\cap \nprec \mathbf{ADM}\{a\theta\phi f\}\nsucc\{b\}.\, f\{bw\}.\, \mathbf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\{c\}.\, f\{cd\}.\, \phi\{wc\}\ .\supset:$ $\phi\{db\}.\mathrm{v}.\,\phi\{wd\}$	
PR	$[abcdw\theta\phi f] \therefore \mathrm{Hp}(9)\ .\supset:$	
10.	$\theta\{cd\}.$	$[8;\ 3;\ T1.1.35]$
11.	$\cap \nprec \mathbf{ADM}\{a\theta\phi f\}\nsucc\{c\}.$	$[7;\ D2.1.07]$

12. $\theta\{c\}$. [2; 4; 11; *T2.1.6*]
13. $\theta\{d\}$. [2; 8; *T2.1.1*]
14. $\theta\{b\}$. [2; 4; 5; *T2.1.6*]
15. $\theta\{w\}$. [2; 6; *T2.1.1*]
16. $\mathbf{T}\langle\theta\rangle \nleftarrow \phi \nrightarrow$. [1; *D1.2.04*]
17. $\theta\{wd\}$: [12; 13; 15; 9; 10; *D1.2.02*]
$\theta\{db\} . \vee . \theta\{wd\}$ [17]

T2.1.41 $[abcdw\theta\phi f] \therefore \mathsf{PO}\langle\theta\rangle \nleftarrow\phi\nrightarrow . \mathsf{IN}\langle\theta\theta\rangle \nleftarrow f \nrightarrow . \theta\{a\} . \cap \nleftarrow \mathsf{ADM}\{_1 a\theta\phi f\}_1 \nrightarrow \{b\} .$
$\mathsf{G1}\{_1 a\theta\phi f\}_1\{b\} . f\{bw\} . \mathsf{B1}\{_2 ba\theta\phi fw\}_2\{c\} . f\{cd\} . \phi\{cb\} . \supset :$
$\phi\{db\} . \vee . \phi\{wd\}$

PR $[abcdw\theta\phi f] \therefore \mathrm{Hp}(9) . \supset :$
10. $\theta\{b\}$. [5; *D2.1.06*]
11. $\cap \nleftarrow \mathsf{ADM}\{_1 a\theta\phi f\}_1 \nrightarrow \{c\}$. [7; *D2.1.07*]
12. $\theta\{c\}$: [2; 3; 11; *T2.1.6*]
13. $\sim(\phi\{bc\}) . \vee . c \circ b$: [9; 1; 12; 10; *T1.2.9*]
$\phi\{db\} . \vee . \phi\{wd\}$ [13; 5; 7; 8; 9; *T2.1.38*; 13; 1; 2; 3; 4; 6; 8; *T2.1.39*]

T2.1.42 $[abcdw\theta\phi f] \therefore \mathsf{PO}\langle\theta\rangle \nleftarrow\phi\nrightarrow . \mathsf{IN}\langle\theta\theta\rangle \nleftarrow f \nrightarrow . \theta\{a\} . f \subset \phi .$
$\cap \nleftarrow \mathsf{ADM}\{_1 a\theta\phi f\}_1 \nrightarrow \{b\} . \mathsf{G1}\{_1 a\theta\phi f\}_1\{b\} . f\{bw\} . \mathsf{B1}\{_2 ba\theta\phi fw\}_2\{c\} . f\{cd\} . \supset :$
$\phi\{db\} . \vee . \phi\{wd\}$

PR $[abcdw\theta\phi f] \therefore \mathrm{Hp}(9) . \supset :$
10. $\phi\{cd\} . \vee . \phi\{wc\}$: [8; *D2.1.07*]
$\phi\{db\} . \vee . \phi\{wd\}$ [10; 1; 2; 3; 5; 6; 7; 8; 9; *T2.1.41*; 10; 1; 2; 3; 4; 5; 6; 7; 8; *T2.1.40*]

T2.1.43 $[abcdw\theta\phi f] : \mathsf{PO}\langle\theta\rangle \nleftarrow\phi\nrightarrow . \mathsf{IN}\langle\theta\theta\rangle \nleftarrow f \nrightarrow . f \subset \phi . \theta\{a\} .$
$\cap \nleftarrow \mathsf{ADM}\{_1 a\theta\phi f\}_1 \nrightarrow \{b\} . \mathsf{G1}\{_1 a\theta\phi f\}_1\{b\} . f\{bw\} . \mathsf{B1}\{_2 ba\theta\phi fw\}_2\{c\} . f\{cd\} . \supset .$
$\mathsf{B1}\{_2 ba\theta\phi fw\}_2\{d\}$

PR $[abcdw\theta\phi f] \therefore \mathrm{Hp}(9) . \supset :$
10. $\cap \nleftarrow \mathsf{ADM}\{_1 a\theta\phi f\}_1 \nrightarrow \{d\}$: [8; 9; *T2.1.37*]
11. $\phi\{db\} . \vee . \phi\{wd\}$: [1; 2; 3; 4; 5; 6; 7; 8; 9; *T2.1.42*]
$\mathsf{B1}\{_2 ba\theta\phi fw\}_2\{d\}$ [10; 11; *D2.1.07*]

T2.1.44 $[abw\theta\phi f] : \mathsf{PO}\langle\theta\rangle \nleftarrow\phi\nrightarrow . \mathsf{IN}\langle\theta\theta\rangle \nleftarrow f \nrightarrow . f \subset \phi . \theta\{a\} .$
$\cap \nleftarrow \mathsf{ADM}\{_1 a\theta\phi f\}_1 \nrightarrow \{b\} . \mathsf{G1}\{_1 a\theta\phi f\}_1\{b\} . f\{bw\} . \supset . \mathsf{C1}\langle \mathsf{B1}\{_2 ba\theta\phi fw\}_2\rangle \nleftarrow f \nrightarrow$
[*T2.1.43*; *D2.1.02*]

T2.1.45 $[abmw\sigma\theta\phi f] : \mathsf{PO}\langle\theta\rangle \nleftarrow\phi\nrightarrow . \mathsf{IN}\langle\theta\theta\rangle \nleftarrow f \nrightarrow . \theta\{a\} . ! \nleftarrow \sigma \nrightarrow .$
$\mathsf{CH} \nleftarrow \mathsf{B1}\{_2 ba\theta\phi fw\}_2 \phi \nrightarrow \nleftarrow \sigma \nrightarrow . \mathsf{LUB} \nleftarrow \sigma\theta\phi \nrightarrow \{m\} . \supset . \cap \nleftarrow \mathsf{ADM}\{_1 a\theta\phi f\}_1 \nrightarrow \{m\}$

PR $[abmw\sigma\theta\phi f] : \mathrm{Hp}(6) . \supset .$
7. $\mathsf{CH} \nleftarrow \cap \nleftarrow \mathsf{ADM}\{_1 a\theta\phi f\}_1 \nrightarrow \phi \nrightarrow \nleftarrow \sigma \nrightarrow$. [1; 2; 3; 5; *T2.1.33*; *T2.1.27*]
$\cap \nleftarrow \mathsf{ADM}\{_1 a\theta\phi f\}_1 \nrightarrow \{m\}$ [1; 4; 7; 6; *T2.1.17*]

T2.1.46 $[abwz\sigma\theta\phi f] \therefore \sigma \subset \mathsf{B1}\{_2 ba\theta\phi fw\}_2 . \sigma\{z\} . \supset : \phi\{zb\} . \vee . \phi\{wz\}$

PR $[abwz\sigma\theta\phi f] \therefore \mathrm{Hp}(2) . \supset :$
3. $\mathsf{B1}\{_2 ba\theta\phi fw\}_2\{z\}$: [1; 2; *T1.1.14*]
$\phi\{zb\} . \vee . \phi\{wz\}$ [3; *D2.1.07*]

T2.1.47 $[abw\sigma\theta\phi f]::\sigma\subset \mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\,.\supset\therefore[z]:\sigma\{z\}\,.\supset.\,\phi\{zb\}:\vee.[\exists z]\,.$
$\sigma\{z\}\,.\,\phi\{wz\}$ [*T2.1.46*]

T2.1.48 $[abw\sigma\theta\phi f]\therefore \mathsf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\{\!\!\{f\}\!\!\}\,.\,\theta\{a\}\,.\cap\langle\!\langle\mathsf{ADM}\{a\theta\phi f\}\rangle\!\rangle\{b\}\;\sigma\subset$
$\mathsf{B1}\underset{2}{\{}a\theta\phi fw\underset{2}{\}}:[z]:\sigma\{z\}\,.\supset.\,\phi\{zb\}:\supset.\,\mathsf{UB}\{\!\!\{\sigma\theta\phi\}\!\!\}\{b\}$

PR $[abwz\sigma\theta\phi f]\therefore \mathrm{Hp}(6):\supset.$
7. $\theta\{b\}.$ [2; 3; 4; *T2.1.6*]
8. $\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\subset\theta\,.$ [2; 3; *T2.1.35*]
9. $\sigma\subset\theta\,.$ [5; 8; *T1.1.13*]
$\mathsf{UB}\{\!\!\{\sigma\theta\phi\}\!\!\}\{b\}$ [1; 7; 9; 6; *D1.2.08*]

T2.1.49 $[abmw\sigma\theta\phi f]\therefore \mathsf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\{\!\!\{f\}\!\!\}\,.\,\theta\{a\}\,.\cap\langle\!\langle\mathsf{ADM}\{a\theta\phi f\}\rangle\!\rangle\{b\}\,.$
$\sigma\subset\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\,.\,\mathsf{LUB}\{\!\!\{\sigma\theta\phi\}\!\!\}\{m\}:[z]:\sigma\{z\}\,.\supset.\,\phi\{zb\}:\supset:$
$\phi\{mb\}\,.\vee.\,\phi\{wm\}$

PR $[abmzw\sigma\theta\phi f]\therefore \mathrm{Hp}(7):\supset:$
8. $\mathsf{UB}\{\!\!\{\sigma\theta\phi\}\!\!\}\{b\}\,.$ [1; 2; 3; 4; 5; 7; *T2.1.48*]
9. $\phi\{mb\}:$ [6; 8; *D1.2.09*; *D1.2.08*]
$\phi\{mb\}\,.\vee.\,\phi\{wm\}$ [9]

T2.1.50 $[abmzw\sigma\theta\phi f]\therefore \mathsf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\{\!\!\{f\}\!\!\}\,.\,\theta\{a\}\,.\cap\langle\!\langle\mathsf{ADM}\{a\theta\phi f\}\rangle\!\rangle\{b\}\,.$
$f\{bw\}\,.\,\sigma\subset\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\,.\,\mathsf{LUB}\{\!\!\{\sigma\theta\phi\}\!\!\}\{m\}\,.\,\sigma\{z\}\,.\,\phi\{wz\}\,.\supset:$
$\phi\{mb\}\,.\vee.\,\phi\{wm\}$

PR $[abmzw\sigma\theta\phi f]\therefore \mathrm{Hp}(9)\,.\supset:$
9. $\mathsf{UB}\{\!\!\{\sigma\theta\phi\}\!\!\}\{m\}:$ [7; *D1.2.09*]
10. $[l]:\sigma\{l\}\,.\supset.\,\phi\{lm\}:$ [9; *D1.2.08*]
11. $\theta\{m\}.$ [9; *D1.2.08*]
12. $\theta\{w\}.$ [4; 5; *T2.1.1*]
13. $\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\subset\theta$ [2; 3; *T2.1.35*]
14. $\sigma\subset\theta\,.$ [6; 13; *T1.1.13*]
15. $\mathbf{T}\langle\theta\rangle\{\!\!\{\phi\}\!\!\}$ [1; *D1.2.04*]
16. $\theta\{z\}.$ [14; 8; *T1.1.14*]
17. $\phi\{zm\}.$ [8; 10]
18. $\phi\{wm\}:$ [11; 12; 16; 9; 17; 15; *D1.2.02*]
$\phi\{mb\}\,.\vee.\,\phi\{wm\}$ [20]

T2.1.51 $[abmw\sigma\theta\phi f]\therefore \mathsf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\{\!\!\{f\}\!\!\}\,.\,\theta\{a\}\,.\cap\langle\!\langle\mathsf{ADM}\{a\theta\phi f\}\rangle\!\rangle\{b\}\,.$
$f\{bw\}\,.\,\sigma\subset\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\,.\,\mathsf{LUB}\{\!\!\{\sigma\theta\phi\}\!\!\}\{m\}\,.\supset:\,\phi\{mb\}\,.\vee.\,\phi\{wm\}$

PR $[abmw\sigma\theta\phi f]::\mathrm{Hp}(7)\,.\supset\therefore$
8. $[z]:\sigma\{z\}\,.\supset.\,\phi\{zb\}:\vee.[\exists z]\,.\,\sigma\{z\}\,.\,\phi\{wz\}\therefore$ [6; *T2.1.47*]
$\phi\{mb\}\,.\vee.\,\phi\{wm\}$
[1; 2; 3; 4; 6; 7; 8; *T2.1.49*; 1; 2; 3; 4; 5; 6; 7; 8; *T2.1.50*]

T2.1.52 $[abmw\sigma\theta\phi f]:\mathsf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\{\!\!\{f\}\!\!\}\,.\,\theta\{a\}\,.\cap\langle\!\langle\mathsf{ADM}\{a\theta\phi f\}\rangle\!\rangle\{b\}\,.$
$f\{bw\}\,.\,!\{\!\!\{\sigma\}\!\!\}\,.\,\mathsf{CH}\langle\!\langle\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\phi\rangle\!\rangle\{\!\!\{\sigma\}\!\!\}\,.\,\mathsf{LUB}\{\!\!\{\sigma\theta\phi\}\!\!\}\{m\}\,.\supset.$
$\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\{m\}$

PR $[abmw\sigma\theta\phi f]\therefore \mathrm{Hp}(8)\,.\supset:$
9. $\sigma\subset\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\,.$ [7; *D1.2.011*]

10. $\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ\{m\}$: [1; 2; 3; 6; 7; 8; *T2.1.45*]
11. $\phi\{mb\}.\mathrm{v}.\phi\{wm\}$: [1; 2; 3; 4; 5; 9; 8; *T2.1.51*]
$\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\{m\}$ [10; 11; *D2.1.07*]

T2.1.53 $[abw\theta\phi f]:\mathsf{PO}\langle\theta\rangle\,\{\!|\phi|\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\,\{\!|f|\!\}\,.\,\theta\{a\}\,.\,\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ\{b\}\,.$
$f\{bw\}\;.\supset.\;\mathsf{C2}\{\!|\theta\phi|\!\}\{\!|\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}|\!\}$ [*T2.1.52*; *D2.1.03*]

T2.1.54 $[abw\theta\phi f]:\mathsf{PO}\langle\theta\rangle\,\{\!|\phi|\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\,\{\!|f|\!\}\,.\,f\subset\phi\,.\,\theta\{a\}\,.$
$\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ\{b\}\,.\,\mathsf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{b\}\,.\,f\{bw\}\;.\supset.$
$\mathsf{ADM}\underset{2}{\{}a\theta\phi f\underset{2}{\}}\{\!|\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}|\!\}$
[*T2.1.35*; *T2.1.36*; *T2.1.44*; *T2.1.53*; *D2.1.04*]

T2.1.55 $[abw\theta\phi f]:\mathsf{PO}\langle\theta\rangle\,\{\!|\phi|\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\,\{\!|f|\!\}\,.\,f\subset\phi\,.\,\theta\{a\}\,.$
$\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ\{b\}\,.\,\mathsf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{b\}\,.\,f\{bw\}\;.\supset.$
$\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}\circ\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ$ [*T2.1.33*; *T2.1.54*; *T1.1.27*]

T2.1.55 establishes that $\mathsf{B1}\underset{2}{\{}ba\theta\phi fw\underset{2}{\}}$, though defined in *D2.1.07* as a subset of $\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ$, is, under the given conditions, equal to it. The penultimate step of this proof involves showing that one more subset of $\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ$ is, under certain conditions, equal to it. *D2.1.08* is the auxiliary definition which defines this subset.

D2.1.08 $[am\theta\phi f]:\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ\{m\}\,.\,\mathsf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{m\}\;.\!\equiv\!.\;\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{m\}$

T2.1.56 $[a\theta\phi f]:\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\subset\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ$ [*D2.1.08*; *D1.1.04*]

T2.1.57 $[am\theta\phi f]:\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{m\}\;.\supset.\;\theta\{m\}$ [*D2.1.08*; *D2.1.06*]

T2.1.58 $[a\theta\phi f]:\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\subset\theta$ [*T2.1.57*; *D1.1.04*]

T2.1.59 $[abcd\theta\phi f]:\mathsf{PO}\langle\theta\rangle\,\{\!|\phi|\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\,\{\!|f|\!\}\,.\,f\subset\phi\,.\,\theta\{a\}\,.$
$\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ\{c\}\,.\sim(\phi\{ac\})\;.\supset.\;\phi\{da\}$
PR $[acd\theta\phi f]\therefore\mathrm{Hp}(6)\;.\supset:$
7. $\phi\{ac\}$. [1; 2; 3; 4; 5; *T2.1.32*]
$\phi\{da\}$ [6; 7]

T2.1.60 $[a\theta\phi f]\therefore\mathsf{PO}\langle\theta\rangle\,\{\!|\phi|\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\,\{\!|f|\!\}\,.\,f\subset\phi\,.\,\theta\{a\}\;.\supset:[cd]:$
$\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ\{c\}\,.\,\phi\{ca\}\,.\sim(\phi\{ac\})\,.\,f\{cd\}\;.\supset.\;\phi\{da\}$ [*T2.4.59*]

T2.1.61 $[a\theta\phi f]:\mathsf{PO}\langle\theta\rangle\,\{\!|\phi|\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\,\{\!|f|\!\}\,.\,f\subset\phi\,.\,\theta\{a\}\;.\supset.\;\mathsf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{a\}$
[*T2.1.60*; *D2.1.06*]

T2.1.62 $[a\theta\phi f]:\mathsf{PO}\langle\theta\rangle\,\{\!|\phi|\!\}\,.\,\mathsf{IN}\langle\theta\theta\rangle\,\{\!|f|\!\}\,.\,f\subset\phi\,.\,\theta\{a\}\;.\supset.\;\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{a\}$
PR $[a\theta\phi f]:\mathrm{Hp}(4)\;.\supset.$
5. $\mathsf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{a\}$. [1; 2; 3; 4; *T2.1.61*]
$\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{a\}$ [*T2.1.9*; 6; *D2.1.08*]

T2.1.63 $[acd\theta\phi f]:\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{c\}\,.\,f\{cd\}\;.\supset.\;\cap \prec\!\mathsf{ADM}\{a\theta\phi f\}\!\succ\{d\}$
PR $[acd\theta\phi f]:\mathrm{Hp}(2)\;.\supset.$

3. $\cap \langle\!\langle \mathbf{ADM}\{\!\{a\theta\phi f\}\!\}\rangle\!\rangle\{c\}.$ [1; *D2.1.08*]
$\cap \langle\!\langle \mathbf{ADM}\{\!\{a\theta\phi f\}\!\}\rangle\!\rangle\{d\}$ [3; 2; *T2.1.11*]

T2.1.64 $[acdlm\theta\phi f] : \mathbf{PO}\langle\theta\rangle \nless\phi\ngtr . f \subset \phi . \mathbf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{c\} . f\{cd\} .$
$\cap \langle\!\langle \mathbf{ADM}\{\!\{a\theta\phi f\}\!\}\rangle\!\rangle\{l\} . \phi\{lc\} . \sim(\phi\{cl\}) . f\{lm\} . \theta\{d\} . \theta\{c\} .$
$\theta\{m\} .\supset. \phi\{md\}$

PR $[acdlm\theta\phi f] \therefore \mathrm{Hp}(11) .\supset:$
12. $[jk] : \cap \langle\!\langle \mathbf{ADM}\{\!\{a\theta\phi f\}\!\}\rangle\!\rangle\{j\} . \phi\{jc\} . \sim(\phi\{cj\}) . f\{jk\} .\supset. \phi\{kc\} :$ [3; *D2.1.06*]
13. $\phi\{mc\} .$ [5; 6; 7; 8; 12]
14. $\phi\{cd\} .$ [2; 4; *T1.1.35*]
15. $\mathbf{T}\langle\theta\rangle \nless\phi\ngtr .$ [1; *D1.2.04*]
$\phi\{md\}$ [9; 10; 11; 13; 14; 15; *D1.2.02*]

T2.1.65 $[cdlm\theta\phi f] : \mathbf{PO}\langle\theta\rangle \nless\phi\ngtr . \mathbf{IN}\langle\theta\theta\rangle \nless f\ngtr . \theta\{m\} . f\{cd\} . f\{lm\} .$
$l \circ c .\supset. \phi\{md\}$

PR $[cdlm\theta\phi f] : \mathrm{Hp}(6) .\supset.$
7. $\mathbf{R}\langle\theta\rangle \nless\phi\ngtr .$ [1; *D1.2.04*]
8. $\phi\{mm\} .$ [7; 3; *D1.2.01*]
9. $\rightrightarrows \nless f\ngtr .$ [2; *D2.1.01*]
10. $f\{cm\} .$ [6; 5; *T1.1.37*]
11. $m \circ d .$ [4; 9; 10; *D1.1.015*]
$\phi\{md\}$ [8; 11; *T1.1.36*]

T2.1.66 $[acdlm\theta\phi f] : \mathbf{PO}\langle\theta\rangle \nless\phi\ngtr . \mathbf{IN}\langle\theta\theta\rangle \nless f\ngtr . f \subset \phi . \theta\{a\} . \mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{c\} .$
$f\{cd\} . \cap \langle\!\langle \mathbf{ADM}\{\!\{a\theta\phi f\}\!\}\rangle\!\rangle\{l\} . \sim(\phi\{dl\}) . f\{lm\} .\supset. \phi\{md\}$

PR $[acdlm\theta\phi f] \therefore (9) .\supset:$
10. $\cap \langle\!\langle \mathbf{ADM}\{\!\{a\theta\phi f\}\!\}\rangle\!\rangle\{c\} .$ [5; *D2.1.08*]
11. $\mathbf{G1}\underset{1}{[}a\theta\phi f\underset{1}{]}\{c\} .$ [5; *D2.1.08*]
12. $\theta\{c\} .$ [11; *D2.1.06*]
13. $\cap \langle\!\langle \mathbf{ADM}\{\!\{a\theta\phi f\}\!\}\rangle\!\rangle\{m\} .$ [7; 9; *T2.1.11*]
14. $\cap \langle\!\langle \mathbf{ADM}\{\!\{a\theta\phi f\}\!\}\rangle\!\rangle\{d\} .$ [10; 6; *T2.1.11*]
15. $\theta\{m\} .$ [2; 4; 13; *T2.1.6*]
16. $\theta\{d\} .$ [2; 4; 14; *T2.1.6*]
17. $\mathbf{B1}\underset{2}{\{}ca\theta\phi fd\underset{2}{\}} \circ \cap \langle\!\langle \mathbf{ADM}\{\!\{a\theta\phi f\}\!\}\rangle\!\rangle .$ [1; 2; 3; 4; 10; 11; 6; *T2.1.55*]
18. $\mathbf{B1}\underset{2}{\{}ca\theta\phi fd\underset{2}{\}}\{l\} :$ [7; 17; *T1.1.10*]
19. $\phi\{lc\} .\mathrm{v}. \phi\{dl\} :$ [18; *D2.1.07*]
20. $\phi\{lc\} .$ [19; 8]
21. $\theta\{l\} :$ [2; 4; 7; *T2.1.6*]
22. $l \circ c .\mathrm{v}. \sim(\phi\{cl\}) :$ [1; 21; 12; 20; *T1.2.9*]
$\phi\{md\}$ [22; 1; 2; 15; 6; 9; *T2.1.65*; 22; 1; 3; 11; 6; 7; 20; 9; 16; 12; 15; *T2.1.64*]

T2.1.67 $[acd\theta\phi f] \therefore \mathbf{PO}\langle\theta\rangle \nless\phi\ngtr . \mathbf{IN}\langle\theta\theta\rangle \nless f\ngtr . f \subset \phi . \theta\{a\} . \mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{c\} .$
$f\{cd\} .\supset: [lm] : \cap \langle\!\langle \mathbf{ADM}\{\!\{a\theta\phi f\}\!\}\rangle\!\rangle\{l\} . \phi\{ld\} . \sim(\phi\{dl\}) .$
$f\{lm\} .\supset. \phi\{md\}$ [*T2.1.66*]

T2.1.68 $[acd\theta\phi f]:\mathbf{PO}\langle\theta\rangle\nprec\phi\nsucc.\mathbf{IN}\langle\theta\theta\rangle\nprec f\nsucc.f\subset\phi.\theta\{a\}.\mathbf{G2}\underset{1}{\{\!\vert}a\theta\phi f\underset{1}{\vert\!\}}\{c\}.f\{cd\}.$
$\supset.\ \mathbf{G1}\underset{1}{\{\!\vert}a\theta\phi f\underset{1}{\vert\!\}}\{d\}$

PR $[acd\theta\phi f]\therefore\mathrm{Hp}(6)\ .\supset:$

7. $\cap\nless\mathbf{ADM}\{\!\vert a\theta\phi f\vert\!\}\ngtr\{c\}.$ [5; *D2.1.08*]

8. $\cap\nless\mathbf{ADM}\{\!\vert a\theta\phi f\vert\!\}\ngtr\{d\}.$ [5; 6; *T2.1.63*]

9. $\theta\{d\}:$ [2; 4; 8; *T2.1.6*]

10. $[lm]:\cap\nless\mathbf{ADM}\{\!\vert a\theta\phi f\vert\!\}\ngtr\{l\}.\phi\{ld\}.\sim(\phi\{dl\}).f\{lm\}\ .\supset.\ \phi\{md\}:$ [1; 2; 3; 4; 5; 6; *T2.1.67*]

$\mathbf{G1}\underset{1}{\{\!\vert}a\theta\phi f\underset{1}{\vert\!\}}\{d\}$ [9; 10; *D2.1.06*]

T2.1.69 $[acd\theta\phi f]:\mathbf{PO}\langle\theta\rangle\nprec\phi\nsucc.\mathbf{IN}\langle\theta\theta\rangle\nprec f\nsucc.f\subset\phi.\theta\{a\}.\mathbf{G2}\underset{1}{\{\!\vert}a\theta\phi f\underset{1}{\vert\!\}}\{c\}.f\{cd\}.$
$\supset.\ \mathbf{G2}\underset{1}{\{\!\vert}a\theta\phi f\underset{1}{\vert\!\}}\{d\}$

PR $[acd\theta\phi f]:\mathrm{Hp}(6)\ .\supset.$

7. $\cap\nless\mathbf{ADM}\{\!\vert a\theta\phi f\vert\!\}\ngtr\{d\}.$ [5; 6; *T2.1.63*]

8. $\mathbf{G1}\underset{1}{\{\!\vert}a\theta\phi f\underset{1}{\vert\!\}}\{d\}.$ [1; 2; 3; 4; 5; 6; *T2.1.68*]

$\mathbf{G2}\underset{1}{[}a\theta\phi f\underset{1}{]}\{d\}$ [7; 8; *D2.1.08*]

T2.1.70 $[a\theta\phi f]:\mathbf{PO}\langle\theta\rangle\nprec\phi\nsucc.\mathbf{IN}\langle\theta\theta\rangle\nprec f\nsucc.f\subset\phi.\theta\{a\}\ .\supset.\ \mathbf{C1}\langle\mathbf{G2}\{\!\vert a\theta\phi f\vert\!\}\rangle\nprec f\nsucc$ [*T2.1.69*; *D2.1.02*]

T2.1.71 $[am\sigma\theta\phi f]:\mathbf{PO}\langle\theta\rangle\nprec\phi\nsucc.\mathbf{IN}\langle\theta\theta\rangle\nprec f\nsucc.\theta\{a\}.!\nprec\sigma\nsucc.$
$\mathbf{CH}\nless\mathbf{G2}\underset{1}{\{\!\vert}a\theta\phi f\underset{1}{\vert\!\}}\phi\ngtr\nprec\sigma\nsucc.\mathbf{LUB}\nprec\sigma\theta\phi\nsucc\{m\}\ .\supset.\ \cap\nless\mathbf{ADM}\{\!\vert a\theta\phi f\vert\!\}\ngtr\{m\}$

PR $[am\sigma\theta\phi f]:\mathrm{Hp}(6)\ .\supset:$

7. $\sigma\subset\mathbf{G2}\underset{1}{[}a\theta\phi f\underset{1}{]}$ [5; *D1.2.011*]

8. $\sigma\subset\cap\nless\mathbf{ADM}\{\!\vert a\theta\phi f\vert\!\}\ngtr.$ [7; *T2.1.56*; *T1.1.13*]

9. $\mathbf{CH}\nless\cap\nless\mathbf{ADM}\{\!\vert a\theta\phi f\vert\!\}\ngtr\phi\ngtr\nprec\sigma\nsucc$ [1; 2; 3; 5; 8; *T2.1.27*]

$\cap\nless\mathbf{ADM}[a\theta\phi f]\ngtr\{m\}$ [1; 4; 9; 6; *T2.1.17*]

T2.1.72 $[acl\sigma\theta\phi f]\therefore\mathbf{PO}\langle\theta\rangle\nprec\sigma\nsucc.\mathbf{IN}\langle\theta\theta\rangle\nprec f\nsucc.f\subset\phi.\theta\{a\}.\sigma\subset\mathbf{G2}\underset{1}{\{\!\vert}a\theta\phi f\underset{1}{\vert\!\}}.$
$\cap\nless\mathbf{ADM}\{\!\vert a\theta\phi f\vert\!\}\ngtr\{c\}:[s]:\sigma\{s\}\ .\supset.\ \sim(\phi\{cs\}):\sigma\{l\}\ .\supset.\ \phi\{lc\}$

PR $[acl\sigma\theta\phi f]::\mathrm{Hp}(8)\ .\supset\therefore$

9. $\sim(\phi\{cl\}).$ [7; 8]

10. $\mathbf{G2}\underset{1}{\{\!\vert}a\theta\phi f\underset{1}{\vert\!\}}\{l\}.$ [8; 5; *T1.1.14*]

11. $\cap\nless\mathbf{ADM}\{\!\vert a\theta\phi f\vert\!\}\ngtr\{l\}.$ [10; *D2.1.08*]

12. $\mathbf{G1}\underset{1}{\{\!\vert}a\theta\phi f\underset{1}{\vert\!\}}\{l\}.$ [10; *D2.1.08*]

13. $\theta\{l\}.$ [12; *D2.1.06*]

14. $\mathbf{D}\nprec f\nsucc\circ\theta.$ [2; *D2.1.01*]

15. $\mathbf{D}\nprec f\nsucc\{l\}.$ [13; 14; *T1.1.10*]

16. $\theta\{c\}.$ [2; 4; 6; *T2.1.6*]

17. $\mathbf{T}\langle\theta\rangle\nprec\phi\nsucc\therefore$ [1; *D1.2.04*]

$[\exists w]\therefore$

18. $f\{lw\}.$ [15; *D1.1.013*]

19. $\cap \prec \mathbf{ADM}\{\!|a\theta\phi f|\!\}\succ\{w\}$. [11; 18; *T2.1.11*]
20. $\theta\{w\}$. [2; 4; 19; *T2.1.6*]
21. $\phi\{lw\}$. [3; 18; *T1.1.35*]
22. $\mathbf{B1}\{\!|_2 la\theta\phi fw|\!\}_2 \circ \cap \prec\mathbf{ADM}\{\!|a\theta\phi f|\!\}\succ$. [2; 3; 4; 11; 12; 18; *T2.1.55*]
23. $\mathbf{B1}\{\!|_1 la\theta\phi fw|\!\}_1\{c\}$: [6; 22; *T1.1.10*]
24. $\phi\{cl\} . \vee . \phi\{wc\}$: [23; *D2.1.07*]
25. $\phi\{wc\} \therefore$ [9; 24]

$\phi\{lc\}$ [21; 25; 13; 16; 20; 17; *D1.2.02*]

T2.1.73 $[ac\sigma\theta\phi f] \therefore \mathbf{PO}\langle\theta\rangle \{\!|\phi|\!\} . \mathbf{IN}\langle\theta\theta\rangle \{\!|f|\!\} . f \subset \phi . \theta\{a\} . \sigma \subset \mathbf{G2}\{\!|_1 a\theta\phi f|\!\}_1 .$
$\cap \prec\mathbf{ADM}\{\!|a\theta\phi f|\!\}\succ\{c\} : [s] : \sigma\{s\} .\supset. \sim(\phi\{cs\}) :\supset: [l] : \sigma\{l\} .\supset. \phi\{lc\}$
[*T2.1.72*]

T2.1.74 $[ac\sigma\theta\phi f] \therefore \mathbf{PO}\langle\theta\rangle \{\!|\phi|\!\} . \mathbf{IN}\langle\theta\theta\rangle \{\!|f|\!\} . f \subset \phi . \theta\{a\} . \sigma \subset \mathbf{G2}\{\!|_1 a\theta\phi f|\!\}_1 .$
$\cap \prec\mathbf{ADM}\{\!|a\theta\phi f|\!\}\succ\{c\} : [s] : \sigma\{s\} .\supset. \sim(\phi\{cs\}) :\supset. \mathbf{UB}\{\!|\sigma\theta\phi|\!\}\{c\}$

PR $[ac\sigma\theta\phi f] \therefore$ Hp(7) $.\supset:$
8. $[l] : \sigma\{l\} .\supset. \phi\{lc\}$: [1; 2; 3; 4; 5; 6; 7; *T2.1.73*]
9. $\phi\{c\}$. [2; 4; 6; *T2.1.6*]
10. $\sigma \subset \theta$. [5; *T1.1.13*; *T2.1.58*]

$\mathbf{UB}\{\!|\sigma\theta\phi|\!\}\{c\}$ [1; 9; 10; 8; *D1.2.08*]

T2.1.75 $[acm\sigma\theta\phi f] \therefore \mathbf{PO}\langle\theta\rangle \{\!|\phi|\!\} . \mathbf{IN}\langle\theta\theta\rangle \{\!|f|\!\} . f \subset \phi . \theta\{a\} . \sigma \subset \mathbf{G2}\{\!|_1 a\theta\phi f|\!\}_1 .$
$\mathbf{LUB}\{\!|\sigma\theta\phi|\!\}\{m\} . \cap \prec\mathbf{ADM}\{\!|a\theta\phi f|\!\}\succ\{c\} . \sim(\phi\{mc\}) : [s] : \sigma\{s\} .\supset.$
$\sim(\phi\{cs\}) :\supset: \sim([s] : \sigma\{s\} .\supset. \sim(\phi\{cs\}))$

PR $[acm\sigma\theta\phi f]$: Hp(9) $.\supset.$
10. $\mathbf{UB}\{\!|\sigma\theta\phi|\!\}\{c\}$. [1; 2; 3; 4; 5; 7; 9; *T2.1.74*]
11. $\phi\{mc\}$. [6; 10; *T1.2.13*]

$\sim([s] : \sigma\{s\} .\supset. \sim(\phi\{cs\}))$ [8; 11]

T2.1.76 $[acm\sigma\theta\phi f] : \mathbf{PO}\langle\theta\rangle \{\!|\phi|\!\} . \mathbf{IN}\langle\theta\theta\rangle \{\!|f|\!\} . f \subset \phi . \theta\{a\} . \sigma \subset \mathbf{G2}\{\!|_1 a\theta\phi f|\!\}_1 .$
$\mathbf{LUB}\{\!|\sigma\theta\phi|\!\}\{m\} . \cap \prec\mathbf{ADM}\{\!|a\theta\phi f|\!\}\succ\{c\} . \sim(\phi\{mc\}) .\supset.$
$\sim([s] : \sigma\{s\} .\supset. \sim(\phi\{cs\}))$ [*T2.1.75*]

T2.1.77 $[acm\sigma\theta\phi f] : \mathbf{PO}\langle\theta\rangle \{\!|\phi|\!\} . \mathbf{IN}\langle\theta\theta\rangle \{\!|f|\!\} . f \subset \phi . \theta\{a\} . \sigma \subset \mathbf{G2}\{\!|_1 a\theta\phi f|\!\}_1 .$
$\mathbf{LUB}\{\!|\sigma\theta\phi|\!\}\{m\} . \cap \prec\mathbf{ADM}\{\!|a\theta\phi f|\!\}\succ\{c\} . \sim(\phi\{mc\}) .\supset.$
$[\exists s] . \sigma\{s\} . \phi\{cs\}$ [*T2.1.76*]

T2.1.78 $[acdms\sigma\theta\phi f] : \mathbf{PO}\langle\theta\rangle \{\!|\phi|\!\} . \mathbf{IN}\langle\theta\theta\rangle \{\!|f|\!\} . \theta\{a\} . \sigma \subset \mathbf{G2}\{\!|_1 a\theta\phi f|\!\}_1 .$
$\mathbf{LUB}\{\!|\sigma\theta\phi|\!\}\{m\} . \cap \prec\mathbf{ADM}\{\!|a\theta\phi f|\!\}\succ\{c\} . f\{cd\} . \sigma\{s\} . \phi\{cs\} .$
$\sim(\phi\{sc\}) .\supset. \phi\{dm\}$

PR $[acdms\sigma\theta\phi f] \therefore$ Hp(10) $.\supset:$
11. $\mathbf{G2}\{\!|_1 a\theta\phi f|\!\}_1\{s\}$. [8; 4; *T1.1.14*]
12. $\theta\{s\}$. [11; *T2.1.57*]
13. $\mathbf{T}\langle\theta\rangle \{\!|\phi|\!\}$. [1; *D1.2.04*]
14. $\theta\{m\}$. [5; *D1.2.09*; *D1.2.08*]
15. $\mathbf{G1}\{\!|_1 a\theta\phi f|\!\}_1\{s\}$: [11; *D2.1.08*]
16. $[jk] : \cap \prec\mathbf{ADM}\{\!|a\theta\phi f|\!\}\succ\{j\} . \phi\{js\} . \sim(\phi\{sj\}) . f\{jk\} .\supset. \phi\{ks\}$:
[15; *D2.1.06*]

17. $\phi\{ds\}$. [6; 7; 9; 10; 16]
18. $\cap \langle\!|\mathsf{ADM}\{a\theta\phi f\}|\!\rangle\{d\}$. [6; 7; *T2.1.11*]
19. $\theta\{d\}$. [2; 3; 18; *T2.1.6*]
20. $\phi\{sm\}$. [8; 5; *D1.2.09*; *D1.2.08*]
$\phi\{dm\}$ [13; 12; 14; 19; 17; 20; *D1.2.02*]

T2.1.79 $[acdms\sigma\theta\phi f] : \mathsf{PO}\langle\theta\rangle \{\!-\phi-\!\} . \mathsf{IN}\langle\theta\theta\rangle \{\!-f-\!\} . f \subset \phi . \theta\{a\} . !\{\!-\sigma-\!\}$.
$\mathsf{CH}\{\!|\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\phi|\!\}\{\!-\sigma-\!\} . \mathsf{LUB}\{\!=\sigma\theta\phi=\!\}\{m\} . \cap \langle\!|\mathsf{ADM}\{a\theta\phi f\}|\!\rangle\{c\}$.
$\sim(\phi\{mc\}) . f\{cd\} . \sigma\{s\} . c \circ s . \supset . \phi\{dm\}$

PR $[acdms\sigma\theta\phi f] \therefore \mathrm{Hp}(12) . \supset :$
13. $\sigma \subset \mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}$. [6; *D1.2.011*]
14. $\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{s\}$. [11; 13; *T1.1.14*]
15. $\mathsf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{s\}$. [14; *D2.1.08*]
16. $\mathsf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{c\}$. [12; 15; *T1.1.4*]
17. $\mathsf{B1}\underset{2}{\{}ca\theta\phi fd\underset{2}{\}} \circ \cap \langle\!|\mathsf{ADM}\{a\theta\phi f\}|\!\rangle$. [1; 2; 3; 4; 8; 16; 10; *T2.1.55*]
18. $\cap \langle\!|\mathsf{ADM}\{a\theta\phi f\}|\!\rangle\{m\}$. [*T2.1.71*; 1; 2; 4; 6; 7]
19. $\mathsf{B1}\underset{2}{\{}ca\theta\phi fd\underset{2}{\}}\{m\}$: [17; 18; *T1.1.10*]
20. $\phi\{mc\} . \vee . \phi\{dm\}$: [19; *D2.1.07*]
$\phi\{dm\}$ [9; 20]

T2.1.80 $[acdm\sigma\theta\phi f] : \mathsf{PO}\langle\theta\rangle \{\!-\phi-\!\} . \mathsf{IN}\langle\theta\theta\rangle \{\!-f-\!\} . f \subset \phi . \theta\{a\} . !\{\!-\sigma-\!\}$.
$\mathsf{CH}\{\!|\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\phi|\!\}\{\!-\sigma-\!\} . \mathsf{LUB}\{\!=\sigma\theta\phi=\!\}\{m\} . \cap \langle\!|\mathsf{ADM}\{a\theta\phi f\}|\!\rangle\{c\}$.
$\phi\{cm\} . \sim(\phi\{mc\}) . f\{cd\} . \supset . \phi\{dm\}$

PR $[acdm\sigma\theta\phi f] :: \mathrm{Hp}(11) . \supset \therefore$
12. $\sigma \subset \mathsf{G2}\underset{1}{[}a\theta\phi f\underset{1}{]} \therefore$ [6; *D1.2.011*]
$[\exists s] \therefore$
13. $\sigma\{s\}$.
14. $\phi\{cs\}$. [1; 2; 3; 4; 6; 7; 8; 10; *T2.1.77*]
15. $\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{s\}$. [12; 13; *T1.1.14*]
16. $\mathsf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{s\}$. [15; *D2.1.08*]
17. $\theta\{s\}$. [16; *D2.1.06*]
18. $\theta\{c\}$: [2; 4; 8; *T2.1.6*]
19. $c \circ s . \vee . \sim(\phi\{sc\}) \therefore$ [1; 14; 17; 18; *T1.2.9*]
$\phi\{dm\}$ [19; 1; 2; 3; 4; 5; 6; 7; 8; 10; 11; 13; *T2.1.79*; 19; 1; 2; 4; 12; 7; 8; 11; 13; 14; *T2.1.78*]

T2.1.81 $[am\sigma\theta\phi f] \therefore \mathsf{PO}\langle\theta\rangle \{\!-\phi-\!\} . \mathsf{IN}\langle\theta\theta\rangle \{\!-f-\!\} . f \subset \phi . \theta\{a\} . !\{\!-\sigma-\!\}$.
$\mathsf{CH}\{\!|\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\phi|\!\}\{\!-\sigma-\!\} . \mathsf{LUB}\{\!=\sigma\theta\phi=\!\}\{m\} . \supset : \{cd\}$:
$\cap \langle\!|\mathsf{ADM}\{a\theta\phi f\}|\!\rangle\{c\} . \phi\{cm\} . \sim(\phi\{mc\}) . f\{cd\} . \supset . \phi\{dm\}$ [*T2.1.80*]

T2.1.82 $[am\sigma\theta\phi f] : \mathsf{PO}\langle\theta\rangle \{\!-\phi-\!\} . \mathsf{IN}\langle\theta\theta\rangle \{\!-f-\!\} . f \subset \phi . \theta\{a\} . !\{\!-\sigma-\!\}$.
$\mathsf{CH}\{\!|\mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\phi|\!\}\{\!-\sigma-\!\} . \mathsf{LUB}\{\!=\sigma\theta\phi=\!\}\{m\} . \supset . \mathsf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{m\}$

PR $[am\sigma\theta\phi f] \therefore \mathrm{Hp}(7) . \supset :$
8. $\sigma \subset \mathsf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}$. [6; *D1.2.011*]

9. $\theta\{m\}$: [7; *D1.2.09*; *D1.2.08*]

10. $[cd] : \cap \prec \mathbf{ADM}\{a\theta\phi f\}\succ\{c\} . \phi\{cm\} . \sim(\phi\{mc\}) . f\{cd\} .\supset. \phi\{dm\}$:
[1; 2; 3; 4; 5; 6; 7; *T2.1.81*]

$\mathbf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{m\}$ [9; 10; *D2.1.06*]

T2.1.83 $[am\sigma\theta\phi f] : \mathbf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\} . \mathbf{IN}\langle\theta\theta\rangle\{\!\!\{f\}\!\!\} . f \subset \phi . \theta\{a\} . !\{\!\!\{\sigma\}\!\!\}$.
$\mathbf{CH}\{\!\!\{\mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\phi\}\!\!\}\{\!\!\{\sigma\}\!\!\} . \mathbf{LUB}\{\!\!\{\sigma\theta\phi\}\!\!\}\{m\} .\supset. \mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{m\}$

PR $[am\sigma\theta\phi f] : \mathrm{Hp}(7) .\supset.$

8. $\cap\prec\mathbf{ADM}\{a\theta\phi f\}\succ\{m\}$. [1; 2; 4; 5; 6; 7; *T2.1.71*]

9. $\mathbf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{m\}$. [1; 2; 3; 4; 5; 6; 7; *T2.1.82*]

$\mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{m\}$ [8; 9; *D2.1.08*]

T2.1.84 $[a\theta\phi f] : \mathbf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\} . \mathbf{IN}\langle\theta\theta\rangle\{\!\!\{f\}\!\!\} . f \subset \phi . \theta\{a\} .\supset.$
$\mathbf{C2}\{\!\!\{\theta\phi\}\!\!\}\{\!\!\{\mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\}\!\!\}$ [*T2.1.82*; *D2.1.03*]

T2.1.85 $[a\theta\phi f] : \mathbf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\} . \mathbf{IN}\langle\theta\theta\rangle\{\!\!\{f\}\!\!\} . f \subset \phi . \theta\{a\} .\supset.$
$\mathbf{ADM}\{a\theta\phi f\}\{\!\!\{\mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\}\!\!\}$
[*T2.1.58*; *T2.1.62*; *T2.1.70*; *T2.1.84*; *D2.1.04*]

T2.1.86 $[a\theta\phi f] : \mathbf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\} \circ \mathbf{IN}\langle\theta\theta\rangle\{\!\!\{f\}\!\!\} . f \subset \phi . \theta\{a\} .\supset.$
$\mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}} \circ \cap\prec\mathbf{ADM}\{a\theta\phi f\}\succ$

PR $[a\theta\phi f] : \mathrm{Hp}(4) .\supset.$

5. $\mathbf{ADM}\{a\theta\phi f\}\{\!\!\{\mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\}\!\!\}$. [1; 2; 3; 4; *T2.1.85*]

$\mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}} \circ \cap\prec\mathbf{ADM}\{a\theta\phi f\}\succ$ [*T2.1.56*; 6; *T1.1.27*]

T2.1.86 establishes the equality, under the g i v e n c o n d i t i o n s, of $\cap\prec\mathbf{ADM}\{a\theta\phi f\}\succ$ and its subset $\mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}$. Using this we can now complete the proof.

T2.1.87 $[cds\theta\phi] \therefore \mathbf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\} . \theta\{s\} . \theta\{c\} . \theta\{d\} . \phi\{sd\} . \phi\{cs\} .\supset: \phi\{cd\} .\vee.$
$\phi\{cd\}$

PR $[cds\theta\phi] \therefore \mathrm{Hp}(6) .\supset:$

7. $\mathbf{T}\langle\theta\rangle\{\!\!\{\phi\}\!\!\}$ [1; *D1.2.04*]

8. $\phi\{cd\}$: [7; 2; 3; 4; 5; 6; *D1.2.02*]

$\phi\{cd\} .\vee. \phi\{dc\}$ [8]

T2.1.88 $[acd\theta\phi f] \therefore \mathbf{PO}\langle\theta\rangle\{\!\!\{\phi\}\!\!\} . \mathbf{IN}\langle\theta\theta\rangle\{\!\!\{f\}\!\!\} . f \subset \phi . \theta\{a\}$.
$\cap\prec\mathbf{ADM}\{a\theta\phi f\}\succ\{c\} . \cap\prec\mathbf{ADM}\{a\theta\phi f\}\succ\{d\} .\supset: \phi\{cd\} .\vee. \phi\{dc\}$

PR $[acd\theta\phi f] :: \mathrm{Hp}(6) .\supset\therefore$

7. $\mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}} \circ \cap\prec\mathbf{ADM}\{a\theta\phi f\}\succ$. [1; 2; 3; 4; *T2.1.86*]

8. $\mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{c\}$. [5; 7; *T1.1.10*]

9. $\mathbf{G2}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{d\}$. [6; 7; *T1.1.10*]

10. $\mathbf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{c\}$. [8; *D2.1.08*]

11. $\mathbf{G1}\underset{1}{\{}a\theta\phi f\underset{1}{\}}\{d\}$. [9; *D2.1.08*]

12. $\theta\{c\}$. [10; *D2.1.06*]

13. $\theta\{d\}$. [11; *D2.1.06*]
14. $\mathsf{D}\langle\!\langle f\rangle\!\rangle\circ\theta$. [2; *D2.1.01*]
15. $\mathsf{D}\langle\!\langle f\rangle\!\rangle\{c\}\therefore$ [12; 14; *T1.1.10*]
$[\exists s]\therefore$
16. $f\{cs\}$. [15; *D1.1.013*]
17. $\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot\{s\}$. [5; 16; *T2.1.11*]
18. $\theta\{s\}$. [2; 4; 17; *T2.1.6*]
19. $\mathsf{B1}\underset{2}{\{}ca\theta\phi fs\underset{2}{\}}\circ\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot$.
[1; 2; 3; 4; 5; 10; 16; *T2.1.55*]
20. $\mathsf{B1}\underset{2}{\{}ca\theta\phi fs\underset{2}{\}}\{d\}$: [6; 19; *T1.1.10*]
21. $\phi\{sd\}.\mathsf{v}.\phi\{dc\}$: [20; *D2.1.07*]
22. $\phi\{cs\}\therefore$ [3; 16; *T1.1.35*]
$\phi\{cd\}.\mathsf{v}.\phi\{dc\}$ [21; 1; 18; 12; 13; 22; *T2.1.87*; 21]

T2.1.89 $[a\theta\phi f]:\mathsf{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.\mathsf{IN}\langle\theta\theta\rangle\langle\!\langle f\rangle\!\rangle.f\subset\phi.\theta\{a\}.\supset.$
$\mathsf{C}\langle\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot\rangle\langle\!\langle\phi\rangle\!\rangle$ [*T2.1.88*; *D1.2.010*]

T2.1.89 establishes that $\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot$ is a connected set under the conditions stated.

T2.1.90 $[a\theta\phi f]:\mathsf{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.\mathsf{IN}\langle\theta\theta\rangle\langle\!\langle f\rangle\!\rangle.f\subset\phi.\theta\{a\}.\supset.$
$\mathsf{CH}\{\!|\theta\phi|\!\}(\!|\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot|\!)$
PR $[a\theta\phi f]:\mathrm{Hp}(4).\supset.$
5. $\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot\subset\theta$. [2; 4; *T2.1.7*]
6. $\mathsf{C}\langle\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot\rangle\langle\!\langle\phi\rangle\!\rangle$. [1; 2; 3; 4; *T2.1.89*]
$\mathsf{CH}\{\!|\theta\phi|\!\}(\!|\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot|\!)$ [1; 5; 6; *D1.2.011*]

T2.1.90 establishes that under the conditions of our desired "fixed point" theorem, $\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot$ is a chain.

T2.1.91 $[ab\theta\phi f]:\mathsf{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.\mathsf{IN}\langle\theta\theta\rangle\langle\!\langle f\rangle\!\rangle.f\subset\phi.\theta\{a\}.$
$\mathsf{LUB}\{\!|\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot\theta\phi|\!\}\{b\}.\supset.\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot\{b\}$
PR $[ab\theta\phi f]\therefore\mathrm{Hp}(5).\supset:$
6. $\mathsf{CH}\{\!|\theta\phi|\!\}(\!|\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot|\!)$. [1; 2; 3; 4; *T2.1.90*]
7. $\mathsf{CH}\{\!|\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot\phi|\!\}(\!|\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot|\!)$. [6; *T1.2.15*]
8. $\mathsf{C2}\{\!|\theta\phi|\!\}(\!|\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot|\!)$: [1; *T2.1.18*]
9. $[\sigma b]:!(\!|\sigma|\!).\mathsf{CH}\{\!|\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot\phi|\!\}(\!|\sigma|\!).\mathsf{LUB}\{\!|\sigma\theta\phi|\!\}\{b\}.\supset.$
$\cap\lessdot\mathsf{ADM}[a\theta\phi f]\gtrdot\{b\}$: [8; *D2.1.03*]
10. $!(\!|\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot|\!)$. [1; 2; 4; *T2.1.9*; *D1.1.08*]
$\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot\{b\}$ [10; 7; 5; 9]

T2.1.92 $[ab\theta\phi f]:\mathsf{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.\mathsf{W}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.\mathsf{IN}\langle\theta\theta\rangle\langle\!\langle f\rangle\!\rangle.f\subset\phi.\theta\{a\}.\supset.$
$[\exists b].\theta\{b\}.f\{bb\}$
PR $[ab\theta\phi f]\therefore\mathrm{Hp}(5).\supset:$
6. $\mathsf{A}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle$. [1; *D1.2.04*]
7. $\mathsf{CH}\{\!|\theta\phi|\!\}(\!|\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot|\!)$: [1; 3; 4; 5; *T2.1.90*]
$[\exists b]$:
8. $\theta\{b\}$.
9. $\mathsf{LUB}\{\!|\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot\theta\phi|\!\}\{b\}$. [2; 7; *D1.2.012*]
10. $\cap\lessdot\mathsf{ADM}\{a\theta\phi f\}\gtrdot\{b\}$. [1; 3; 4; 5; 9; *T2.1.91*]

11.	$\mathsf{D}\langle\!\langle f\rangle\!\rangle \circ \theta .$	[3; *D2.1.01*]
12.	$\mathsf{D}\langle\!\langle f\rangle\!\rangle \{b\}.$	[8; 11; *T1.1.10*]
	$[\exists w].$	
13.	$f\{bw\}.$	[12; *D1.1.013*]
14.	$\cap \langle\mathsf{ADM}\{a\theta\phi f\}\rangle \{w\}.$	[10; 13; *T2.1.11*]
15.	$\theta\{w\}.$	[3; 5; 14; *T2.1.6*]
16.	$\phi\{wb\}.$	[9; 15; *D1.2.09*; *D1.2.08*]
17.	$\phi\{bw\}.$	[4; 13; *T1.1.35*]
18.	$w \circ b .$	[6; 8; 15; 16; 17; *D1.2.03*]
19.	$f\{bb\}:$	[13; 18; *T1.1.36*]
	$[\exists b].\theta\{b\}.f\{bb\}$	[8; 19]
T2.1.93	$[\theta\phi f]:!\langle\theta\rangle.\mathsf{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.\mathsf{W}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.\mathsf{IN}\langle\theta\theta\rangle\langle\!\langle f\rangle\!\rangle.f \subset \phi .\supset.$ $[\exists b].\theta\{b\}.f\{bb\}$	
PR	$[\theta\phi f]:\mathrm{Hp}(5) .\supset.$	
	$[\exists a].$	
6.	$\theta\{a\}.$	[1; *D1.1.08*]
	$[\exists b].\theta\{b\}.f\{bb\}$	[2; 3; 4; 5; 6; *T2.1.92*]

Hence, the desired "fixed point" theorem is established.

2.2 *The Generalized Principle of Choice and the Kuratowski-Zorn Lemma* In this section, we show that **ACF** implies **KZL**. The "fixed point" theorem (*T2.1.93*) of the previous section, plays a crucial role in the proof. For reference, both **ACF** and **KZL** are repeated below.

ACF $[\xi]::!\langle\xi\rangle_{1}.\mathsf{A}\langle\xi\rangle_{1}.\supset\therefore[\exists\eta]\therefore\rightrightarrows\langle\!\langle\eta\rangle\!\rangle_{1}.\mathsf{D}\langle\!\langle\eta\rangle\!\rangle_{1}\circ\xi:[a\theta]:\eta\{\theta a\}_{1}.\supset.\theta\{a\}$

KZL $[\theta\phi]:\mathsf{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.!\langle\theta\rangle.\mathsf{W}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.\supset.[\exists a].\mathsf{MAXL}\{\theta\phi\}\{a\}$

The proof is basically a reductio argument and begins with the following definitions.

D2.2.01 $[ab\theta\phi]:\theta\{b\}.\phi\{ab\}.\sim(\phi\{ba\}).\equiv.\mathsf{S1}\{a\theta\phi\}_{3}\{b\}$

$\mathsf{S1}\{a\theta\phi\}_{3}$ is the set of ϕ-successors of a in θ.

D2.2.02 $[\mu\theta\phi]:[\exists a].\theta\{a\}.\circ\langle\mu\,\mathsf{S1}\{a\theta\phi\}_{3}\rangle.\equiv.\mathsf{S2}\langle\theta\phi\rangle\langle\mu\rangle$

$\mathsf{S2}\langle\theta\phi\rangle$ defines a family of sets each of which is the set of successors of an element in θ.

T2.2.1	$[\theta\phi]:!\langle\theta\rangle.\supset.!\langle\mathsf{S2}\langle\theta\phi\rangle\rangle_{1}$	
PR	$[\theta\phi]:\mathrm{Hp}(1).\supset.$	
	$[\exists a].$	
2.	$\theta\{a\}.$	[1; *D1.1.08*]
3.	$\mathsf{S2}\langle\theta\phi\rangle\langle\mathsf{S1}\{a\theta\phi\}_{3}\rangle.$	[2; *T1.1.7*; *D2.2.02*]
	$!\langle\mathsf{S2}\langle\theta\phi\rangle\rangle_{1}$	[3; *D1.5.01*]
T2.2.2	$[\mu\theta\phi]:\mathsf{PO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.\sim([\exists a].\mathsf{MAXL}\{\theta\phi\}\{a\}).\mathsf{S2}\langle\theta\phi\rangle\langle\mu\rangle.\supset.!\langle\mu\rangle$	
PR	$[\mu\theta\phi]\therefore\mathrm{Hp}(3).\supset:$	

4.	$[a].\sim(\mathbf{MAXL}\,\{\theta\phi\}\{a\}):$	[2]
	$[\exists b]:$	
5.	$\quad\theta\{b\}.$	[3; *D2.2.02*]
6.	$\quad\mu\circ\mathbf{S1}\,\underset{3}{\{}b\theta\phi\underset{3}{\}}.$	
7.	$\quad\sim(\mathbf{MAXL}\,\{\theta\phi\}\{b\}).$	[4]
	$\quad[\exists c].$	
8.	$\qquad\theta\{c\}.$	[7; 1; 5; *D1.2.05*]
9.	$\qquad\phi\{bc\}.$	
10.	$\qquad\sim(b\circ c).$	
11.	$\qquad\sim(\phi\{cb\}).$	[1; 5; 8; 9; 10; *T1.2.10*]
12.	$\qquad\mathbf{S1}\,\underset{3}{\{}b\theta\phi\underset{3}{\}}\{c\}.$	[8; 9; 11; *D2.2.01*]
13.	$\qquad\mu\{c\}:$	[6; 12; *T1.1.10*]
	$!\langle\mu\rangle$	[13; *D1.1.08*]
T2.2.3	$[\theta\phi]:\mathbf{PO}\langle\theta\rangle\langle\phi\rangle.\sim([\exists a].\mathbf{MAXL}\,\{\theta\phi\}\{a\})\ .\supset.\ \curlywedge\underset{1}{\langle}\mathbf{S2}\{\theta\phi\}\underset{1}{\rangle}$	[*T2.2.2*; *D1.5.02*]
D2.2.03	$[bc\theta\phi\eta]\therefore\theta\{b\}:[\exists\mu]:\circ\underset{1}{\langle}\mu\,\mathbf{S1}\,\underset{3}{\{}b\theta\phi\underset{3}{\}}\underset{1}{\rangle}.\eta\underset{1}{\{}\mu c\underset{1}{\}}\ :\equiv.\ \mathbf{R1}\,\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\{bc\}$	

$\mathbf{R1}\,\underset{2}{\{}\theta\phi\eta\underset{2}{\}}$ is a special connection which, we show subsequently, satisfies the conditions required for application of the "fixed point" theorem.

T2.2.4	$[abc\theta\phi\eta]:\rightrightarrows\underset{1}{\langle}\eta\underset{1}{\rangle}.\mathbf{R1}\,\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\{ab\}.\mathbf{R1}\,\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\{ac\}\ .\supset.\ b\circ c$	
PR	$[abc\theta\phi\eta]\therefore \mathrm{Hp}(3)\ .\supset:$	
4.	$\theta\{a\}:$	[2; *D2.2.03*]
	$[\exists\mu]:$	
5.	$\quad\mu\circ\mathbf{S1}\,\underset{3}{\{}a\theta\phi\underset{3}{\}}.$	[2; *D2.2.03*]
6.	$\quad\eta\underset{1}{\{}\mu b\underset{1}{\}}.$	
	$\quad[\exists\rho].$	
7.	$\qquad\rho\circ\mathbf{S1}\,\underset{3}{\{}a\theta\phi\underset{3}{\}}.$	[3; *D2.2.03*]
8.	$\qquad\eta\underset{1}{\{}\rho c\underset{1}{\}}.$	
9.	$\qquad\mu\circ\rho.$	[5; 7; *T1.1.6*; *T1.1.8*]
10.	$\qquad\eta\underset{1}{\{}\mu c\underset{1}{\}}:$	[8; 9; *T1.5.4*]
	$b\circ c$	[1; 6; 10; *D1.5.09*]
T2.2.5	$[\theta\phi\eta]:\rightrightarrows\underset{1}{\langle}\eta\underset{1}{\rangle}\ .\supset.\ \rightrightarrows\langle\mathbf{R1}\,\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\rangle$	[*T2.2.4*; *D1.1.015*]
T2.2.6	$[b\theta\phi\eta]:\mathbf{D}\langle\mathbf{R1}\,\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\rangle\{b\}\ .\supset.\ \theta\{b\}$	
PR	$[b\theta\phi\eta]:\mathrm{Hp}(1)\ .\supset.$	
	$[\exists c].$	
2.	$\quad\mathbf{R1}\,\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\{bc\}.$	[1; *D1.1.013*]
	$\theta\{b\}$	[2; *D2.2.03*]
T2.2.7	$[b\theta\phi\eta]:\mathbf{D}\underset{1}{\langle}\eta\underset{1}{\rangle}\circ\mathbf{S2}\{\theta\phi\}.\theta\{b\}\ .\supset.\ \mathbf{D}\langle\mathbf{R1}\,\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\rangle\{b\}$	
PR	$[b\theta\phi\eta]\therefore \mathrm{Hp}(2)\ .\supset:$	

3. $\mathsf{S2}\{\theta\phi\}\{\mathsf{S1}\underset{3}{\{}b\theta\phi\underset{3}{\}}\}$. [2; *T1.1.7*; *D2.2.02*]

4. $\mathsf{D}\underset{1}{\{}\eta\underset{1}{\}}\{\mathsf{S1}\underset{3}{\{}b\theta\phi\underset{3}{\}}\}$: [1; 3; *T1.5.2*]

$[\exists c]$:

5. $\quad \eta\underset{1}{\{}\mathsf{S1}\underset{3}{\{}b\theta\phi\underset{3}{\}}c\underset{1}{\}}$. [4; *D1.5.07*]

6. $\quad [\exists\mu].\ \mu\circ\mathsf{S1}\underset{3}{\{}b\theta\phi\underset{3}{\}}.\eta\underset{1}{\{}\mu c\underset{1}{\}}$. [5; *T1.1.7*]

7. $\quad \mathsf{R1}\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\{bc\}$: [2; 6; *D2.2.03*]

$\mathsf{D}\{\mathsf{R1}\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\}\{b\}$ [7; *D1.1.013*]

T2.2.8 $[\theta\phi\eta]:\mathsf{D}\underset{1}{\{}\eta\underset{1}{\}}\circ\mathsf{S2}\{\theta\phi\}.\supset.\ \mathsf{D}\{\mathsf{R1}\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\}\subset\theta.\ \theta\subset\mathsf{D}\{\mathsf{R1}\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\}$

[*T2.2.6*; *T2.2.7*; *D1.1.04*]

T2.2.9 $[\theta\phi\eta]:\mathsf{D}\underset{1}{\{}\eta\underset{1}{\}}\circ\mathsf{S2}\{\theta\phi\}.\supset.\ \mathsf{D}\{\mathsf{R1}\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\}\circ\theta$ [*T2.2.8*; *T1.1.15*]

T2.2.10 $[a\theta\phi\eta]\therefore[\mu b]:\eta\underset{1}{\{}\mu b\underset{1}{\}}.\supset.\ \mu\{b\}:\mathsf{C}\{\mathsf{R1}\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\}\{a\}:\supset.\ \theta\{a\}$

PR $[a\theta\phi\eta]\therefore \mathrm{Hp}(2)\ .\supset:$

$[\exists b]$:

3. $\quad \mathsf{R1}\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\{ba\}$. [2; *D1.1.014*]

$\quad [\exists\mu]$.

4. $\qquad \mu\circ\mathsf{S1}\underset{3}{\{}b\theta\phi\underset{3}{\}}$.

5. $\qquad \eta\underset{1}{\{}\mu a\underset{1}{\}}$. [*D2.2.03*; 3]

6. $\qquad \mu\{a\}$. [1; 5]

7. $\quad \mathsf{S1}\underset{3}{\{}b\theta\phi\underset{3}{\}}\{a\}$: [4; 6; *T1.1.10*]

$\theta\{a\}$ [7; *D2.2.01*]

T2.2.11 $[\theta\phi\eta]\therefore[\mu b]:\eta\underset{1}{\{}\mu b\underset{1}{\}}.\supset.\ \mu\{b\}:\supset.\ \mathsf{C}\{\mathsf{R1}\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\}\subset\theta$

[*T2.2.10*; *D1.1.04*]

T2.2.12 $[a\theta\phi\eta]\therefore[\mu b]:\eta\underset{1}{\{}\mu b\underset{1}{\}}.\supset.\ \mu\{b\}:\mathsf{R1}\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\{ab\}:\supset.$

$\mathsf{S1}\underset{3}{\{}a\theta\phi\underset{3}{\}}\{b\}.\phi\{ab\}.\theta\{b\}$

PR $[a\theta\phi\eta]:\mathrm{Hp}(2)\ .\supset.$

$[\exists\mu]$.

3. $\quad \mu\circ\mathsf{S1}\underset{3}{\{}a\theta\phi\underset{3}{\}}$.

4. $\quad \eta\underset{1}{\{}\mu b\underset{1}{\}}$. [2; *D2.2.03*]

5. $\quad \mu\{b\}$. [1; 4]

6. $\mathsf{S1}\underset{3}{\{}a\theta\phi\underset{3}{\}}\{b\}$. [3; 5; *T1.1.10*]

7. $\phi\{ab\}.\theta\{b\}$. [6; *D2.2.01*]

$\mathsf{S1}\underset{3}{\{}a\theta\phi\underset{3}{\}}\{b\}.\phi\{ab\}.\theta\{b\}$ [6; 7]

T2.2.13 $[\theta\phi\eta]\therefore[\mu b]:\eta\underset{1}{\{}\mu b\underset{1}{\}}.\supset.\ \mu\{b\}:\supset.\ \mathsf{R1}\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\subset\phi$.

[*T2.2.12*; *D1.1.012*; *D1.1.014*; *D1.1.04*]

T2.2.14 $[\theta\phi\eta]\therefore\rightrightarrows\underset{1}{\{}\eta\underset{1}{\}}.\mathsf{D}\underset{1}{\{}\eta\underset{1}{\}}\circ\mathsf{S2}\{\theta\phi\}:[\mu a]:\eta\underset{1}{\{}\mu a\underset{1}{\}}.\supset.\ \mu\{a\}:\supset.$

$\mathsf{IN}\langle\theta\theta\rangle\{\mathsf{R1}\underset{2}{\{}\theta\phi\eta\underset{2}{\}}\}$ [*T2.2.5*; *T2.2.9*; *T2.2.13*; *D2.1.01*]

T2.2.15 $[\theta\phi]\because[\xi]::!\,{}_{1}\{\xi\}_{1}\,.\,\curlywedge\,{}_{1}\{\xi\}_{1}\,.\supset\therefore[\exists\eta]\therefore\rightrightarrows{}_{1}\{\eta\}_{1}\,.\,\mathsf{D}\,{}_{1}\{\eta\}_{1}\circ\xi:[\mu a]:$
$\eta{}_{1}\{\mu a\}_{1}\,.\supset.\ \mu\{a\}::\mathsf{PO}\langle\theta\rangle\{\phi\}\,.\,!\{\theta\}\,.\,\mathsf{W}\langle\theta\rangle\{\phi\}\,.\sim([\exists a]\,.$
$\mathsf{MAXL}\{\theta\phi\}\{a\})\ ::\supset.\ [\exists a]\,.\,\mathsf{MAXL}\{\theta\phi\}\{a\}$

PR $[\theta\phi]::\mathrm{Hp}(5)\ .\supset\therefore$

6. $!\,{}_{1}\{\mathsf{S2}\{\theta\phi\}\}_{1}:$ [3; *T2.2.1*]

7. $\curlywedge\,{}_{1}\{\mathsf{S2}\{\theta\phi\}\}_{1}\therefore$ [2; 5; *T2.2.3*]

$[\exists\eta]\therefore$

8. $\rightrightarrows{}_{1}\{\eta\}_{1}\,.$

9. $\mathsf{D}\,{}_{1}\{\eta\}_{1}\circ\mathsf{S2}\{\theta\phi\}:$ [1; 6; 7]

10. $[\mu a]:\eta{}_{1}\{\mu a\}_{1}\,.\supset.\ \mu\{a\}:$

11. $\mathsf{IN}\langle\theta\theta\rangle\{\mathsf{R1}\,{}_{2}\{\theta\phi\eta\}_{2}\}:$ [8; 9; 10; *T2.2.14*]

12. $\mathsf{R1}\,{}_{2}\{\theta\phi\eta\}_{2}\subset\phi:$ [10; *T2.2.13*]

$[\exists b]\,.$

13. $\mathsf{R1}\,{}_{2}\{\theta\phi\eta\}_{2}\{bb\}.$ [2; 3; 4; 11; 12; *T2.1.93*]

14. $\mathsf{S1}\,{}_{3}\{b\theta\phi\}_{3}\{b\}.$ [10; 13; *T2.2.12*]

15. $\phi\{bb\}\,.\sim(\phi\{bb\})\therefore$ [15; *D2.2.01*]

$[\exists a]:\mathsf{MAXL}\{\theta\phi\}\{a\}$ [15]

Note the use of the "fixed point" theorem (*T2.1.93*) in the above proof.

T2.2.16 $[\xi]\therefore\ !\,{}_{1}\{\xi\}_{1}\,.\,\curlywedge\,{}_{1}\{\xi\}_{1}\ .\supset.\ [\exists\eta]\,.\rightrightarrows{}_{1}\{\eta\}_{1}\,.\,\mathsf{D}\,{}_{1}\{\eta\}_{1}\circ\xi:[\mu a]:\eta{}_{1}\{\mu a\}_{1}\,.\supset.$
$\mu\{a\}\therefore\supset:[\theta\phi]:\mathsf{PO}\langle\theta\rangle\{\phi\}\,.\,!\{\theta\}\,.\,\mathsf{W}\langle\theta\rangle\{\phi\}\ .\supset.$
$[\exists a]\,.\,\mathsf{MAXL}\{\theta\phi\}\{a\}$ [*T2.2.15*]

Hence, **ACF** implies **KZL**.

2.3 *The Kurtowski-Zorn Lemma and the Well Ordering Principle* In this section we investigate the relationship between the Kuratowski-Zorn Lemma and the Well Ordering Principle in the field of Ontology. We know of no way of proving that **KZL** as stated in section **2.0** (i.e., stated for semantic category S/n) implies **WO** (i.e., the Well Ordering Principle stated for semantic category S/n). However, we show below that the Kuratowski-Zorn Lemma stated for semantic category S/(S/(S/n)(S/nn)) (denoted below as **KZL***) does yield **WO**.

KZL* $[\theta\Pi]:\mathsf{PO}{}_{2}\langle\theta\rangle_{2}\,{}_{2}\{\Pi\}_{2}\,.\,!\,{}_{2}\{\theta\}_{2}\,.\,\mathsf{W}{}_{2}\langle\theta\rangle_{2}\,{}_{2}\{\Pi\}_{2}\ .\supset.\ [\exists\Gamma]\,.\,\mathsf{MAXL}\,{}_{2}\{\theta\Pi\}_{2}\,{}_{2}\{\Gamma\}_{2}$

The following two auxiliary definitions play central roles in the proof.

D2.3.01 $[\theta\Delta]:[\exists\rho\phi]\,.\,\mathsf{WO}\langle\rho\rangle\{\phi\}\,.\,\rho\subset\theta\,.\,\Delta\circ\downarrow{}_{4}\langle\rho\,\mathsf{T}\{\phi\rho\rho\}\rangle_{4}\ :\equiv.\ \mathsf{WOS}{}_{3}\langle\theta\rangle_{3}\,{}_{2}\{\Delta\}_{2}$

$\mathsf{WOS}{}_{3}\langle\theta\rangle_{3}$ denotes a family of "ordered pairs", the first element of which is a well ordered subset of some given set (the arbitrary set for which a well ordering relation is claimed to exist) and the second element of which is the restriction to the subset of the connection which well orders it.

D2.3.02 $[\Gamma\Delta] :: [\rho\mu\phi\psi] \therefore \Gamma \circ \underset{4}{\downarrow}\langle\rho\phi\underset{4}{\rangle} . \Delta \circ \underset{4}{\downarrow}\langle\mu\psi\underset{4}{\rangle} .\supset: \rho \subset \mu . \phi \subset \psi :$
$[ab] : \rho\{a\} . \mu\{b\} . \sim(\rho\{b\}) .\supset. \psi\{ab\} \therefore\equiv. \mathbf{R2}\underset{2}{\{}\Gamma\Delta\underset{2}{\}}$

R2 designates a special connection between "ordered pairs" which, as is established in *T2.3.11*, induces a partial ordering on $\mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}$. In fact, the basic strategy of the proof is to show, first, that $\mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}$ and **R2** satisfy the conditions of **KZL***, and then, that the maximal element predicted by **KZL*** is an "ordered pair" whose first element is the arbitrary set for which it is claimed a well ordering connection exists and whose second element is a connection which well orders the set.

T2.3.1 $[\Sigma] . \underset{2}{\mathbf{R}}\underset{2}{\langle}\Sigma\underset{2}{\rangle}\ddagger\mathbf{R2}\underset{2}{\ddagger}$ [*T1.6.11*; *T1.6.7*; *T1.1.15*; *T1.1.14*; *D2.3.02*; *D1.6.07*]

T2.3.2 $[\theta\Gamma\Delta] : \mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\underset{2}{\{}\Gamma\underset{2}{\}} . \mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\underset{2}{\{}\Delta\underset{2}{\}} . \mathbf{R2}\underset{2}{\{}\Gamma\Delta\underset{2}{\}} . \mathbf{R2}\underset{2}{\{}\Delta\Gamma\underset{2}{\}} .\supset. \Gamma\circ\Delta$

PR $[\theta\Gamma\Delta] \therefore \mathrm{Hp}(4) .\supset:$
$[\exists\rho\phi] :$

5. $\Gamma \circ \underset{4}{\downarrow}\langle\rho\, \mathbf{T} \ddagger\phi\rho\rho\ddagger\underset{4}{\rangle} .$ [1; *D2.3.01*]
$[\exists\mu\psi] .$
6. $\Delta \circ \underset{4}{\downarrow}\langle\mu\, \mathbf{T}\ddagger\psi\mu\mu\ddagger\underset{4}{\rangle} .$ [2; *D2.3.01*]
7. $\rho \subset \mu .$ [5; 6; 3; *D2.3.02*]
8. $\mathbf{T}\ddagger\phi\rho\rho\ddagger \subset \mathbf{T}\ddagger\psi\mu\mu\ddagger .$ [5; 6; 3; *D2.3.02*]
9. $\mu \subset \rho .$ [5; 6; 4; *D2.3.02*]
10. $\mathbf{T}\ddagger\psi\mu\mu\ddagger \subset \mathbf{T}\ddagger\phi\rho\rho\ddagger .$ [5; 6; 4; *D2.3.02*]
11. $\rho \circ \mu .$ [7; 9; *T1.1.15*]
12. $\mathbf{T}\ddagger\phi\rho\rho\ddagger \circ \mathbf{T}\ddagger\psi\mu\mu\ddagger .$ [8; 10; *T1.1.34*]
13. $\underset{4}{\downarrow}\langle\rho\, \mathbf{T}\ddagger\phi\rho\rho\ddagger\underset{4}{\rangle} \circ \underset{4}{\downarrow}\langle\mu\, \mathbf{T}\ddagger\psi\mu\mu\ddagger\underset{4}{\rangle} :$ [11; 12; *T1.6.11*]

$\Gamma\circ\Delta$ [5; 6; 13; *T1.6.5*]

T2.3.3 $[\theta] . \mathbf{A}\underset{2}{\langle}\mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\underset{2}{\rangle}\ddagger\mathbf{R2}\underset{2}{\ddagger}$ [*T2.3.2*; *D1.6.09*]

T2.3.4 $[ab\lambda\mu\rho\xi\psi] \therefore \rho\{a\} . \sim(\rho\{b\}) . \lambda\{b\} : [ab] : \rho\{a\} . \lambda\{b\} . \sim(\rho\{b\}) .\supset.$
$\mathbf{T}\ddagger\xi\lambda\lambda\ddagger\{ab\} : \mathbf{T}\ddagger\xi\lambda\lambda\ddagger \subset \mathbf{T}\ddagger\psi\mu\mu\ddagger :\supset. \mathbf{T}\ddagger\psi\mu\mu\ddagger\{ab\}$ [*T1.1.35*]

T2.3.5 $[ab\lambda\rho\mu\psi] \therefore \rho \subset \lambda . \rho\{a\} . \mu\{b\} . \sim(\lambda\{b\}) : [ab] : \lambda\{a\} . \mu\{b\} .$
$\sim(\lambda\{b\}) .\supset. \mathbf{T}\ddagger\psi\mu\mu\ddagger\{ab\} :\supset. \mathbf{T}\ddagger\psi\mu\mu\ddagger\{ab\}$ [*T1.1.14*]

T2.3.6 $[ab\theta\sigma\delta\Phi\Psi\Gamma\Delta\Upsilon] : \mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\underset{2}{\{}\Gamma\underset{2}{\}} . \mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\underset{2}{\{}\Delta\underset{2}{\}} . \mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\underset{2}{\{}\Upsilon\underset{2}{\}} . \mathbf{R2}\underset{2}{\{}\Gamma\Delta\underset{2}{\}} .$
$\mathbf{R2}\underset{2}{\{}\Delta\Gamma\underset{2}{\}} . \Gamma\circ\underset{4}{\downarrow}\langle\sigma\Phi\underset{4}{\rangle} . \Upsilon\circ\underset{4}{\downarrow}\langle\delta\Psi\underset{4}{\rangle} . \sigma\{a\} . \delta\{b\} . \sim(\sigma\{b\}) .\supset. \Psi\{ab\}$

PR $[ab\theta\sigma\delta\Phi\Psi\Gamma\Delta\Upsilon] :: \mathrm{Hp}(10) .\supset::$
$[\exists\rho\phi] ::$

11. $\Gamma\circ\underset{4}{\downarrow}\langle\rho\, \mathbf{T}\ddagger\phi\rho\rho\ddagger\underset{4}{\rangle} ::$ [1; *D2.3.01*]
$[\exists\lambda\xi] ::$
12. $\Delta\circ\underset{4}{\downarrow}\langle\lambda\, \mathbf{T}\ddagger\xi\lambda\lambda\ddagger\underset{4}{\rangle} \therefore$ [2; *D2.3.01*]
$[\exists\mu\psi] \therefore$
13. $\Upsilon\circ\underset{4}{\downarrow}\langle\mu\, \mathbf{T}\ddagger\psi\mu\mu\ddagger\underset{4}{\rangle} .$ [3; *D2.3.01*]

14. $\downarrow_4\langle\sigma\Phi\rangle_4 \circ \downarrow_4\langle\rho\,\mathbf{T}\nmid\phi\rho\rho\nmid\rangle_4\,.$ [6; 11; *T1.6.1*; *T1.6.3*]

15. $\downarrow_4\langle\delta\Phi\rangle_4 \circ \downarrow_4\langle\mu\,\mathbf{T}\nmid\psi\mu\mu\nmid\rangle_4\,.$ [7; 13; *T1.6.1*; *T1.6.3*]

16. $\sigma\circ\rho\,.$ [14; *T1.6.8*]

17. $\delta\circ\mu\,.$ [15; *T1.6.8*]

18. $\Psi\circ\mathbf{T}\nmid\psi\mu\mu\nmid\,.$ [15; *T1.6.8*]

19. $\rho\{a\}.$ [8; 16; *T1.1.10*]

20. $\mu\{b\}.$ [9; 17; *T1.1.10*]

21. $\sim(\rho\{b\})\,.$ [10; 16; *T1.1.11*]

22. $\rho\subset\lambda:$ [4; 11; 12; *D2.3.02*]

23. $[ab]:\rho\{a\}.\lambda\{b\}.\sim(\rho\{b\}\,.\supset.\ \mathbf{T}\nmid\xi\lambda\lambda\nmid\{ab\}:$ [4; 11; 12; *D2.3.02*]

24. $\mathbf{T}\nmid\xi\lambda\lambda\nmid\subset\mathbf{T}\nmid\psi\mu\mu\nmid:$ [5; 12; 13; *D2.3.02*]

25. $[ab]:\lambda\{a\}:\mu\{b\}.\sim(\lambda\{b\})\,.\supset.\ \mathbf{T}\nmid\psi\mu\mu\nmid\{ab\}:$ [5; 12; 13; *D2.3.02*]

26. $\lambda\{b\}.\mathrm{v}.\sim(\lambda\{b\}):$ [**PC**]

27. $\mathbf{T}\nmid\psi\mu\mu\nmid\{ab\}\,\because$ [26; 19; 21; 23; 24; *T2.3.4*; 26; 22; 19; 20; 25; *T2.3.5*]

$\Psi\{ab\}$ [27; 18; *T1.1.31*]

T2.3.7 $[\theta\sigma\delta\Phi\Psi\Gamma\Delta\Upsilon]\therefore\mathbf{WOS}_{3}\langle\theta\rangle_{3}\{\Gamma\}_{2}{}_{2}.\,\mathbf{WOS}_{3}\langle\theta\rangle_{3}\{\Delta\}_{2}{}_{2}.\,\mathbf{WOS}_{3}\langle\theta\rangle_{3}\{\Upsilon\}_{2}{}_{2}.\,\mathbf{R2}_{2}\{\Gamma\Delta\}_{2}.$ $\mathbf{R2}_{2}\{\Delta\Upsilon\}_{2}.\,\Gamma\circ\downarrow_4\langle\sigma\Phi\rangle_4\,.\,\Upsilon\circ\downarrow_4\langle\delta\Psi\rangle_4\,.\supset:[ab]:\sigma\{a\}.\,\delta\{b\}.\sim(\sigma\{b\})\,.\supset.\ \Psi\{ab\}$ [*T2.3.6*]

T2.3.8 $[\sigma\delta\Phi\Psi\Gamma\Delta\Upsilon]:\mathbf{R2}_{2}\{\Gamma\Delta\}_{2}.\,\mathbf{R2}_{2}\{\Delta\Upsilon\}_{2}.\,\Gamma\circ\downarrow_4\langle\sigma\Phi\rangle_4\,.\,\Upsilon\circ\downarrow_4\langle\delta\Psi\rangle_4\,.\supset.\ \sigma\subset\delta\,.$ $\Phi\subset\Psi$ [*D2.3.02*; *T1.1.13*; *T1.1.33*]

T2.3.9 $[\theta\Gamma\Delta\Upsilon]:\mathbf{WOS}_{3}\langle\theta\rangle_{3}\{\Gamma\}_{2}{}_{2}.\,\mathbf{WOS}_{3}\langle\theta\rangle_{3}\{\Delta\}_{2}{}_{2}.\,\mathbf{WOS}_{3}\langle\theta\rangle_{3}\{\Upsilon\}_{2}{}_{2}.\,\mathbf{R2}_{2}\{\Gamma\Delta\}_{2}.\,\mathbf{R2}_{2}\{\Delta\Upsilon\}_{2}\,.\supset.$ $\mathbf{R2}_{2}\{\Gamma\Upsilon\}_{2}$ [*T2.3.7*; *T2.3.8*; *D2.3.02*]

T2.3.10 $[\theta]\,.\,\mathbf{T}_{2}\langle\mathbf{WOS}_{3}\langle\theta\rangle_{3}\rangle_{2}\,\nleftarrow\mathbf{R2}\nrightarrow_{2}$ [*T2.3.9*; *D1.6.08*]

T2.3.11 $[\theta]\,.\,\mathbf{PO}_{2}\langle\mathbf{WOS}_{3}\langle\theta\rangle_{3}\rangle_{2}\,\nleftarrow\mathbf{R2}\nrightarrow_{2}$ [*T2.3.1*; *T2.3.3*; *T2.3.10*; *D1.6.010*]

The next series of theses are directed at establishing that every **R2** chain in $\mathbf{WOS}_{3}\langle\theta\rangle_{3}$ has a least upper bound. This goal requires the following definitions.

D2.3.03 $[\rho\Sigma]:[\exists\phi\Gamma]\,.\,\Sigma\{\Gamma\}_{2}{}_{2}.\,\Gamma\circ\downarrow_4\langle\rho\phi\rangle_4\,.\equiv.\ \mathbf{C3}_4\nmid\Sigma\nmid_4\prec\rho\succ$

This is an auxiliary definition which identifies the set ρ as an element of an "ordered pair" which is itself an element of Σ.

D2.3.04 $[\psi\Sigma]:[\exists\rho\Gamma]\,.\,\Sigma\{\Gamma\}_{2}{}_{2}.\,\Gamma\circ\downarrow_4\langle\rho\psi\rangle_4\,.\equiv.\ \mathbf{C4}_4\nmid\Sigma\nmid_4\nleftarrow\psi\nrightarrow$

This definition identifies the connection ψ as an element of an "ordered pair" itself an element of Σ.

Our strategy is to show that if Σ is an **R2**-chain in $\mathbf{WOS}_{3}\langle\theta\rangle_{3}$, then the "ordered pair" whose first element is the generalized union of all sets satisfying $\mathbf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}$ and whose second element is the generalized union of all connections satisfying $\mathbf{C4}\underset{4}{\{}\Sigma\underset{4}{\}}$ is the required least upper bound. This will be established at *T2.3.56.*

T2.3.12 $[\theta\rho\psi\Gamma]:\mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\underset{2}{\{}\Gamma\underset{2}{\}}.\Gamma\circ{\downarrow}\underset{4}{\langle}\rho\psi\underset{4}{\rangle}.\supset.\rho\subset\theta$

PR $[\theta\rho\psi\Gamma]:\mathrm{Hp}(2).\supset.$

$[\exists\mu\phi].$

3. $\Gamma\circ{\downarrow}\underset{4}{\langle}\mu\,\mathbf{T}\{\phi\mu\mu\}\underset{4}{\rangle}.$

4. $\mu\subset\theta.$ [1; *D2.3.01*]

5. $\mu\circ\rho.$ [2; 3; *T1.6.1*; *T1.6.3*; *T1.6.8*]

$\rho\subset\theta$ [4; 5; *T1.1.16*]

T2.3.13 $[a\theta\Sigma]:\mathbf{CH}\underset{2}{\{}\mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\mathbf{R2}\underset{2}{\}}\underset{2}{\{}\Sigma\underset{2}{\}}.\cup\{\mathbf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\}\{a\}.\supset.\theta\{a\}$

PR $[a\theta\Sigma]\therefore\mathrm{Hp}(2).\supset:$

$[\exists\rho]:$

3. $\mathbf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\{\rho\}.$

4. $\rho\{a\}.$ [2; *D1.1.09*]

$[\exists\Gamma\phi].$

5. $\Sigma\underset{2}{\{}\Gamma\underset{2}{\}}.$

6. $\Gamma\circ{\downarrow}\underset{4}{\langle}\rho\phi\underset{4}{\rangle}.$ [3; *D2.3.03*]

7. $\mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\underset{2}{\{}\Gamma\underset{2}{\}}.$ [1; 5; *T1.6.18*]

8. $\rho\subset\theta:$ [6; 7; *T2.3.12*]

$\theta\{a\}$ [4; 8; *T1.1.14*]

T2.3.14 $[\theta\Sigma]:\mathbf{CH}\underset{2}{\{}\mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\mathbf{R2}\underset{2}{\}}\underset{2}{\{}\Sigma\underset{2}{\}}.\supset.\cup\{\mathbf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\}\subset\theta$ [*T2.3.13*; *D1.1.04*]

T2.3.15 $[\rho\theta\Sigma]:\mathbf{CH}\underset{2}{\{}\mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\mathbf{R2}\underset{2}{\}}\underset{2}{\{}\Sigma\underset{2}{\}}.\mathbf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\{\rho\}.\supset.[\exists\psi\Gamma].\Sigma\underset{2}{\{}\Gamma\underset{2}{\}}.$
$\mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\underset{2}{\{}\Gamma\underset{2}{\}}.\mathbf{WO}\langle\rho\rangle\{\psi\}.\rho\subset\theta.\Gamma\circ{\downarrow}\underset{4}{\langle}\rho\,\mathbf{T}\{\psi\rho\rho\}\underset{4}{\rangle}$

PR $[\rho\theta\Sigma]\therefore\mathrm{Hp}(2).\supset:$

$[\exists\phi\Gamma]:$

3. $\Sigma\underset{2}{\{}\Gamma\underset{2}{\}}.$

4. $\Gamma\circ{\downarrow}\underset{4}{\langle}\rho\phi\underset{4}{\rangle}.$ [2; *D2.3.03*]

5. $\mathbf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\underset{2}{\{}\Gamma\underset{2}{\}}.$ [1; 3; *T1.6.19*]

6. $\rho\subset\theta.$ [4; 5; *T2.3.12*]

$[\exists\mu\psi].$

7. $\mathbf{WO}\langle\mu\rangle\{\psi\}.$

8. $\Gamma\circ{\downarrow}\underset{4}{\langle}\mu\,\mathbf{T}\{\psi\mu\mu\}\underset{4}{\rangle}.$ [5; *D2.3.01*]

9. $\mu\circ\rho.$ [4; 8; *T1.6.12*]

10. $\Gamma\circ{\downarrow}\underset{4}{\langle}\rho\,\mathbf{T}\{\psi\rho\rho\}\underset{4}{\rangle}.$ [8; 9; *T1.6.18*]

11. $\mathbf{WO}\langle\rho\rangle\{\psi\}:$ [7; 9; *T1.2.12*]

$[\exists\psi\Gamma] . \Sigma\{_2\Gamma\}_2 . \mathbf{WOS}\langle_3\theta\rangle_3\{_2\Gamma\}_2 . \mathbf{WO}\langle\rho\rangle \{\!|\psi|\!\} . \rho\subset\theta . \Gamma\circ\downarrow\langle_4\rho\ \mathbf{T}\{\!\|\psi\rho\rho\|\!\}\rangle_4$ [3; 5; 6; 10; 11]

T2.3.16 $[a\theta\Sigma] : \mathbf{CH}\{_2\mathbf{WOS}\langle_3\theta\rangle_3 \mathbf{R2}\}_2 \{\!|\Sigma|\!\}_2 . \cup \not\langle\mathbf{C3}\{_4\Sigma\}_4\rangle\{a\} .\supset. \cup \not\langle\!\langle\mathbf{C4}\{_4\Sigma\}_4\rangle\!\rangle\{aa\}$

PR $[a\theta\Sigma] \therefore \mathrm{Hp}(2) .\supset:$

$[\exists\rho] :$

3. $\mathbf{C3}\{_4\Sigma\}_4\{\!|\rho|\!\}.$
4. $\rho\{a\}.$ [2; *D1.1.09*]

$[\exists\psi\Gamma].$

5. $\mathbf{WO}\langle\rho\rangle\{\!|\psi|\!\}.$
6. $\Gamma\circ\downarrow\langle_4\rho\ \mathbf{T}\{\!\|\psi\rho\rho\|\!\}\rangle_4 .$ [1; 3; *T2.3.15*]
7. $\Sigma\{_2\Gamma\}_2.$ [1; 3; *T2.3.15*]
8. $\mathbf{R}\langle\rho\rangle\{\!|\psi|\!\}.$ [5; *D1.2.07*; *D1.2.04*]
9. $\psi\{aa\}.$ [4; 8; *D1.2.01*]
10. $\mathbf{T}\{\!\|\psi\rho\rho\|\!\}\{aa\}.$ [4; 9; *D1.1.016*]
11. $\mathbf{C4}\{_4\Sigma\}_4\{\!|\mathbf{T}\{\!\|\psi\rho\rho\|\!\}|\!\} :$ [6; 7; *D2.3.04*]

$\cup \not\langle\!\langle\mathbf{C4}\{_4\Sigma\}_4\rangle\!\rangle\{aa\}$ [10; 11; *D1.1.017*]

T2.3.17 $[\theta\Sigma] : \mathbf{CH}\{_2\mathbf{WOS}\langle_3\theta\rangle_3 \mathbf{R2}\}_2 \{\!|\Sigma|\!\}_2 . \mathbf{R}\langle\cup \not\langle\mathbf{C3}\{_4\Sigma\}_4\rangle\rangle .\supset. \{\!|\cup \not\langle\!\langle\mathbf{C4}\{_4\Sigma\}_4\rangle\!\rangle|\!\}$ [*T2.3.16*; *D1.2.01*]

T2.3.18 $[\theta\Sigma\phi] : \mathbf{CH}\{_2\mathbf{WOS}\langle_3\theta\rangle_3 \mathbf{R2}\}_2 \{\!|\Sigma|\!\}_2 . \mathbf{C4}\{_4\Sigma\}_4\{\!|\phi|\!\} .\supset. [\exists\Gamma\rho\Phi] . \Sigma\{_2\Gamma\}_2 .$
$\Gamma\circ\downarrow\langle_4\rho\phi\rangle_4 . \phi\circ\mathbf{T}\{\!\|\Phi\rho\rho\|\!\} . \mathbf{WO}\langle\rho\rangle\{\!|\Phi|\!\} . \mathbf{WOS}\langle_3\theta\rangle_3\{_2\Gamma\}_2 . \rho\subset\theta$

PR $[\theta\Sigma\phi] \therefore \mathrm{Hp}(2) .\supset:$

$[\exists\rho\Gamma] :$

3. $\Sigma\{_2\Gamma\}_2.$
4. $\Gamma\circ\downarrow\langle_4\rho\phi\rangle_4 .$ [2; *D2.3.04*]
5. $\mathbf{WOS}\langle_3\theta\rangle_3\{_2\Gamma\}_2.$ [1; 3; *T1.6.19*]

$[\exists\mu\Phi].$

6. $\mathbf{WO}\langle\mu\rangle\{\!|\Phi|\!\}.$
7. $\Gamma\circ\downarrow\langle_4\mu\ \mathbf{T}\{\!\|\Phi\mu\mu\|\!\}\rangle_4 .$ [5; *D2.3.01*]
8. $\rho\subset\theta.$ [4; 5; *T2.3.12*]
9. $\rho\circ\mu .$ [4; 7; *T1.6.12*]
10. $\phi\circ\mathbf{T}\{\!\|\Phi\mu\mu\|\!\}.$ [4; 7; *T1.6.12*]
11. $\mathbf{WO}\langle\rho\rangle\{\!|\Phi|\!\}.$ [6; 9; *T1.1.6*; *T1.2.12*]
12. $\phi\circ\mathbf{T}\{\!\|\Phi\rho\rho\|\!\} :$ [10; 9; *T1.1.1*; *T1.1.28*; *T1.1.30*; *T1.1.39*]

$[\exists\Gamma\rho\Phi] . \Sigma\{_2\Gamma\}_2 . \Gamma\circ\downarrow\langle_4\rho\phi\rangle_4 . \phi\circ\mathbf{T}\{\!\|\Phi\rho\rho\|\!\} . \mathbf{WO}\langle\rho\rangle\{\!|\Phi|\!\} . \mathbf{WOS}\langle_3\theta\rangle_3\{_2\Gamma\}_2 .$
$\rho\subset\theta$ [3; 4; 11; 12; 5; 8]

T2.3.19 $[abc\rho\mu\phi\psi\Psi\Gamma\Delta\Sigma] : \Gamma\circ\downarrow\langle_4\rho\phi\rangle_4 . \Delta\circ\downarrow\langle_4\mu\psi\rangle_4 . \phi\{ab\} . \psi\{bc\} . \psi\circ\mathbf{T}\{\!\|\Psi\mu\mu\|\!\} .$
$\mathbf{WO}\langle\mu\rangle\{\!|\Psi|\!\} . \Sigma\{_2\Delta\}_2 . \mathbf{R2}\{_2\Gamma\Delta\}_2 .\supset. \cup \not\langle\mathbf{C4}\{_4\Sigma\}_4\rangle\{ac\} . \psi\{ac\}$

PR $[abc\rho\mu\phi\psi\Psi\Gamma\Delta\Sigma] : \mathrm{Hp}(8)\ .\supset.$

9. $\phi \subset \psi\,.$ [8; 1; 2; *D2.3.02*]

10. $\psi\{ab\}\,.$ [3; 9; *T1.1.35*]

11. $\mathsf{T}\lbrace\!\!\vert\Psi\mu\mu\vert\!\!\rbrace\{ab\}\,.$ [10; 5; *T1.1.31*]

12. $\mathsf{T}\lbrace\!\!\vert\Psi\mu\mu\vert\!\!\rbrace\{bc\}\,.$ [4; 5; *T1.1.31*]

13. $\Psi\{ab\}\,.$ [11; *D1.1.016*]

14. $\Psi\{bc\}\,.$ [12; *D1.1.016*]

15. $\mu\{a\}\,.$ [11; *D1.1.016*]

16. $\mu\{b\}\,.$ [11; *D1.1.016*]

17. $\mu\{c\}\,.$ [12; *D1.1.016*]

18. $\mathbf{T}\langle\mu\rangle \dashv\!\Psi\!\vdash\,.$ [6; *D1.2.07*; *D1.2.04*]

19. $\Psi\{ac\}\,.$ [18; 15; 16; 17; 13; 14; *D1.2.02*]

20. $\mathsf{T}\lbrace\!\!\vert\Psi\mu\mu\vert\!\!\rbrace\{ac\}\,.$ [15; 17; 19; *D1.1.016*]

21. $\psi\{ac\}\,.$ [20; 5; *T1.1.28*; *T1.1.31*]

22. $\mathsf{C4}\lbrace\!\!\vert_4\Sigma\vert\!\!\rbrace_4 \dashv\!\psi\!\vdash\,.$ [2; 7; *D2.3.04*]

23. $\cup\lbrace\!\!\vert\!\!\vert\mathsf{C4}\lbrace\!\!\vert_4\Sigma\vert\!\!\rbrace_4\vert\!\!\vert\!\!\rbrace\{ac\}\,.$ [21; 22; *D1.1.017*]

$\cup\lbrace\!\!\vert\!\!\vert\mathsf{C4}\lbrace\!\!\vert_4\Sigma\vert\!\!\rbrace_4\vert\!\!\vert\!\!\rbrace\{ac\}\,.\,\psi\{ac\}$ [21; 23]

T2.3.20 $[abc\rho\mu\phi\psi\Phi\Gamma\Delta\Sigma] : \Gamma\circ{\downarrow}\langle_4\rho\phi\rangle_4\,.\,\Delta\circ{\downarrow}\langle_4\mu\psi\rangle_4\,.\,\phi\{ab\}\,.\,\psi\{bc\}\,.\,\phi\circ\mathsf{T}\lbrace\!\!\vert\Phi\rho\rho\vert\!\!\rbrace\,.$
$\mathsf{WO}\langle\rho\rangle\dashv\!\Phi\!\vdash\,.\,\Sigma\{_2\Gamma\}_2\,.\,\mathsf{R2}\{_2\Delta\Gamma\}_2\ .\supset.\ \cup\lbrace\!\!\vert\!\!\vert\mathsf{C4}\lbrace\!\!\vert_4\Sigma\vert\!\!\rbrace_4\vert\!\!\vert\!\!\rbrace\{ac\}\,.\,\phi\{ac\}$

PR $[abc\rho\mu\phi\psi\Phi\Gamma\Delta\Sigma] : \mathrm{Hp}(8)\ .\supset.$

9. $\psi \subset \phi\,.$ [8; 1; 2; *D2.3.02*]

10. $\phi\{bc\}\,.$ [4; 9; *T1.1.35*]

11. $\mathsf{T}\lbrace\!\!\vert\Phi\rho\rho\vert\!\!\rbrace\{ab\}\,.$ [3; 5; *T1.1.31*]

12. $\mathsf{T}\lbrace\!\!\vert\Phi\rho\rho\vert\!\!\rbrace\{bc\}\,.$ [10; 5; *T1.1.31*]

13. $\Phi\{ab\}\,.$ [11; *D1.1.016*]

14. $\Phi\{bc\}\,.$ [12; *D1.1.016*]

15. $\rho\{a\}\,.$ [11; *D1.1.016*]

16. $\rho\{b\}\,.$ [11; *D1.1.016*]

17. $\rho\{c\}\,.$ [12; *D1.1.016*]

18. $\mathbf{T}\langle\rho\rangle \dashv\!\Phi\!\vdash\,.$ [6; *D1.2.07*; *D1.2.04*]

19. $\Phi\{ac\}\,.$ [15; 16; 17; 13; 14; *D1.2.02*]

20. $\mathsf{T}\lbrace\!\!\vert\Phi\rho\rho\vert\!\!\rbrace\{ac\}\,.$ [15; 17; 19; *D1.1.016*]

21. $\phi\{ac\}\,.$ [5; 20; *T1.1.28*; *T1.1.31*]

22. $\mathsf{C4}\lbrace\!\!\vert_4\Sigma\vert\!\!\rbrace_4 \dashv\!\phi\!\vdash\,.$ [1; 7; *D2.3.04*]

23. $\cup\lbrace\!\!\vert\!\!\vert\mathsf{C4}\lbrace\!\!\vert_4\Sigma\vert\!\!\rbrace_4\vert\!\!\vert\!\!\rbrace\{ac\}\,.$ [21; 22; *D1.1.017*]

$\cup\lbrace\!\!\vert\!\!\vert\mathsf{C4}\lbrace\!\!\vert_4\Sigma\vert\!\!\rbrace_4\vert\!\!\vert\!\!\rbrace\{ac\}\,.\,\phi\{ac\}$ [21; 23]

T2.3.21 $[abc\theta\Sigma] : \mathsf{CH}\lbrace\!\!\vert_2\mathsf{WOS}\langle_3\theta\rangle_3\,\mathsf{R2}\vert\!\!\rbrace_2 \dashv_2\Sigma\vdash_2\,.\,\cup\lbrace\!\!\vert\!\!\vert\mathsf{C4}\lbrace\!\!\vert_2\Sigma\vert\!\!\rbrace_2\vert\!\!\vert\!\!\rbrace\{ab\}\,.$
$\cup\lbrace\!\!\vert\!\!\vert\mathsf{C4}\lbrace\!\!\vert_4\Sigma\vert\!\!\rbrace_4\vert\!\!\vert\!\!\rbrace\{bc\}\ .\supset.\ \cup\lbrace\!\!\vert\!\!\vert\mathsf{C4}\lbrace\!\!\vert_4\Sigma\vert\!\!\rbrace_4\vert\!\!\vert\!\!\rbrace\{ac\}$

PR $[abc\theta\Sigma] \because \mathrm{Hp}(3)\ .\supset ::$

$[\exists\phi] ::$

4. $\mathsf{C4}\lbrace\!\!\vert_4\Sigma\vert\!\!\rbrace_4 \dashv\!\phi\!\vdash\,.$

5. $\phi\{ab\} \therefore$ [2; *D1.1.017*]

$[\exists \Gamma \rho \Phi] \because\!\cdot$

6. $\Sigma \{_2 \Gamma \}_2 .$
7. $\Gamma \circ \downarrow \langle_4 \rho \phi \rangle_4 .$
8. $\phi \circ \mathsf{T} ⦃ \Phi \rho \rho ⦄ .$ [1; 4; *T2.3.18*]
9. $\mathsf{WO} \langle \rho \rangle ⦃ \Phi ⦄ .$
10. $\mathsf{WOS} \langle_3 \theta \rangle_3 \{_2 \Gamma \}_2 ::$

$[\exists \psi] ::$

11. $\mathsf{C4} ⦃_4 \Sigma ⦄_4 ⦃ \psi ⦄ .$ [3; *D1.1.017*]
12. $\psi \{bc\} \therefore$

$[\exists \Delta \mu \Psi] \therefore$

13. $\Sigma \{_2 \Delta \}_2 .$
14. $\Delta \circ \downarrow \langle_4 \mu \psi \rangle_4 .$
15. $\psi \circ \mathsf{T} ⦃ \Psi \mu \mu ⦄ .$ [1; 11; *T2.3.18*]
16. $\mathsf{WO} \langle \mu \rangle ⦃ \Psi ⦄ .$
17. $\mathsf{WOS} \langle_3 \theta \rangle_3 \{_2 \Delta \}_2 .$
18. $\mathsf{C} \langle_2 \Sigma \rangle_2 ⦃_2 \mathsf{R2} ⦄_2 :$ [1; *D1.6.015*]
19. $\mathsf{R2} \{_2 \Gamma \Delta \}_2 . \mathrm{v} . \mathsf{R2} \{_2 \Delta \Gamma \}_2 ::$

[6; 13; 18; *D1.6.014*]

$\cup ⦃ \mathsf{C4} ⦃_4 \Sigma ⦄_4 ⦄ \{ac\}$ [19; 7; 14; 5; 12; 15; 16; 19; *T2.3.19*; 19; 7; 14; 5; 12; 8; 9; 6; *T2.3.20*]

T2.3.22 $[\theta \Sigma] : \mathsf{CH} ⦃_2 \mathsf{WOS} \langle_3 \theta \rangle_3 \mathsf{R2} ⦄_2 ⦃_2 \Sigma ⦄_2 . \supset . \mathbf{T} \langle \cup ⦃ \mathsf{C3} ⦃_4 \Sigma ⦄_4 ⦄ \rangle ⦃ \cup ⦃ \mathsf{C4} ⦃_4 \Sigma ⦄_4 ⦄ ⦄$

[*T2.3.21*; *D1.6.08*]

T2.3.23 $[ab\rho \mu \phi \psi \Psi \Gamma \Delta] : \Gamma \circ \downarrow \langle_4 \rho \phi \rangle_4 . \Delta \circ \downarrow \langle_4 \mu \psi \rangle_4 . \phi \{ab\} . \psi \{ba\} . \psi \circ \mathsf{T} ⦃ \Psi \mu \mu ⦄ .$
$\mathsf{WO} \langle \mu \rangle ⦃ \Psi ⦄ . \mathsf{R2} \{_2 \Gamma \Delta \}_2 . \supset . a \circ b$

PR $[ab\rho \mu \phi \psi \Psi \Gamma \Delta] : \mathrm{Hp}(7) . \supset .$

8. $\phi \subset \psi .$ [7; 1; 2; *D2.3.02*]
9. $\psi \{ab\} .$ [3; 8; *T1.1.35*]
10. $\mathsf{T} ⦃ \Psi \mu \mu ⦄ \{ab\} .$ [9; 5; *T1.1.31*]
11. $\mathsf{T} ⦃ \Psi \mu \mu ⦄ \{ba\} .$ [4; 5; *T1.1.31*]
12. $\Psi \{ab\} .$ [10; *D1.1.016*]
13. $\Psi \{ba\} .$ [11; *D1.1.016*]
14. $\mu \{a\} .$ [11; *D1.1.016*]
15. $\mu \{b\} .$ [11; *D1.1.016*]
16. $\mathsf{A} \langle \mu \rangle ⦃ \Psi ⦄ .$ [6; *D1.2.07*; *D1.2.04*]

$a \circ b$ [12; 13; 14; 15; *D1.2.03*]

T2.3.24 $[ab\theta \Sigma] : \mathsf{CH} ⦃_2 \mathsf{WOS} \langle_3 \theta \rangle_3 \mathsf{R2} ⦄_2 ⦃_2 \Sigma ⦄_2 . \cup ⦃ \mathsf{C4} ⦃_4 \Sigma ⦄_4 ⦄ \{ab\} .$
$\cup ⦃ \mathsf{C4} ⦃_4 \Sigma ⦄_4 ⦄ \{ba\} . \supset . a \circ b$

PR $[ab\theta \Sigma] :: \mathrm{Hp}(3) . \supset ::$

$[\exists \phi] ::$

4.	$\mathbf{C4}\{\Sigma\}_{4\,4}\{\phi\}$.	[2; *D1.1.017*]
5.	$\phi\{ab\}$ ∴	
	$[\exists \Gamma\rho\Phi]$ ∴	
6.	$\Gamma \circ {\downarrow}\langle\rho\phi\rangle_{4\,4}$.	[1; 4; *T2.3.18*]
7.	$\phi \circ \mathbf{T}\{\Phi\rho\rho\}$.	
8.	$\mathbf{WO}\langle\rho\rangle\{\Phi\}$.	
9.	$\Sigma\{\Gamma\}_{2\,2}$ ::	
	$[\exists\psi]$ ::	
10.	$\mathbf{C4}\{\Sigma\}_{4\,4}\{\psi\}$	[3; *D1.1.017*]
11.	$\psi\{ba\}$ ∴	
	$[\exists\Delta\mu\Psi]$ ∴	
12.	$\Delta \circ {\downarrow}\langle\mu\psi\rangle_{4\,4}$.	[1; 10; *T2.3.18*]
13.	$\psi \circ \mathbf{T}\{\Psi\mu\mu\}$.	
14.	$\mathbf{WO}\langle\mu\rangle\{\Psi\}$.	
15.	$\Sigma\{\Delta\}_{2\,2}$.	
16.	$\mathbf{C}\langle\Sigma\rangle_{2\,2\,2}\{\mathbf{R2}\}_{2}$:	[1; *D1.6.015*]
17.	$\mathbf{R2}\{\Gamma\Delta\}_{2\,2}$ .v. $\mathbf{R2}\{\Delta\Gamma\}_{2\,2}$ ∷	[9; 15; *D1.6.014*]

$a \circ b$ [17; 6; 12; 5; 11; 13; 14; *T2.3.23*; 17; 6; 12; 5; 11; 7; 8; *T2.3.23*; *T1.1.1*]

T2.3.25 $[\theta\Sigma]$: $\mathbf{CH}\{\mathbf{WOS}\langle\theta\rangle_{3\,3}\,\mathbf{R2}\}_{2\,2\,2}\{\Sigma\}_{2}$.⊃. $\mathbf{A}\langle \cup \{\mathbf{C3}\{\Sigma\}_{4\,4}\}\rangle\{\cup\{\mathbf{C4}\{\Sigma\}_{4\,4}\}\}$ [*T2.3.24*; *D1.2.03*]

T2.3.26 $[\theta\Sigma]$: $\mathbf{CH}\{\mathbf{WOS}\langle\theta\rangle_{3\,3}\,\mathbf{R2}\}_{2\,2\,2}\{\Sigma\}_{2}$.⊃. $\mathbf{PO}\langle \cup \{\mathbf{C3}\{\Sigma\}_{4\,4}\}\rangle\{\cup\{\mathbf{C4}\{\Sigma\}_{4\,4}\}\}$ [*T2.3.17*; *T2.3.22*; *T2.3.25*; *D1.2.04*]

T2.3.27 $[bc\rho\mu\sigma\phi\psi\Gamma\Delta\Sigma]$: $\Sigma\{\Gamma\}_{2\,2}$. $\Gamma \circ {\downarrow}\langle\rho\,\mathbf{T}\{\psi\rho\rho\}\rangle_{4\,4}$. $\mathbf{P}\{\cap\{\rho\mu\}\psi\}\{b\}$. $\mu\{c\}$. $\sigma\{c\}$. $\Delta \circ {\downarrow}\langle\sigma\phi\rangle_{4\,4}$. $\mathbf{R2}\{\Delta\Gamma\}_{2\,2}$.⊃. $\cup\{\mathbf{C4}\{\Sigma\}_{4\,4}\}\{bc\}$

PR	$[bc\rho\mu\sigma\phi\psi\Gamma\Delta\Sigma]$ :: Hp(7) .⊃∴	
8.	$\sigma \subset \rho$.	[2; 6; 7; *D2.3.02*]
9.	$\rho\{c\}$.	[5; 8; *T1.1.14*]
10.	$\cap\{\rho\mu\}\{c\}$:	[9; 4; *D1.1.06*]
11.	$[c]$: $\cap\{\rho\mu\}\{c\}$.⊃. $\psi\{bc\}$:	[3; *D1.2.06*]
12.	$\psi\{bc\}$.	[10; 11]
13.	$\cap\{\rho\mu\}\{b\}$.	[3; *D1.2.06*]
14.	$\rho\{b\}$.	[13; *D1.1.06*]
15.	$\mathbf{T}\{\psi\rho\rho\}\{bc\}$.	[12; 14; 9; *D1.1.016*]
16.	$\mathbf{C4}\{\Sigma\}_{4\,4}\{\mathbf{T}\{\psi\rho\rho\}\}$ ∴	[1; 2; *D2.3.04*]
	$\cup\{\mathbf{C4}\{\Sigma\}_{4\,4}\}\{bc\}$	[15; 16; *D1.1.017*]

T2.3.28 $[bc\rho\mu\psi\Gamma\Sigma]$: $\Sigma\{\Gamma\}_{2\,2}$. $\Gamma \circ {\downarrow}\langle\rho\,\mathbf{T}\{\psi\rho\rho\}\rangle_{4\,4}$. $\mathbf{P}\{\cap\{\rho\mu\}\psi\}\{b\}$. $\mu\{c\}$. $\rho\{c\}$.⊃. $\cup\{\mathbf{C4}\{\Sigma\}_{4\,4}\}\{bc\}$

PR $[bc\rho\mu\psi\Gamma\Sigma] :: \mathrm{Hp}(5) \;.\supset\therefore$

6. $\cap \langle \rho\mu \rangle \{c\}:$ [4; 5; *D1.1.06*]

7. $[c] : \cap \langle \rho\mu \rangle \{c\} \;.\supset.\; \psi\{bc\}:$ [3; *D1.2.06*]

8. $\psi\{bc\}.$ [6; 7]

9. $\cap \langle \rho\mu \rangle \{b\}.$ [3; *D1.2.06*]

10. $\rho\{b\}.$ [9; *D1.1.06*]

11. $\mathbf{T}\{\psi\rho\rho\}\{bc\}.$ [5; 10; 8; *D1.1.016*]

12. $\mathbf{C4}\{_4\Sigma\}_4 \langle \mathbf{T}\{\psi\rho\rho\} \rangle \therefore$ [1; 2; *D2.3.04*]

$\cup \langle \mathbf{C4}\{_4\Sigma\}_4 \rangle \{bc\}$ [11; 12; *D1.1.017*]

T2.3.29 $[bc\rho\mu\sigma\phi\psi\Delta\Gamma\Sigma] : \mathbf{P}\{\cap \langle \rho\mu \rangle \psi\}\{b\} \,.\, \Gamma \circ {\downarrow}\langle_4 \rho\, \mathbf{T}\{\psi\rho\rho\} \rangle_4 \,.$
$\Delta \circ {\downarrow}\langle_4 \sigma\, \mathbf{T}\{\phi\sigma\sigma\} \rangle_4 \,.\, \sigma\{c\} \,.\, \Sigma\{_2\Delta\}_2 \,.\, \mathbf{R2}\{_2\Gamma\Delta\}_2 \,.\, \sim(\rho\{c\}) \;.\supset.\; \cup \langle \mathbf{C4}\{_4\Sigma\}_4 \rangle \{bc\}$

PR $[bc\rho\mu\sigma\phi\psi\Delta\Gamma\Sigma] : \mathrm{Hp}(7) \;.\supset:$

9. $[bc] : \rho\{c\} \,.\, \sigma\{c\} \,.\, \sim(\rho\{c\}) \;.\supset.\; \mathbf{T}\{\phi\sigma\sigma\}\{bc\}:$ [2; 3; 6; *D2.3.02*]

10. $\cap \langle \rho\mu \rangle \{b\}.$ [1; *D1.2.06*]

11. $\rho\{b\}.$ [11; *D1.1.06*]

12. $\mathbf{T}\{\phi\sigma\sigma\}\{bc\}.$ [9; 11; 4; 8]

13. $\mathbf{C4}\{_4\Sigma\}_4 \langle \mathbf{T}\{\phi\sigma\sigma\} \rangle :$ [3; 5; *D2.3.04*]

$\cup \langle \mathbf{C4}\{_4\Sigma\}_4 \rangle \{bc\}$ [13; 12; *D1.1.017*]

T2.3.30 $[bc\rho\mu\sigma\phi\psi\Gamma\Delta\Sigma] : \Sigma\{_2\Gamma\}_2 \,.\, \Sigma\{_2\Delta\}_2 \,.\, \Gamma \circ {\downarrow}\langle_4 \rho\, \mathbf{T}\{\psi\rho\rho\} \rangle_4 . \Delta \circ {\downarrow}\langle_4 \sigma\, \mathbf{T}\{\phi\sigma\sigma\} \rangle_4 \,.$
$\mathbf{P}\{\cap \langle \rho\mu \rangle \psi\}\{b\} \,.\, \sigma\{c\} \,.\, \mu\{c\} \,.\, \mathbf{R2}\{_2\Gamma\Delta\}_2 \;.\supset.\; \cup \langle \mathbf{C4}\{_4\Sigma\}_4 \rangle \{bc\}.$
[*T2.3.28*; *T2.3.29*]

T2.3.31 $[bc\mu\rho\theta\psi\Gamma\Sigma] : \mathbf{CH}\{_2\mathbf{WOS}\langle_3\theta\rangle_3\, \mathbf{R2}\}_2 \langle_2 \Sigma \rangle_2 \,.\, \mu \subset \cup \langle \mathbf{C3}\{_4\Sigma\}_4 \rangle \,.\, \Sigma\{_2\Gamma\}_2 \,.$
$\Gamma \circ {\downarrow}\langle_4 \rho\, \mathbf{T}\{\psi\rho\rho\} \rangle_4 \,.\, \mathbf{P}\{\cap \langle \rho\mu \rangle \psi\}\{b\} \,.\, \mu\{c\} \;.\supset.\; \cup \langle \mathbf{C4}\{_4\Sigma\}_4 \rangle \{bc\}$

PR $[bc\mu\rho\theta\psi\Gamma\Sigma] :: \mathrm{Hp}(6) \;.\supset\therefore$

7. $\cup \langle \mathbf{C3}\{_4\Sigma\}_4 \rangle \{c\} \therefore$ [2; 6; *T1.1.14*]

$[\exists\sigma]:$

8. $\mathbf{C3}\{_4\Sigma\}_4 \langle \sigma \rangle \,.$

9. $\sigma\{c\}:$ [7; *D1.1.09*]

$[\exists\phi\Delta]:$

10. $\Sigma\{_2\Delta\}_2 \,.$

11. $\Delta \circ {\downarrow}\langle_4 \sigma\, \mathbf{T}\{\phi\sigma\sigma\} \rangle_4 \,.$ [1; 8; *T2.3.15*]

12. $\mathbf{C}\langle_2 \Sigma \rangle_2 \langle_2 \mathbf{R2} \rangle_2 :$ [1; *D1.6.015*]

13. $\mathbf{R2}\{_2\Gamma\Delta\}_2 . \mathrm{v} . \mathbf{R2}\{_2\Delta\Gamma\}_2 \therefore$ [3; 10; 12; *D1.6.014*]

$\cup \langle \mathbf{C4}\{_4\Sigma\}_4 \rangle \{bc\}$ [13; 3; 10; 4; 11; 5; 9; 6; *T2.3.30*; 13; 3; 4; 5; 6; 9; 11; *T2.3.27*]

T2.3.32 $[b\mu\rho\theta\psi\Gamma\Sigma] \therefore \mathbf{CH}\{_2\mathbf{WOS}\langle_3\theta\rangle_3\, \mathbf{R2}\}_2 \langle_2 \Sigma \rangle_2 \,.\, \mu \subset \cup \langle \mathbf{C3}\{_4\Sigma\}_4 \rangle \,.\, \Sigma\{_2\Gamma\}_2 \,.$
$\Gamma \circ {\downarrow}\langle_4 \rho\, \mathbf{T}\{\psi\rho\rho\} \rangle_4 \,.\, \mathbf{P}\{\cap \langle \rho\mu \rangle \psi\}\{b\} :\supset: [c] : \mu\{c\} \;.\supset.$
$\cup \langle \mathbf{C4}\{_4\Sigma\}_4 \rangle \{bc\}$ [*T2.3.31*]

T2.3.33 $[b\theta\mu\rho\psi\Gamma\Sigma]:\mathbf{CH}\{_2\mathbf{WOS}\langle_3\theta\rangle_3\mathbf{R2}\}_2\{_2\Sigma\}_2.\ \mu\subset\cup\{\mathbf{C3}\{_4\Sigma\}_4\}.\ \Sigma\{_2\Gamma\}_2.$
$\Gamma\circ\downarrow\langle_4\rho\ \mathbf{T}\{\psi\rho\rho\}\rangle_4.\ \mathbf{P}\{\cap\{\rho\mu\}\psi\}\{b\}\ .\supset.\ \mathbf{P}\{\mu\cup\{\mathbf{C4}\{_4\Sigma\}_4\}\}\{b\}$

PR $[b\mu\rho\theta\psi\Gamma\Sigma]\therefore \mathrm{Hp}(5)\ .\supset:$

6. $\cap\{\rho\mu\}\{b\}.$ [5; *D1.2.06*]

7. $\mu\{b\}.$ [6; *D1.1.06*]

8. $\mathbf{PO}\langle\cup\{\mathbf{C3}\{_4\Sigma\}_4\}\rangle\{\cup\{\mathbf{C4}\{_4\Sigma\}_4\}\}.$ [1; *T2.3.26*]

9. $\mathbf{PO}\langle\mu\rangle\{\cup\{\mathbf{C4}\{_4\Sigma\}_4\}\}:$ [2; 8; *T1.2.28*]

10. $[c]:\mu\{c\}\ .\supset.\ \cup\{\mathbf{C4}\{_4\Sigma\}_4\}\{bc\}:$ [1; 2; 3; 4; 5; *T2.3.32*]

$\mathbf{P}\{\mu\cup\{\mathbf{C4}\{_4\Sigma\}_4\}\}\{b\}$ [9; 7; 10; *D1.2.06*]

T2.3.34 $[\theta\mu\Sigma]:\mathbf{CH}\{_2\mathbf{WOS}\langle_3\theta\rangle_3\mathbf{R2}\}_2\{_2\Sigma\}_2.\ \mu\subset\cup\{\mathbf{C3}\{_4\Sigma\}_4\}.\ !\{\mu\}\ .\supset.\ [\exists b].$
$\mathbf{P}\{\mu\cup\{\mathbf{C4}\{_4\Sigma\}_4\}\}\{b\}$

PR $[\theta\mu\Sigma]::\mathrm{Hp}(3)\ .\supset::$

$[\exists a]::$

4. $\mu\{a\}.$ [3; *D1.1.08*]

5. $\cup\{\mathbf{C3}\{_4\Sigma\}_4\}\{a\}\therefore$ [4; 2; *T1.1.14*]

$[\exists\rho]\therefore$

6. $\mathbf{C3}\{_4\Sigma\}_4\{\rho\}.$

7. $\rho\{a\}:$ [5; *D1.1.09*]

$[\exists\Gamma\psi]:$

8. $\Sigma\{_2\Gamma\}_2.$

9. $\Gamma\circ\downarrow\langle_4\rho\ \mathbf{T}\{\psi\rho\rho\}\rangle_4.$ [1; 6; *T2.3.15*]

10. $\mathbf{WO}\langle\rho\rangle\{\psi\}.$

11. $\cap\{\rho\mu\}\{a\}.$ [4; 7; *D1.1.06*]

12. $\cap\{\rho\mu\}\subset\rho.$ [*T1.1.21*]

13. $!\{\cap\{\rho\mu\}\}.$ [11; *D1.1.08*]

$[\exists b].$

14. $\mathbf{P}\{\cap\{\rho\mu\}\psi\}\{b\}.$ [10; 12; 13; *D1.2.07*]

15. $\mathbf{P}\{\mu\cup\{\mathbf{C4}\{_4\Sigma\}_4\}\}\{b\}::$ [1; 2; 8; 9; 14; *T2.3.33*]

$[\exists b].\ \mathbf{P}\{\mu\cup\{\mathbf{C4}\{_4\Sigma\}_4\}\}\{b\}$ [15]

T2.3.35 $[\theta\Sigma]:\mathbf{CH}\{_2\mathbf{WOS}\langle_3\theta\rangle_3\mathbf{R2}\}_2\{_2\Sigma\}_2\ .\supset:[\mu]:\mu\subset\cup\{\mathbf{C3}\{_4\Sigma\}_4\}.\ !\{\mu\}\ .\supset.$
$[\exists b].\ \mathbf{P}\{\mu\cup\{\mathbf{C4}\{_4\Sigma\}_4\}\}\{b\}$ [*T2.3.34*]

T2.3.36 $[\theta\Sigma]:\mathbf{CH}\{_2\mathbf{WOS}\langle_3\theta\rangle_3\mathbf{R2}\}_2\{_2\Sigma\}_2\ .\supset.\ \mathbf{WO}\langle\cup\{\mathbf{C3}\{_4\Sigma\}_4\}\rangle\{\cup\{\mathbf{C4}\{_4\Sigma\}_4\}\}$
[*T2.3.26*; *T2.3.35*; *D1.2.07*]

T2.3.37 $[\theta\Sigma]:\mathbf{CH}\{_2\mathbf{WOS}\langle_3\theta\rangle_3\mathbf{R2}\}_2\{_2\Sigma\}_2\ .\supset.$
$\mathbf{WOS}\langle_3\theta\rangle_3\{_2\downarrow\langle_4\cup\{\mathbf{C3}\{_4\Sigma\}_4\}\ \mathbf{T}\{\cup\{\mathbf{C4}\{_4\Sigma\}_4\}\cup\{\mathbf{C3}\{_4\Sigma\}_4\}\cup\{\mathbf{C3}\{_4\Sigma\}_4\}\}\rangle_4\}_2$
[*T2.3.36*; *T2.3.14*; *T1.6.2*; *D2.3.01*]

T2.3.38 $[\rho\mu\phi\psi\Gamma\Sigma]:\Sigma\{_2\Gamma\}_2 . \Gamma \circ \downarrow\langle_4 \rho\phi\rangle_4 .$
$\downarrow\langle_4 \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4 \gtrdot \mathbf{T}\{\cup \lessdot \mathbf{C4}\{_4\Sigma\}_4\gtrdot \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot_4\}_4\rangle_4 \circ \downarrow\langle_4 \mu\psi\rangle_4 .\supset.$
$\rho \subset \mu$

PR $[\rho\mu\phi\psi\Gamma\Sigma]:\mathrm{Hp}(3) .\supset.$

4. $\mu \circ \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot .$ [3; *T1.6.8*]

5. $\mathbf{C3}\{_4\Sigma\}_4 \{\rho\} .$ [1; 2; *D2.3.03*]

6. $\rho \subset \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot .$ [5; *T1.1.24*]

$\rho \subset \mu$ [6; 4; *T1.1.16*]

T2.3.39 $[ab\rho\mu\theta\phi\psi\Gamma\Sigma]:\mathbf{CH}\lessdot_2 \mathbf{WOS}\langle_3\theta\rangle_3 \mathbf{R2}\gtrdot_2 \{_2\Sigma\}_2 . \Sigma\{_2\Gamma\}_2 . \Gamma \circ \downarrow\langle_4 \rho\phi\rangle_4 .$
$\downarrow\langle \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4 \gtrdot \mathbf{T}\{\cup \lessdot \mathbf{C4}\{_4\Sigma\}_4\gtrdot \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot_4\}_4\rangle \circ \downarrow\langle_4 \mu\psi\rangle_4 .$
$\phi\{ab\} .\supset. \psi\{ab\}$

PR $[ab\rho\mu\theta\phi\psi\Gamma\Sigma]:\mathrm{Hp}(5) .\supset.$

6. $\mathbf{C4}\{_4\Sigma\}_4 \{\phi\} .$ [2; 3; *D2.3.04*]

7. $\cup \lessdot \mathbf{C4}\{_4\Sigma\}_4\gtrdot \{ab\} .$ [5; 6; *D1.1.017*]

8. $\mathbf{T}\{\cup \lessdot \mathbf{C4}\{_4\Sigma\}_4\gtrdot \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot\} \circ \psi .$ [4; *T1.6.8*]

$[\exists\Delta\sigma\Phi] .$

9. $\Sigma\{_2\Delta\}_2 .$

10. $\Delta \circ \downarrow\langle_4 \sigma\Phi\rangle_4 .$ [1; 6; *T2.3.18*]

11. $\phi \circ \mathbf{T}\{\Phi\sigma\sigma\} .$

12. $\mathbf{T}\{\Phi\sigma\sigma\}\{ab\} .$ [5; 11; *T1.1.31*]

13. $\sigma\{a\} .$ [12; *D1.1.016*]

14. $\sigma\{b\} .$ [12; *D1.1.016*]

15. $\mathbf{C3}\{_4\Sigma\}_4 \{\sigma\} .$ [9; 10; *D2.3.03*]

16. $\cup \lessdot \mathbf{C3}\{_4\Sigma\}_4 \gtrdot \{a\} .$ [15; 13; *D1.1.09*]

17. $\cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot \{b\} .$ [15; 14; *D1.1.09*]

18. $\mathbf{T}\{\cup \lessdot \mathbf{C4}\{_4\Sigma\}_4\gtrdot \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot\}\{ab\} .$
[7; 16; 17; *D1.1.016*]

$\psi\{ab\}$ [8; 18; *T1.1.31*]

T2.3.40 $[\rho\mu\theta\phi\psi\Gamma\Sigma]:\mathbf{CH}\lessdot_2 \mathbf{WOS}\langle_3\theta\rangle_3 \mathbf{R2}\gtrdot_2 \{_2\Sigma\}_2 . \Sigma\{_2\Gamma\}_2 . \Gamma \circ \downarrow\langle_4 \rho\phi\rangle_4 .$
$\downarrow\langle \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4 \gtrdot \mathbf{T}\{\cup \lessdot \mathbf{C4}\{_4\Sigma\}_4\gtrdot \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot \cup \lessdot \mathbf{C3}\{_4\Sigma\}_4\gtrdot\}\rangle \circ \downarrow\langle_4 \mu\psi\rangle_4 .$
$\supset. \phi \subset \psi$ [*T2.3.39*; *D1.1.012*]

T2.3.41 $[ab\rho\sigma\phi\psi\Gamma\Delta]:\Gamma \circ \downarrow\langle_4 \rho\phi\rangle_4 . \Delta \circ \downarrow\langle_4 \sigma\psi\rangle_4 . \sigma\{b\} . \sim(\rho\{b\}) . \mathbf{R2}\{_2\Delta\Gamma\}_2 .\supset.$
$\psi\{ab\}$

PR $[ab\rho\sigma\phi\psi\Gamma\Delta]:\mathrm{Hp}(5) .\supset.$

6. $\sigma \subset \rho .$ [1; 2; 5; *D2.3.02*]

7. $\rho\{b\} .$ [3; 6; *T1.1.14*]

$\psi\{ab\}$ [4; 7]

T2.3.42 $[ab\rho\sigma\phi\psi\Psi\Gamma\Delta\Sigma] : \rho\{a\} . \sim(\rho\{b\}) . \psi \circ \mathbf{T} \lVert \cup \lll \mathbf{C4} \{_4 \Sigma \}_4 \ggg \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot$
$\cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \rVert . \sigma\{b\} . \mathbf{C3} \{_4 \Sigma \}_4 \prec\sigma\succ . \Sigma\{_2 \Delta \}_2 . \Gamma \circ \downarrow\langle_4 \rho\phi \rangle_4 . \Delta \circ \downarrow\langle_4 \sigma\, \mathbf{T} \lVert \Psi\sigma\sigma \rVert \rangle_4 .$
$\mathbf{R2} \{_2 \Gamma\Delta \}_2 . \supset . \psi\{ab\}$

PR $[ab\rho\sigma\phi\psi\Psi\Gamma\Delta\Sigma] \therefore \mathrm{Hp}(9) . \supset :$

10. $\rho \subset \sigma .$ [7; 8; 9; *D2.3.02*]
11. $\phi \subset \mathbf{T} \lVert \Psi\sigma\sigma \rVert :$ [7; 8; 9; *D2.3.02*]
12. $[ab] : \rho\{a\} . \sigma\{b\} . \sim(\rho\{b\}) . \supset . \mathbf{T} \lVert \Psi\sigma\sigma \rVert \{ab\} :$ [7; 8; 9; *D2.3.02*]
13. $\mathbf{T} \lVert \Psi\sigma\sigma \rVert \{ab\} .$ [1; 2; 4; 12]
14. $\mathbf{C4} \{_4 \Sigma \}_4 \prec \mathbf{T} \lVert \Psi\sigma\sigma \rVert \succ .$ [6; 8; *D2.3.04*]
15. $\cup \lll \mathbf{C4} \{_4 \Sigma \}_4 \ggg \{ab\} .$ [13; 14; *D1.1.017*]
16. $\sigma\{a\} .$ [13; *D1.1.016*]
17. $\sigma\{b\} .$ [13; *D1.1.016*]
18. $\cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \{a\} .$ [5; 16; *D1.1.09*]
19. $\cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \{b\} .$ [5; 17; *D1.1.09*]
20. $\mathbf{T} \lVert \cup \lessdot \mathbf{C4} \{_4 \Sigma \}_4 \gtrdot \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \rVert \{ab\} .$
[15; 18; 19; *D1.1.016*]

$\psi\{ab\}$ [3; 20; *T1.1.31*]

T2.3.43 $[ab\rho\mu\theta\phi\psi\Gamma\Sigma] : \mathbf{CH} \lVert_2 \mathbf{WOS}\langle_3 \theta \rangle_3 \mathbf{R2} \rVert_2 \prec_2 \Sigma \succ . \Sigma\{_2 \Gamma \}_2 . \Gamma \circ \downarrow\langle_4 \rho\phi \rangle_4 .$
$\downarrow\langle_4 \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \mathbf{T} \lVert \cup \lll \mathbf{C4} \{_4 \Sigma \}_4 \ggg \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \rVert \rangle_4 \circ \downarrow\langle_4 \mu\psi \rangle_4 .$
$\rho\{a\} . \mu\{b\} . \sim(\rho\{b\}) . \supset . \psi\{ab\}$

PR $[ab\rho\mu\theta\phi\psi\Gamma\Sigma] :: \mathrm{Hp}(7) . \supset ::$

8. $\mu \circ \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot .$ [4; *T1.6.8*]
9. $\psi \circ \mathbf{T} \lVert \cup \lll \mathbf{C4} \{_4 \Sigma \}_4 \ggg \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \rVert .$ [4; *T1.6.8*; *T1.1.28*]
10. $\cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \{b\} ::$ [8; 6; *T1.1.10*]

$[\exists\sigma] ::$

11. $\mathbf{C3} \{_4 \Sigma \}_4 \prec\sigma\succ .$ [10; *D1.1.09*]
12. $\sigma\{b\} \therefore$

$[\exists\Psi\Delta] \therefore$

13. $\Sigma\{_2 \Delta \}_2 .$ [1; 11; *T2.3.13*]
14. $\Delta \circ \downarrow\langle_4 \sigma\, \mathbf{T} \lVert \Psi\sigma\sigma \rVert \rangle_4 .$
15. $\mathbf{C} \langle_2 \Sigma \rangle_2 \prec_2 \mathbf{R2} \succ_2 :$ [1; *D1.6.015*]
16. $\mathbf{R2} \{_2 \Gamma\Delta \}_2 . \vee . \mathbf{R2} \{_2 \Delta\Gamma \}_2 ::$ [2; 13; 15; *D1.6.014*]

$\psi\{ab\}$ [16; 5; 7; 9; 12; 11; 13; 3; 14; *T2.3.42*; 16; 3; 14; 12; 7; *T2.3.41*]

T2.3.44 $[\rho\mu\theta\phi\psi\Gamma\Sigma] \therefore \mathbf{CH} \lVert_2 \mathbf{WOS}\langle_3 \theta \rangle_3 \mathbf{R2} \rVert_2 \prec_2 \Sigma \succ . \Sigma\{_2 \Gamma \}_2 . \Gamma \circ \downarrow\langle_4 \rho\phi \rangle_4 .$
$\downarrow\langle_4 \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \mathbf{T} \lVert \cup \lll \mathbf{C4} \{_4 \Sigma \}_4 \ggg \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \cup \lessdot \mathbf{C3} \{_4 \Sigma \}_4 \gtrdot \rVert \rangle_4 \circ \downarrow\langle_4 \mu\psi \rangle_4 .$
$\supset : [ab] : \rho\{a\} . \mu\{b\} . \sim(\rho\{b\}) . \supset . \psi\{ab\}$ [*T2.3.43*]

T2.3.45 $[\rho\mu\theta\phi\psi\Gamma\Sigma] \therefore \mathrm{CH}\{_2\mathrm{WOS}\langle_3\theta\rangle_3\,\mathrm{R2}\}_2\,(_2\Sigma)_2\,.\,\Sigma\{_2\Gamma\}_2\,.\,\Gamma\circ\downarrow\langle_4\rho\phi\rangle_4\,.$
$\downarrow\langle_4\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\,\mathrm{T}\{\cup\langle\mathrm{C4}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle_4\}\rangle_4\circ\downarrow\langle_4\mu\psi\rangle_4\,.$
$\supset: \rho\subset\mu\,.\,\phi\subset\psi:[ab]:\rho\{a\}\,.\,\mu\{b\}\,.\sim(\rho\{b\})\,.\supset.\,\psi\{ab\}$
[*T2.3.38*; *T2.3.40*; *T2.3.44*]

T2.3.46 $[\theta\Sigma] \therefore \mathrm{CH}\{_2\mathrm{WOS}\langle_3\theta\rangle_3\,\mathrm{R2}\}_2\,(_2\Sigma)_2\,.\supset:[\Gamma]:\Sigma\{_2\Gamma\}_2\,.\supset.$
$\mathrm{R2}\{_2\Gamma\downarrow\langle_4\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\,\mathrm{T}\{\cup\langle\mathrm{C4}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\}\rangle_4\}_2$
[*T2.3.45*; *D2.3.02*]

T2.3.47 $[\theta\Sigma]:\mathrm{CH}\{_2\mathrm{WOS}\langle_3\theta\rangle_3\,\mathrm{R2}\}_2\,(_2\Sigma)_2\,.\supset.$
$\mathrm{UB}\{_2\Sigma\ \mathrm{WOS}\langle_3\theta\rangle_3\,\mathrm{R2}\}_2\{_2\downarrow\langle_4\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\,\mathrm{T}\{\cup\langle\mathrm{C4}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\cup$
$\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\}\rangle_4\}_2$ [*T2.3.11*; *T2.3.37*; *D1.6.015*; *T2.3.46*; *D1.6.012*]

T2.3.48 $[a\mu\rho\theta\phi\psi\Gamma\Delta\Sigma]:\mathrm{UB}\{_2\Sigma\ \mathrm{WOS}\langle_3\theta\rangle_3\,\mathrm{R2}\}_2\{_2\Delta\}_2\,.$
$\downarrow\langle_4\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\,\mathrm{T}\{\cup\langle\mathrm{C4}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle_4\}\rangle_4\circ\downarrow\langle_4\rho\phi\rangle_4\,.$
$\Delta\circ\downarrow\langle_4\mu\psi\rangle_4\,.\,\rho\{a\}\,.\supset.\ \mu\{a\}$

PR $[a\mu\rho\theta\phi\psi\Gamma\Delta\Sigma]::\mathrm{Hp}(4)\,.\supset\therefore$

5. $\rho\circ\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\,.$ [2; *T1.6.8*]
6. $\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\{a\}\therefore$ [4; 5; *T1.1.10*]
7. $[\Gamma]:\Sigma\{_2\Gamma\}_2\,.\supset.\,\mathrm{R2}\{_2\Gamma\Delta\}_2:$ [1; *D1.6.012*]

$[\exists\sigma]:$

8. $\mathrm{C3}\{_4\Sigma\}_4\,(\sigma)\,.$
9. $\sigma\{a\}\,.$ [6; *D1.1.09*]

$[\exists\Gamma\Phi]\,.$

10. $\Sigma\{_2\Gamma\}_2\,.$
11. $\Gamma\circ\downarrow\langle_4\sigma\Phi\rangle_4\,.$ [8; *D2.3.03*]
12. $\mathrm{R2}\{_2\Gamma\Delta\}_2\,.$ [7; 10]
13. $\sigma\subset\mu\therefore$ [3; 11; 12; *D2.3.02*]

$\mu\{a\}$ [9; 13; *T1.1.14*]

T2.3.49 $[\rho\mu\theta\phi\psi\Gamma\Delta\Sigma]:\mathrm{UB}\{_2\Sigma\ \mathrm{WOS}\langle_3\theta\rangle_3\,\mathrm{R2}\}_2\{_2\Delta\}_2\,.$
$\downarrow\langle_4\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\,\mathrm{T}\{\cup\langle\mathrm{C4}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle_4\}\rangle_4\circ\downarrow\langle_4\rho\phi\rangle_4\,.$
$\Delta\circ\downarrow\langle_4\mu\psi\rangle_4\,.\supset.\ \rho\subset\mu$ [*T2.3.48*; *D1.1.04*]

T2.3.50 $[ab\mu\rho\theta\phi\psi\Delta\Sigma]:\mathrm{UB}\{_2\Sigma\ \mathrm{WOS}\langle_3\theta\rangle_3\,\mathrm{R2}\}_2\{_2\Delta\}_2\,.$
$\downarrow\langle_4\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\,\mathrm{T}\{\cup\langle\mathrm{C4}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle_4\}\rangle_4\circ\downarrow\langle_4\rho\phi\rangle_4\,.$
$\Delta\circ\downarrow\langle_4\mu\psi\rangle_4\,.\,\phi\{ab\}\,.\supset.\ \psi\{ab\}$

PR $[ab\mu\rho\theta\phi\psi\Delta\Sigma]::\mathrm{Hp}(4)\,.\supset\therefore$

5. $\phi\circ\mathrm{T}\{\cup\langle\mathrm{C4}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\}\,.$ [2; *T1.6.8*]
6. $\mathrm{T}\{\cup\langle\mathrm{C4}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\cup\langle\mathrm{C3}\{_4\Sigma\}_4\rangle\}\{ab\}:$ [4; 5; *T1.1.31*]

7.	$[\Gamma]:\Sigma\underset{2}{\{}\Gamma\underset{2}{\}}\ .\supset.\ \mathsf{R2}\underset{2}{\{}\Gamma\Delta\underset{2}{\}}:$	[1; *D1.6.012*]
8.	$\cup\,\not\Lleftarrow\!\mathsf{C4}\underset{4}{\{\!\!\{}\Sigma\underset{4}{\}\!\!\}}\!\not\Rrightarrow\{ab\}:$	[6; *D1.1.016*]
	$[\exists\Phi]:$	
9.	$\quad\mathsf{C4}\underset{4}{\{\!\!\{}\Sigma\underset{4}{\}\!\!\}}\ \not\prec\Phi\not\succ.$	[8; *D1.1.017*]
10.	$\quad\Phi\{ab\}.$	
	$\quad[\exists\sigma\Gamma].$	
11.	$\qquad\Sigma\underset{2}{\{}\Gamma\underset{2}{\}}.$	[9; *D2.3.04*]
12.	$\qquad\Gamma\circ\downarrow\underset{4}{\langle}\sigma\Phi\underset{4}{\rangle}.$	
13.	$\qquad\mathsf{R2}\underset{2}{\{}\Gamma\Delta\underset{2}{\}}.$	[7; 11]
	$\quad\Phi\subset\psi\therefore$	[12; 3; 13; *D2.3.02*]
	$\psi\{ab\}$	[10; 14; *T1.1.35*]
T2.3.51	$[\mu\rho\theta\phi\psi\Delta\Sigma]:\mathsf{UB}\underset{2}{\{\!\!\{}\Sigma\ \mathsf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\ \mathsf{R2}\underset{2}{\}\!\!\}}\underset{2}{\{}\Delta\underset{2}{\}}.$	
	$\downarrow\underset{4}{\langle}\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\mathsf{T}\{\!\!\{\cup\not\Lleftarrow\mathsf{C4}\underset{4}{\{\!\!\{}\Sigma\underset{4}{\}\!\!\}}\not\Rrightarrow\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\}\!\!\}\underset{4}{\rangle}\circ\downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}.$	
	$\Delta\circ\downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}\ .\supset.\ \phi\subset\psi$	[*T2.3.50*; *D1.1.012*]
T2.3.52	$[b\Sigma]:\sim(\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\{b\})\ .\equiv.\ [\rho].\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\ \not\prec\rho\not\succ\ .\supset.\ \sim(\rho\{b\})$	[*D1.1.09*]
T2.3.53	$[ab\rho\mu\theta\phi\psi\Delta\Sigma]:\mathsf{UB}\underset{2}{\{\!\!\{}\Sigma\ \mathsf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\ \mathsf{R2}\underset{2}{\}\!\!\}}\underset{2}{\{}\Delta\underset{2}{\}}.$	
	$\downarrow\underset{4}{\langle}\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\mathsf{T}\{\!\!\{\cup\not\Lleftarrow\mathsf{C4}\underset{4}{\{\!\!\{}\Sigma\underset{4}{\}\!\!\}}\not\Rrightarrow\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\}\!\!\}\underset{4}{\rangle}\circ\downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}.$	
	$\Delta\circ\downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}.\rho\{a\}.\ \mu\{b\}.\sim(\rho\{b\})\ .\supset.\ \psi\{ab\}$	
PR	$[ab\rho\mu\theta\phi\psi\Delta\Sigma]::\mathrm{Hp}(6)\ .\supset::$	
7.	$\rho\circ\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ.$	[2; *T1.6.8*]
8.	$\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\{a\}:$	[4; 7; *T1.1.10*]
9.	$[\Gamma]:\Sigma\underset{2}{\{}\Gamma\underset{2}{\}}\ .\supset.\ \mathsf{R2}\underset{2}{\{}\Gamma\Delta\underset{2}{\}}:$	[1; *D1.6.012*]
10.	$\sim(\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\{b\})\therefore$	[6; 7; *T1.1.11*]
	$[\exists\sigma]\therefore$	
11.	$\quad\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\ \not\prec\sigma\not\succ.$	[8; *D1.1.09*]
12.	$\quad\sigma\{a\}:$	
	$\quad[\exists\Phi\Gamma]:$	
13.	$\qquad\Sigma\underset{2}{\{}\Gamma\underset{2}{\}}.$	[11; *D2.1.03*]
14.	$\qquad\Gamma\circ\downarrow\underset{4}{\langle}\sigma\Phi\underset{4}{\rangle}.$	
15.	$\qquad\mathsf{R2}\underset{2}{\{}\Gamma\Delta\underset{2}{\}}:$	[9; 13]
16.	$\quad[ab]:\sigma\{a\}.\ \mu\{b\}.\sim(\sigma\{b\})\ .\supset.\ \psi\{ab\}:$	[14; 3; *D2.3.02*]
17.	$\quad\sim(\sigma\{b\})::$	[10; 11; *T2.3.52*]
	$\psi\{ab\}$	[16; 12; 5; 17]
T2.3.54	$[\rho\mu\theta\phi\psi\Delta\Sigma]\therefore\mathsf{UB}\underset{2}{\{\!\!\{}\Sigma\ \mathsf{WOS}\underset{3}{\langle}\theta\underset{3}{\rangle}\ \mathsf{R2}\underset{2}{\}\!\!\}}\underset{2}{\{}\Delta\underset{2}{\}}.$	
	$\downarrow\underset{4}{\langle}\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\mathsf{T}\{\!\!\{\cup\not\Lleftarrow\mathsf{C4}\underset{4}{\{\!\!\{}\Sigma\underset{4}{\}\!\!\}}\not\Rrightarrow\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\cup\not\prec\mathsf{C3}\underset{4}{\{}\Sigma\underset{4}{\}}\not\succ\}\!\!\}\underset{4}{\rangle}\circ\downarrow\underset{4}{\langle}\rho\phi\underset{4}{\rangle}.$	
	$\Delta\circ\downarrow\underset{4}{\langle}\mu\psi\underset{4}{\rangle}\ :\supset:\ [ab].\rho\{a\}.\ \mu\{b\}.\sim(\rho\{b\})\ .\supset.\ \psi\{ab\}$	[*T2.3.53*]

T2.3.55 $[\theta\Delta\Sigma]:\mathbf{UB}\{\!\!\{_2\,\Sigma\ \mathbf{WOS}\langle_3\theta\rangle_3\,\mathbf{R2}\}\!\!\}_2\{_2\Delta\}_2\ .\supset.$

$\mathbf{R2}\{_2\downarrow\langle_4\cup\langle\!\langle_4\mathbf{C3}\{\Sigma\}\rangle\!\rangle_4\,\mathbf{T}\{\!\!\{\cup\langle\!\langle_4\mathbf{C4}\{\!\!\{\Sigma\}\!\!\}\rangle\!\rangle_4\cup\langle\!\langle_4\mathbf{C3}\{\Sigma\}\rangle\!\rangle_4\cup\langle\!\langle_4\mathbf{C3}\{\Sigma\}\rangle\!\rangle_4\}\!\!\}\rangle_4\,\Delta\}_2$ [*T2.3.49*; *T2.3.51*; *T2.3.54*; *D2.3.03*]

T2.3.56 $[\theta\Sigma]:\mathbf{CH}\{\!\!\{_2\mathbf{WOS}\langle_3\theta\rangle_3\,\mathbf{R2}\}\!\!\}_2\{\!\!\{_2\Sigma\}\!\!\}_2\ .\supset.$

$\mathbf{LUB}\{\!\!\{_2\,\Sigma\ \mathbf{WOS}\langle_3\theta\rangle_3\,\mathbf{R2}\}\!\!\}_2\{_2\downarrow\langle_4\cup\langle\!\langle_4\mathbf{C3}\{\Sigma\}\rangle\!\rangle_4\,\mathbf{T}\{\!\!\{\cup\langle\!\langle_4\mathbf{C4}\{\!\!\{\Sigma\}\!\!\}\rangle\!\rangle_4\cup\langle\!\langle_4\mathbf{C3}\{\Sigma\}\rangle\!\rangle_4$
$\cup\langle\!\langle_4\mathbf{C3}\{\Sigma\}\rangle\!\rangle_4\}\!\!\}\rangle_4\}_2$ [*T2.3.47*; *T2.3.55*]

T2.3.57 $[\theta\Sigma]:\mathbf{CH}\{\!\!\{_2\mathbf{WOS}\langle_3\theta\rangle_3\,\mathbf{R2}\}\!\!\}_2\{\!\!\{_2\Sigma\}\!\!\}_2\ .\supset.\ [\exists\Gamma].\,\mathbf{WOS}\langle_3\theta\rangle_3\{_2\Gamma\}_2.$

$\mathbf{LUB}\{\!\!\{_2\,\Sigma\ \mathbf{WOS}\langle_3\theta\rangle_3\,\mathbf{R2}\}\!\!\}_2\{_2\Gamma\}_2$ [*T2.3.37*; *T2.3.56*]

T2.3.58 $[\theta].\,\mathbf{W}\langle_2\mathbf{WOS}\langle_3\theta\rangle_3\rangle_2\{\!\!\{_2\mathbf{R2}\}\!\!\}_2$ [*T2.3.57*; *D1.6.016*]

T2.3.58 completes the series begun at *T2.3.12* and establishes that every **R2**-chain in $\mathbf{WOS}\langle_3\theta\rangle_3$ has a least upper bound. Next we show that the last condition for application of **KZL*** is satisfied by $\mathbf{WOS}\langle_3\theta\rangle_3$ and **R2**.

T2.3.59 $[\theta]:\,!\{\!\!\{\theta\}\!\!\}\ .\supset.\ !\{\!\!\{_2\mathbf{WOS}\langle_3\theta\rangle_3\}\!\!\}_2$

PR $[\theta]:\mathrm{Hp}(1)\ .\supset:$
$[\exists a].$

2. $\theta\{a\}.$ [1; *D1.1.08*]
3. $\downarrow\langle_1 a\rangle_1\subset\theta.$ [2; *T1.3.6*]
4. $\mathbf{WO}\langle\downarrow\langle_1 a\rangle_1\rangle\{\!\!\{\circ\}\!\!\}.$ [*T1.3.11*]
5. $\mathbf{WOS}\langle_3\theta\rangle_3\{_2\downarrow\langle_4\downarrow\langle_1 a\rangle_1\,\mathbf{T}\{\!\!\{\circ\downarrow\langle_1 a\rangle_1\downarrow\langle_1 a\rangle_1\}\!\!\}\rangle_4\}_2.$ [3; 4; 5; *T1.6.2*; *D2.3.01*]

$!\{\!\!\{_2\mathbf{WOS}\langle_3\theta\rangle_3\}\!\!\}_2$ [5; *D1.6.06*]

Thus, under the condition that θ is unempty, we can apply **KZL*** to $\mathbf{WOS}\langle_3\theta\rangle_3$ and **R2** on the basis of *T2.3.11*, *T2.3.58*, and *T2.3.59.* We continue by showing that the maximal element predicted by **KZL*** is, in fact, an "ordered pair" consisting of θ and a connection which well orders it.

T2.3.60 $[m\rho\theta\phi]:\mathbf{WO}\langle\rho\rangle\{\!\!\{\phi\}\!\!\}.\,\rho\subset\theta\,.\,\theta\{m\}.\sim(\rho\{m\})\ .\supset.$

$\mathbf{WOS}\langle_3\theta\rangle_3\{_2\downarrow\langle_4\cup\langle\!\langle\rho\downarrow\langle_1 m\rangle_1\rangle\!\rangle\,\mathbf{T}\{\!\!\{\mathbf{E1}\{\!\!\{_1 m\rho\phi\}\!\!\}_1\cup\langle\!\langle\rho\downarrow\langle_1 m\rangle_1\rangle\!\rangle\cup\langle\!\langle\rho\downarrow\langle_1 m\rangle_1\rangle\!\rangle\}\!\!\}\rangle_4\}_2$

PR $[m\rho\theta\phi]:\mathrm{Hp}(4)\ .\supset.$

5. $\mathbf{WO}\langle\cup\langle\!\langle\rho\downarrow\langle_1 m\rangle_1\rangle\!\rangle\{\!\!\{\mathbf{E1}\{\!\!\{_1 m\rho\phi\}\!\!\}_1\}\!\!\}.$ [1; 4; *T1.4.39*]
6. $\cup\langle\!\langle\rho\downarrow\langle_1 m\rangle_1\rangle\!\rangle\subset\theta.$ [2; 3; *T1.3.9*]

$\mathbf{WOS}\langle_3\theta\rangle_3\{_2\downarrow\langle_4\cup\langle\!\langle\rho\downarrow\langle_1 m\rangle_1\rangle\!\rangle\,\mathbf{T}\{\!\!\{\mathbf{E1}\{\!\!\{_1 m\rho\phi\}\!\!\}_1\cup\langle\!\langle\rho\downarrow\langle_1 m\rangle_1\rangle\!\rangle\cup\langle\!\langle\rho\downarrow\langle_1 m\rangle_1\rangle\!\rangle\}\!\!\}\rangle_4\}_2$
[5; 6; *T1.6.2*; *D2.3.01*]

In the above, we finally use the major lemma of section **1.4**, viz., *T1.4.39*.

T2.3.61 $[m\rho\sigma\mu\phi\Phi\psi\Psi\Gamma]:\Gamma\circ\downarrow\langle_4\rho\phi\rangle_4\,.\,\Gamma\circ\downarrow\langle_4\sigma\Phi\rangle_4\,.$

$\downarrow\langle_4\cup\langle\!\langle\rho\downarrow\langle_1 m\rangle_1\rangle\!\rangle\psi\rangle_4\circ\downarrow\langle_4\mu\psi\rangle_4\ .\supset.\ \sigma\subset\mu$

PR $[m\rho\sigma\mu\phi\Phi\psi\Psi\Gamma] : \mathrm{Hp}(3) \;.\!\supset.$

4. $\rho \circ \sigma .$ [1; 2; *T1.6.12*]

5. $\cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \circ \mu .$ [3; *T1.6.8*; *T1.1.6*]

6. $\sigma \subset \rho .$ [4; *T1.1.6*; *T1.1.15*]

7. $\cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \subset \mu .$ [5; *T1.1.15*]

$\sigma \subset \mu$ [6; 7; *T1.1.17*; *T1.1.13*]

T2.3.62 $[abm\sigma\mu\rho\phi\Phi\psi\Gamma] : \Gamma \circ \downarrow \underset{4}{\langle} \rho\ \mathbf{T}\,\{\!\!\{\phi\rho\rho\}\!\!\} \underset{4}{\rangle} . \Gamma \circ \downarrow \underset{4}{\langle} \sigma\Phi \underset{4}{\rangle} .$

$\downarrow \underset{4}{\langle} \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle\ \mathbf{T}\,\{\!\!\{ \mathbf{E1}\,\underset{1}{\{\!\!\{} m\rho\phi \underset{1}{\}\!\!\}} \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \}\!\!\} \underset{4}{\rangle} \circ \downarrow \underset{4}{\langle} \mu\psi \underset{4}{\rangle} .$

$\Phi\{ab\} \;.\!\supset.\; \psi\{ab\}$

PR $[abm\sigma\mu\rho\phi\Phi\psi\Gamma] : \mathrm{Hp}(4) \;.\!\supset.$

5. $\Phi \circ \mathbf{T}\,\{\!\!\{\phi\rho\rho\}\!\!\} .$ [1; 2; *T1.6.12*]

6. $\mathbf{T}\,\{\!\!\{\phi\rho\rho\}\!\!\}\{ab\} .$ [4; 5; *T1.1.31*]

7. $\mathbf{E1}\,\underset{1}{\{\!\!\{} m\rho\phi \underset{1}{\}\!\!\}}\{ab\} .$ [6; *D1.4.01*]

8. $\rho\{a\} .$ [6; *D1.1.016*]

9. $\rho\{b\} .$ [6; *D1.1.016*]

10. $\cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \{a\} .$ [8; *T1.1.17*; *T1.1.14*]

11. $\cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \{b\} .$ [9; *T1.1.17*; *T1.1.14*]

12. $\mathbf{T}\,\{\!\!\{ \mathbf{E1}\,\underset{1}{\{\!\!\{} m\rho\phi \underset{1}{\}\!\!\}} \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \}\!\!\}\{ab\} .$ [7; 10; 11; *D1.1.016*]

13. $\mathbf{T}\,\{\!\!\{ \mathbf{E1}\,\{\!\!\{ m\rho\phi \}\!\!\} \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \}\!\!\} \circ \psi .$ [3; *T1.6.8*]

$\psi\{ab\}$ [12; 13; *T1.1.31*]

T2.3.63 $[m\sigma\mu\rho\phi\Phi\psi\Gamma] : \Gamma \circ \downarrow \underset{4}{\langle} \rho\ \mathbf{T}\,\{\!\!\{\phi\rho\rho\}\!\!\} \underset{4}{\rangle} . \Gamma \circ \downarrow \underset{4}{\langle} \sigma\Phi \underset{4}{\rangle} .$

$\downarrow \underset{4}{\langle} \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle\ \mathbf{T}\,\{\!\!\{ \mathbf{E1}\,\underset{1}{\{\!\!\{} m\rho\phi \underset{1}{\}\!\!\}} \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \}\!\!\} \underset{4}{\rangle} \circ \downarrow \underset{4}{\langle} \mu\psi \underset{4}{\rangle} \;.\!\supset.$

$\Phi \subset \psi$ [*T2.3.62*; *D1.1.012*]

T2.3.64 $[abm\mu\rho\sigma\phi\Phi\psi\Gamma] : \Gamma \circ \downarrow \underset{4}{\langle} \rho\ \mathbf{T}\,\{\!\!\{\phi\rho\rho\}\!\!\} \underset{4}{\rangle} . \Gamma \circ \downarrow \underset{4}{\langle} \sigma\Phi \underset{4}{\rangle} .$

$\downarrow \underset{4}{\langle} \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle\ \mathbf{T}\,\{\!\!\{ \mathbf{E1}\,\underset{1}{\{\!\!\{} m\rho\phi \underset{1}{\}\!\!\}} \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \}\!\!\} \underset{4}{\rangle} \circ \downarrow \underset{4}{\langle} \mu\psi \underset{4}{\rangle} . \sigma\{a\} .$

$\mu\{b\} . \sim(\sigma\{b\}) \;.\!\supset.\; \psi\{ab\}$

PR $[abm\mu\rho\sigma\phi\Phi\psi\Gamma] : \mathrm{Hp}(6) \;.\!\supset.$

7. $\mu \circ \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle .$ [3; *T1.6.8*]

8. $\cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \{b\} .$ [7; 5; *T1.1.14*]

9. $\sigma \circ \rho .$ [1; 2; *T1.6.12*]

10. $\sim(\rho\{b\}) .$ [6; 9; *T1.1.11*]

11. $\rho\{a\} .$ [4; 9; *T1.1.10*]

12. $m \circ b .$ [8; 10; *T1.3.7*]

13. $\mathbf{E1}\,\underset{1}{\{\!\!\{} m\rho\phi \underset{1}{\}\!\!\}}\{ab\} .$ [11; 12; *T1.1.1*; *D1.4.01*]

14. $\cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \{a\} .$ [11; *T1.1.17*; *T1.1.14*]

15. $\mathbf{T}\,\{\!\!\{ \mathbf{E1}\,\underset{1}{\{\!\!\{} m\rho\phi \underset{1}{\}\!\!\}} \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \}\!\!\}\{ab\} .$ [13; 14; 8; *D1.1.016*]

16. $\mathbf{T}\,\{\!\!\{ \mathbf{E1}\,\underset{1}{\{\!\!\{} m\rho\phi \underset{1}{\}\!\!\}} \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \cup \langle\!\langle \rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \rangle\!\rangle \}\!\!\} \circ \psi .$ [3; *T1.6.8*]

$\psi\{ab\}$ [15; 16; *T1.1.31*]

T2.3.65 $[m\mu\rho\sigma\phi\Phi\psi\Gamma] \therefore \Gamma \circ \downarrow\langle_4 \rho\, \mathbf{T} \ddagger\phi\rho\rho\ddagger \rangle_4 . \Gamma \circ \downarrow\langle_4 \sigma\Phi\rangle_4 .$
$\downarrow\langle_4 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \mathbf{T} \ddagger \mathbf{E1} \ddagger_1 m\rho\phi \ddagger_1 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \ddagger \rangle_4 \circ \downarrow\langle_4 \sigma\Phi\rangle_4 . \supset :$
$[ab] : \sigma\{a\} . \mu\{b\} . \sim(\sigma\{b\}) . \supset . \psi\{ab\}$ [*T2.3.64*]

T2.3.66 $[m\rho\phi\Gamma] :: \Gamma \circ \downarrow\langle_4 \rho\, \mathbf{T} \ddagger\phi\rho\rho\ddagger \rangle_4 . \supset \therefore [\sigma\mu\Phi\psi] \therefore \Gamma \circ \downarrow\langle_4 \sigma\Phi\rangle_4 .$
$\downarrow\langle_4 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \mathbf{T} \ddagger \mathbf{E1} \ddagger_1 m\rho\phi \ddagger_1 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \ddagger \rangle_4 \circ \downarrow\langle_4 \mu\psi\rangle_4 . \supset :$
$\sigma \subset \mu . \Phi \subset \psi : [ab] : \sigma\{a\} . \mu\{b\} . \sim(\sigma\{b\}) . \supset . \psi\{ab\}$
[*T2.3.61*; *T2.3.63*; *T2.3.65*]

T2.3.67 $[m\rho\phi\Gamma] : \Gamma \circ \downarrow\langle_4 \rho\, \mathbf{T} \ddagger\phi\rho\rho\ddagger \rangle_4 . \supset .$
$\mathbf{R2}\{_2 \Gamma \downarrow\langle_4 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \mathbf{T} \ddagger \mathbf{E1} \ddagger_1 m\rho\phi \ddagger_1 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \ddagger \rangle_4 \}_2$
[*T2.3.66*; *D2.3.02*]

T2.3.68 $[\rho\theta] : \rho \subset \theta . \sim(\rho \circ \theta) . \sim([\exists m] . \theta\{m\} . \sim(\rho\{m\})) . \supset . [\exists m] . \theta\{m\} .$
$\sim(\rho\{m\})$

PR $[\rho\theta] \therefore \mathrm{Hp}(3) . \supset :$
4. $[m] : \theta\{m\} . \supset . \rho\{m\} :$ [3]
5. $\theta \subset \rho .$ [4; *D1.1.04*]
6. $\theta \circ \rho .$ [1; 5; *T1.1.15*]
$[\exists m] . \theta\{m\} . \sim(\rho\{m\})$ [2; 6; *T1.1.6*]

T2.3.69 $[\rho\theta] : \rho \subset \theta . \sim(\rho \circ \theta) . \supset . [\exists m] . \theta\{m\} . \sim(\rho\{m\})$ [*T2.3.68*]

T2.3.70 $[\theta\lambda\psi\Gamma] : \mathbf{MAXL} \ddagger_2 \mathbf{WOS}\langle_3 \theta\rangle_3 \mathbf{R2} \ddagger_2 \{_2 \Gamma\}_2 . \Gamma \circ \downarrow\langle_4 \lambda\psi\rangle_4 . \sim(\lambda \circ \theta) . \supset . \lambda \circ \theta$

PR $[\theta\lambda\psi\Gamma] :: \mathrm{Hp}(3) . \supset ::$
4. $\mathbf{WOS}\langle_3 \theta\rangle_3 \{_2 \Gamma\}_2 ::$ [1; *D1.6.011*]
$[\exists\rho\phi] ::$
5. $\mathbf{WO}\langle\rho\rangle \dagger\phi\dagger .$ [4; *D2.3.01*]
6. $\rho \subset \theta .$
7. $\Gamma \circ \downarrow\langle_4 \rho\, \mathbf{T} \ddagger\phi\rho\rho\ddagger \rangle_4 .$
8. $\lambda \circ \rho .$ [2; 7; *T1.6.12*]
9. $\sim(\rho \circ \theta) \therefore$ [3; 8; *T1.1.6*; *T1.1.9*]
$[\exists m] \therefore$
10. $\theta\{m\} .$ [6; 9; *T2.3.69*]
11. $\sim(\rho\{m\}) :$
12. $[\Delta] : \mathbf{WOS}\langle_3 \theta\rangle_3 \{_2 \Delta\}_2 . \mathbf{R2}\{_2 \Gamma\Delta\}_2 . \supset . \Gamma \circ \Delta :$ [1; *D1.6.011*]
13. $\mathbf{WOS}\langle_3 \theta\rangle_3 \{_2 \downarrow\langle \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \mathbf{T} \ddagger \mathbf{E1} \ddagger_1 m\rho\phi \ddagger_1 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1$
$\cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \ddagger \rangle_4 \}_2 .$ [5; 6; 10; 11; *T2.3.60*]
14. $\mathbf{R2}\{_2 \Gamma \downarrow\langle_4 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \mathbf{T} \ddagger \mathbf{E1} \ddagger_1 m\rho\phi \ddagger_1$
$\cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \ddagger \rangle_4 \}_2 .$ [7; *T2.3.67*]
15. $\Gamma \circ \downarrow\langle_4 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \mathbf{T} \ddagger \mathbf{E1} \ddagger_1 m\rho\phi \ddagger_1$
$\cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 \ddagger \rangle_4 .$ [12; 13; 14]
16. $\rho \circ \cup \lessdot_1 \rho \downarrow\langle m\rangle \gtrdot_1 .$ [7; 15; *T1.6.12*]

17.	$\cup \not\{\rho \downarrow \underset{1}{\langle} m \underset{1}{\rangle} \not\} \subset \rho\,.$	[16; *T1.1.15*]
18.	$\downarrow \underset{1}{\langle} m \underset{1}{\rangle} \subset \rho\,.$	[17; *T1.1.20*]
19.	$\rho\{m\}::$	[18; *T1.3.5*]
	$\lambda \circ \theta$	[11; 19]
T2.3.71	$[\theta\Gamma] \therefore \mathsf{MAXL} \underset{2}{\not\{} \mathsf{WOS} \underset{3}{\langle} \theta \underset{3}{\rangle} \mathsf{R2} \underset{2}{\not\}} \underset{2}{\{} \Gamma \underset{2}{\}} .\supset: [\lambda\psi] : \Gamma \circ \downarrow \underset{4}{\langle} \lambda\psi \underset{4}{\rangle} .\supset. \lambda \circ \theta$	[*T2.3.70*]
T2.3.72	$[\theta] \therefore [\theta\pi] : \mathsf{PO} \underset{2}{\langle} \theta \underset{2}{\rangle} \underset{2}{\not\not\{} \pi \underset{2}{\not\not\}} . ! \underset{2}{\not\{} \theta \underset{2}{\not\}} . \mathsf{W} \underset{2}{\langle} \theta \underset{2}{\rangle} \underset{2}{\not\{} \pi \underset{2}{\not\}} .\supset. [\exists\Gamma] . \mathsf{MAXL} \underset{2}{\not\{} \theta\pi \underset{2}{\not\}} \underset{1}{\{} \Gamma \underset{1}{\}} :$ $! \not\{\theta\not\} .\supset. [\exists\phi] . \mathsf{WO} \langle\theta\rangle \not\not\{\phi\not\not\}$	
PR	$[\theta] \therefore \mathrm{Hp}(2) .\supset:$	
3.	$\mathsf{PO} \underset{2}{\langle} \mathsf{WOS} \underset{3}{\langle} \theta \underset{3}{\rangle} \underset{2}{\rangle} \underset{2}{\not\not\{} \mathsf{R2} \underset{2}{\not\not\}} .$	[*T2.3.11*]
4.	$! \underset{2}{\not\{} \mathsf{WOS} \underset{3}{\langle} \theta \underset{3}{\rangle} \underset{2}{\not\}} .$	[2; *T2.3.59*]
5.	$\mathsf{W} \underset{2}{\langle} \mathsf{WOS} \underset{3}{\langle} \theta \underset{3}{\rangle} \underset{2}{\rangle} \underset{2}{\not\not\{} \mathsf{R2} \underset{2}{\not\not\}} :$	[*T2.3.58*]
	$[\exists\Gamma]:$	
6.	$\mathsf{MAXL} \underset{2}{\not\{} \mathsf{WOS} \underset{3}{\langle} \theta \underset{3}{\rangle} \mathsf{R2} \underset{2}{\not\}} \underset{2}{\{} \Gamma \underset{2}{\}} .$	[1; 3; 4; 5]
7.	$\mathsf{WOS} \underset{3}{\langle} \theta \underset{3}{\rangle} \underset{2}{\{} \Gamma \underset{2}{\}} .$	[6; *D1.6.011*]
	$[\exists\rho\phi] .$	
8.	$\mathsf{WO} \langle\rho\rangle \not\not\{\phi\not\not\} .$	[7; *D2.3.01*]
9.	$\Gamma \circ \downarrow \underset{4}{\langle} \rho\, \mathsf{T} \not\not\not\{\phi\rho\rho\not\not\not\} \underset{4}{\rangle} .$	[7; *D2.3.01*]
10.	$\theta \circ \rho .$	[6; 9; *T2.3.71*]
11.	$\mathsf{WO} \langle\theta\rangle \not\not\{\phi\not\not\} :$	[8; 10; *T1.2.12*]
	$[\exists\phi] . \mathsf{WO} \langle\theta\rangle \not\{\phi\not\}$	[11]

T2.3.72 establishes that every unempty set can be well ordered. We complete the work of this section by showing that every empty set can likewise be well ordered.

T2.3.73	$[\theta] : \sim (! \not\{\theta\not\}) .\supset. [a] . \sim (\theta\{a\})$	[*D1.1.08*]
T2.3.74	$[ap\theta\phi] : \sim (! \not\{\theta\not\}) . \theta\{a\} .\supset. p$	[*T2.3.73*]
T2.3.75	$[\theta] : \sim (! \not\{\theta\not\}) .\supset. [\phi] . \mathsf{R} \langle\theta\rangle \not\not\{\phi\not\not\}$	[*T2.3.74*; *D1.2.02*]
T2.3.76	$[\theta] : \sim (! \not\{\theta\not\}) .\supset. [\phi] . \mathbf{T} \langle\theta\rangle \not\not\{\phi\not\not\}$	[*T2.3.74*; *D1.2.03*]
T2.3.77	$[\theta] : \sim (! \not\{\theta\not\}) .\supset. [\phi] . \mathsf{A} \langle\theta\rangle \not\not\{\phi\not\not\}$	[*T2.3.74*; *D1.2.04*]
T2.3.78	$[\theta] : \sim (! \not\{\theta\not\}) .\supset. [\phi] . \mathsf{PO} \langle\theta\rangle \not\not\{\phi\not\not\}$	[*T2.3.75*; *T2.3.76*; *T2.3.77*; *D1.2.04*]
T2.3.79	$[\theta] : \sim (! \not\{\theta\not\}) .\supset: [\rho\phi] : \rho \subset \theta . ! \not\{\rho\not\} .\supset. [\exists a] . \mathsf{P} \not\{\theta\phi\not\} \{a\}$	[*T2.3.74*; *D1.1.08*; *T1.1.14*]
T2.3.80	$[\theta] : \sim (! \not\{\theta\not\}) .\supset. [\phi] . \mathsf{WO} \langle\theta\rangle \not\not\{\phi\not\not\}$	[*T2.3.78*; *T2.3.79*; *D1.2.07*]

The above establishes that an empty set is well ordered by any relation.

T2.3.81 $[\theta] : \sim (! \not\{\theta\not\}) .\supset. [\exists\phi] . \mathsf{WO} \langle\theta\rangle \not\not\{\phi\not\not\}$ [*T2.3.80*]

T2.3.81 together with *T2.3.72* yield:

T2.3.82 $[\theta\pi] \therefore \mathsf{PO} \underset{2}{\langle} \theta \underset{2}{\rangle} \underset{2}{\not\not\{} \pi \underset{2}{\not\not\}} . ! \underset{2}{\not\{} \theta \underset{2}{\not\}} . \mathsf{W} \underset{2}{\langle} \theta \underset{2}{\rangle} \underset{2}{\not\not\{} \pi \underset{2}{\not\not\}} .\supset. [\exists\Gamma] . \mathsf{MAXL} \underset{2}{\not\{} \theta\pi \underset{2}{\not\}} \underset{2}{\{} \Gamma \underset{2}{\}} :\supset$
$:[\theta] . [\exists\phi] . \mathsf{WO} \langle\theta\rangle \not\not\{\phi\not\not\}$ [*T2.3.72*; *T2.3.81*]

Hence, we have the result that **KZL*** yields **WO**.

2.4 *The Well Ordering Principle and the Generalized Principle of Choice* In this section we show that **WO** implies **ACF**. Again, for convenience we repeat both formulas below.

WO $[\theta]:[\exists\phi].\mathbf{WO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle$

ACF $[\xi]::\,!\underset{1}{\langle}\xi\underset{1}{\rangle}.\curlywedge\underset{1}{\langle}\xi\underset{1}{\rangle}\,.\supset\therefore[\exists\eta].\rightrightarrows\underset{1}{\langle\!\langle}\eta\underset{1}{\rangle\!\rangle}.\mathbf{D}\underset{1}{\langle\!\langle}\eta\underset{1}{\rangle\!\rangle}\circ\xi:[a\theta]:\eta\underset{1}{\{}\theta a\underset{1}{\}}\,.\supset.\,\theta\{a\}$

We begin the proof by defining a special relation.

D2.4.01 $[a\theta\xi\phi]:\xi\langle\theta\rangle.\mathbf{P}\{\theta\phi\}\{a\}\,.\equiv.\,\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\{\theta a\}$

The proof proceeds by showing that such a relation is the required choice relation.

T2.4.1 $[ab\theta\xi\phi]:\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\underset{1}{\{}\theta a\underset{1}{\}}.\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\underset{1}{\{}\theta b\underset{1}{\}}\,.\supset.\,a\circ b$

PR $[ab\theta\xi\phi]:\mathrm{Hp}(2)\,.\supset.$

3\. $\mathbf{P}\{\theta\phi\}\{a\}.$ [1; *D2.4.01*]

4\. $\mathbf{P}\{\theta\phi\}\{b\}.$ [2; *D2.4.01*]

$a\circ b$ [3; 4; *T1.2.11*]

T2.4.2 $[\xi\phi].\rightrightarrows\underset{1}{\langle\!\langle}\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\underset{1}{\rangle\!\rangle}$ [*T2.4.1*; *D1.5.09*]

T2.4.3 $[a\theta\xi\phi]:\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\underset{1}{\{}\theta a\underset{1}{\}}\,.\supset.\,\theta\{a\}$ [*D2.4.01*; *D1.2.06*]

T2.4.4 $[\theta\xi\phi]:\mathbf{D}\underset{1}{\langle\!\langle}\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\underset{1}{\rangle\!\rangle}\langle\theta\rangle\,.\supset.\,\xi\langle\theta\rangle$ [*D2.4.01*; *D1.5.07*]

T2.4.5 $[\xi\phi]:\mathbf{D}\underset{1}{\langle\!\langle}\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\underset{1}{\rangle\!\rangle}\subset\xi$ [*T2.4.4*; *D1.5.04*]

T2.4.6 $[\theta\xi\phi]:\mathbf{WO}\langle\cup\langle\!\langle\xi\rangle\!\rangle\rangle\langle\!\langle\phi\rangle\!\rangle.\curlywedge\underset{1}{\langle}\xi\underset{1}{\rangle}.\xi\langle\theta\rangle\,.\supset.\,\mathbf{D}\underset{1}{\langle\!\langle}\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\underset{1}{\rangle\!\rangle}\langle\theta\rangle$

PR $[\theta\xi\phi]:\mathrm{Hp}(3)\,.\supset.$

4\. $!\langle\theta\rangle$ [2; 3; *D1.5.02*]

5\. $\theta\subset\cup\langle\!\langle\xi\rangle\!\rangle.$ [3; *T1.1.24*]

$[\exists a].$

6\. $\mathbf{P}\{\theta\phi\}\{a\}.$ [1; 4; 5; *D1.2.07*]

7\. $\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\underset{1}{\{}\theta a\underset{1}{\}}.$ [3; 6; *D2.4.01*]

$\mathbf{D}\underset{1}{\langle\!\langle}\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\underset{1}{\rangle\!\rangle}\langle\theta\rangle$ [7; *D1.5.07*]

T2.4.7 $[\xi\phi]:\mathbf{WO}\langle\cup\langle\!\langle\xi\rangle\!\rangle\rangle\langle\!\langle\phi\rangle\!\rangle.\curlywedge\underset{1}{\langle}\xi\underset{1}{\rangle}\,.\supset.\,\xi\subset\mathbf{D}\underset{1}{\langle\!\langle}\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\underset{1}{\rangle\!\rangle}$ [*T2.4.6*; *D1.5.04*]

T2.4.8 $[\xi\phi]:\mathbf{WO}\langle\cup\langle\!\langle\xi\rangle\!\rangle\rangle\langle\!\langle\phi\rangle\!\rangle.\curlywedge\underset{1}{\langle}\xi\underset{1}{\rangle}\,.\supset.\,\xi\circ\mathbf{D}\underset{1}{\langle\!\langle}\mathbf{E2}\underset{3}{\{\!|}\xi\phi\underset{3}{|\!\}}\underset{1}{\rangle\!\rangle}$ [*T2.4.5*; *T2.4.7*; *T1.5.3*]

T2.4.9 $[\theta\xi\phi]\therefore\mathbf{WO}\langle\cup\langle\!\langle\xi\rangle\!\rangle\rangle\langle\!\langle\phi\rangle\!\rangle.!\underset{1}{\langle}\xi\underset{1}{\rangle}.\curlywedge\underset{1}{\langle}\xi\underset{1}{\rangle}\,.\supset.\,[\exists\eta].\rightrightarrows\underset{1}{\langle\!\langle}\eta\underset{1}{\rangle\!\rangle}.$
$\mathbf{D}\underset{1}{\langle\!\langle}\eta\underset{1}{\rangle\!\rangle}\circ\xi:[\theta a]:\eta\underset{1}{\{}\theta a\underset{1}{\}}\,.\supset.\,\theta\{a\}$ [*T2.4.2*; *T2.4.8*; *T2.4.3*]

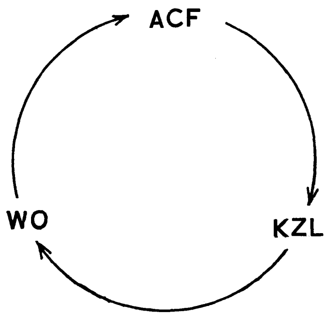
T2.4.10 $[\xi]\therefore[\theta]:[\exists\phi].\mathbf{WO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle.!\underset{1}{\langle}\xi\underset{1}{\rangle}.\curlywedge\underset{1}{\langle}\xi\underset{1}{\rangle}\,.\supset:[\exists\eta].$
$\rightrightarrows\underset{1}{\langle\!\langle}\eta\underset{1}{\rangle\!\rangle}.\mathbf{D}\underset{1}{\langle\!\langle}\eta\underset{1}{\rangle\!\rangle}\circ\xi:[\theta a]:\eta\underset{1}{\{}\theta a\underset{1}{\}}.\eta\underset{1}{\{}\theta a\underset{1}{\}}\,.\supset.\,\theta\{a\}$ [*T2.4.9*]

T2.4.11 $[\theta]:[\exists\phi].\mathbf{WO}\langle\theta\rangle\langle\!\langle\phi\rangle\!\rangle\,:\supset::[\xi]\therefore\,!\underset{1}{\langle}\xi\underset{1}{\rangle}.\curlywedge\underset{1}{\langle}\xi\underset{1}{\rangle}\,.\supset:$
$[\exists\eta].\rightrightarrows\underset{1}{\langle\!\langle}\eta\underset{1}{\rangle\!\rangle}.\mathbf{D}\underset{1}{\langle\!\langle}\eta\underset{1}{\rangle\!\rangle}\circ\xi:[\theta a]:\eta\underset{1}{\{}\theta a\underset{1}{\}}\,.\supset.\,\theta\{a\}$ [*T2.4.10*]

Hence, **WO** yields **ACF**.

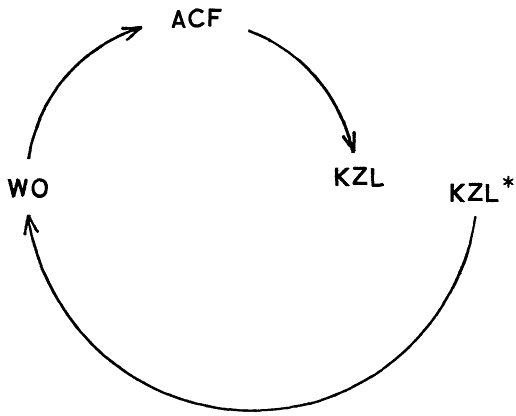
2.5 *Rule Equivalence* Given the results of the preceding sections, the rule equivalence within Ontology of the Axiom of Choice, the Kuratowski-Zorn Lemma, and the Well Ordering Principle is easily established. Rule equivalence is a direct consequence of the preceding results plus the metatheorem mentioned in section **1.0** regarding the reproducibility of results of any lower semantic category.

To make this more explicit, assume for the moment that we had been able to show that **ACF** implies **KZL**, **KZL** implies **WO**, and **WO** implies **ACF**. This would have established a chain of proofs which could be represented by the following diagram.



This situation would have been sufficient to establish the direct equivalence of any one of the three formulas to another. Given that for any theorem established for some semantic category there are direct analogies of that theorem for each higher semantic category, rule equivalence follows trivially, for we are assured that if, e.g., the Axiom of Choice is stated for some higher semantic category we can derive the Kuratowski-Zorn Lemma for that higher category from it and so forth. Thus, a rule allowing the addition of the Axiom of Choice for each semantic category would have the same effect as a rule allowing the addition of the Kuratowski-Zorn Lemma for each semantic category.

The fact that we were not able to prove the direct equivalence of **ACF** and **KZL** complicates the actual situation somewhat but does not essentially change it. The actual situation regarding the implication relationships shown in this chapter can be represented by the diagram on the following page. There we have a kind of spiral rather than a closed figure. Now consider modifying Ontology by the addition of a rule permitting the addition of the Axiom of Choice for each semantic category. If we add to the system the Axiom of Choice stated for semantic category S/n, i.e., **ACF**, then we can derive the Kuratowski-Zorn Lemma as stated for semantic category S/n, i.e., **KZL**, but we cannot at this stage derive the Well Ordering Principle stated for semantic category S/n, i.e., **WO**. However, the metatheorem mentioned assures us we can reproduce the derivation of **KZL** from **ACF** for any higher semantic category. Hence, if we introduce the Axiom of Choice as stated for semantic category $S/(S/(S/n)(S/nn))$, call this **ACF***, then we can derive **KZL*** and from **KZL*** derive **WO**. Thus, the rule equivalence of the three is established.



CHAPTER III: THE RULE OF ONTOLOGY AND THE AXIOM OF CHOICE

3.0 *Introduction* The results of the preceding chapter make clear that in order to extend Ontology so that it includes the Axiom of Choice, a modification to the Rule of Ontology is required rather than the addition of a single new axiom. Since the Rule of Ontology consists of a set of seven interrelated directives, the required modification will take the form of an additional directive which will permit the addition of formulas expressing the Axiom of Choice or some other equivalent principle for every semantic category. In this chapter we formulate the needed directive.

Three features of our formulation need explanation. In the first place, the formalization of any deductive theory requires that each of the rules or directives of that theory be stated in purely syntactic terms. These terms are not themselves part of the object language of the theory but are metalinguistic terms which refer to elements of the object language. As such they need clarification and explanation themselves. In order to provide this, we shall make use of the work of V. F. Rickey who in [21] presented two systems, **M** and **MP**, which use the techniques of formal axiomatic systems to define syntactical terms. We shall adopt Rickey's system **M** and extend it to a system which we call **MO**. System **MO** will contain the syntactical terms necessary for formulating the additional directive for the Rule of Ontology as described above. Second, though a modified Peano-Russell system of notation has been used in the preceding two chapters, the terms of **MO** will be descriptive of a system of Ontology which uses Leśniewski's original symbolism. (See note 4.) Hence the modification of Ontology's Rule presented below presuppose Leśniewski's original symbolism.

Finally, the directive we formulate does not make explicit provisions for adding to Ontology any of the formulas investigated in Chapter Two. Instead we capitalize on the results of C. C. Davis. In [9] Davis showed that an elegantly simple formula which he discovered is rule equivalent in the field of Ontology to more familiar forms of the Axiom of Choice including the Generalized Choice Principle (**ACF**) investigated in Chapter Two. For the semantic category S/n, Davis' formula is:

$$[\exists f] \therefore [a\theta] : \theta\{a\} \;.\supset.\; \theta\{f\langle\theta\rangle\}$$

In Leśniewski's symbolism this formula (referred to hereafter as **AC***) is:

$$\vdash(\llcorner f \lrcorner \ulcorner \vdash (\llcorner a\theta \lrcorner \ulcorner \varphi(\theta\{a\}\ \theta\{f\langle\theta\rangle\})\urcorner)\urcorner)$$

The syntactic simplicity of **AC*** recommends itself and so the directive we present will insure that formulas similar to **AC*** are available for each semantic category. Since adding formulas similar to **AC*** is rule equivalent to adding formulas similar to **ACF**, the goal of extending Ontology with the Axiom of Choice will have been achieved.

3.1 *Axiomatic Inscriptional Syntax* In this section we present the essential features of Rickey's system for general syntax, system **M**, and extend it to system **MO**. System **MO** will contain all the syntactical terms necessary for formulating the directive which incorporates the Axiom of Choice into Ontology. It is not our intent to provide a detailed description of Rickey's system, for complete details can be found in [22] and [23]. What follows presupposes some familiarity with these articles.

System **M** is a system of axiomatic inscriptional syntax which employs formal axiomatic techniques to define terms required for describing the syntax of other formal systems—hence the name axiomatic syntax. The qualification "inscriptional" is intended to indicate that the syntactic terms defined in system **M** refer to actual tokens rather than abstract types. System **M** is constructed by extending a logical base through the addition of new primitive terms and a set of proper axioms which implicitly define them. Other syntactical terms are then explicitly defined in terms of the primitives. In system **M** the extensions or denotations of syntactical terms are delimited axiomatically. Only those inscriptions which satisfy or provide a model for system **M** are elements of the extensions of those terms.

As its purely logical base, system **M** uses Leśniewski's Ontology and presupposes some suitable single axiom and a number of definitions.[12] These are listed below. In all of what follows it is also presupposed that the range of nominal variables is restricted to names of inscriptions. An inscription is understood to be a (not necessarily sequential) collection of one or more printed or written tokens.

Ax0 $[Aa] \therefore A \,\varepsilon\, a \;.\equiv:\; [\exists B] . B \,\varepsilon\, A : [BC] : B \,\varepsilon\, A \,.\, C \,\varepsilon\, A \;.\supset.\; B \,\varepsilon\, C : [B] : B \,\varepsilon\, A \;.\supset.\; B \,\varepsilon\, a$

D01 $[a] : !\{a\} \;.\equiv.\; [\exists A] . A \,\varepsilon\, a$

a is *unempty*.[13]

D02 $[AB] : A = B \;.\equiv.\; A \,\varepsilon\, B \,.\, B \,\varepsilon\, A$

A is *identical* with B.

D03 $[ab] \therefore a \subset b \;.\!\equiv\!: [A] : A \,\varepsilon\, a \;.\!\supset\!.\; A \,\varepsilon\, b$

a is *contained* in b.

D04 $[ab] \therefore a \circ b \;.\!\equiv\!: [A] : A \,\varepsilon\, a \;.\!\equiv\!.\; A \,\varepsilon\, b$

a equals b.

D05 $[Aab] \therefore A \,\varepsilon\, a \cup b \;.\!\equiv\!: A \,\varepsilon A : A \,\varepsilon\, a \,.\!\vee\!.\, A \,\varepsilon\, b$

A is *a or b.*

D06 $[Aab] \therefore A \,\varepsilon\, a \cap b \;.\!\equiv\!: A \,\varepsilon\, a \,.\, A \,\varepsilon\, b$

A is *a and b.*

D07 $[Aa] : A \,\varepsilon \sim\!\![a] \;.\!\equiv\!.\; A \,\varepsilon A \,.\, \sim(A \,\varepsilon\, a)$

A is *non-a*.

D08 $[\phi] \therefore \rightleftarrows \nleftarrow\!\phi\!\nrightarrow \;.\!\equiv\!: [abc] : \phi\{ab\} \,.\, \phi\{ac\} \;.\!\supset\!.\; b \circ c : [abc] : \phi\{ab\} \,.\, \phi\{cb\} \,.\!\supset\!.$ $a \circ c$

ϕ is a *1-1 binary connection.*

D09 $[ab] :: a \infty b \;.\!\equiv\!\therefore [\exists\phi] \therefore \rightleftarrows \nleftarrow\!\phi\!\nrightarrow : [A] : A \,\varepsilon\, a \;.\!\equiv\!.\; [\exists B] \,.\, \phi\{AB\} \,.\, B \,\varepsilon\, b :$ $[A] : A \,\varepsilon\, b \;.\!\equiv\!.\; [\exists B] \,.\, \phi\{BA\} \,.\, B \,\varepsilon\, a$

a is *equinumerous* with b.

D010 $[ab] \therefore a \prec b \;.\!\equiv\!: [\exists c] \,.\, c \subset b \,.\, c \infty a : \sim(a \infty b)$

a is less numerous than b.

D011 $[ab] \therefore a \precsim b \;.\!\equiv\!: a \prec b \,.\!\vee\!.\, a \infty b$

a is equi- or less-numerous than b.

D012 $[a\theta] \therefore \mathbf{Irr}\langle\theta\rangle\{a\} \;.\!\equiv\!: \theta\{a\} : [b] : b \subset a \,.\, \theta\{b\} \;.\!\supset\!.\; a \circ b$

a is *minimal with respect* to θ.

D013 $[a] :: \mathbf{Fin}\{a\} \equiv\therefore [\theta b] \therefore \theta\{b\} : [d] : \theta\{d\} \;.\!\supset\!.\; d \subset a :\!\supset\!.\; [\exists c] \,.\, \mathbf{Irr}\langle\theta\rangle\{c\}$

a is finite.

System **M** itself has three primitive terms:

$A \,\varepsilon\, \mathbf{vrb}\ (B)$	A is *a word in B*
$A \,\varepsilon\, \mathbf{pr}\ (B)$	A is *a word preceding B*
$A \,\varepsilon\, \mathbf{cnf}\ (B)$	A is *equiform* with B

These primitive terms are implicitly defined by system **M**'s eight proper axioms.

A1 $[AB] \therefore A = B \;.\!\equiv\!: [\exists C] \,.\, C \,\varepsilon\, \mathbf{vrb}(A) : [C] : C \,\varepsilon\, \mathbf{vrb}(A) \;.\!\equiv\!.\; C \,\varepsilon\, \mathbf{vrb}(B)$

A2 $[AB] \therefore A \,\varepsilon\, \mathbf{vrb}(B) \;.\!\equiv\!: A \,\varepsilon A : [C] : C \,\varepsilon\, \mathbf{vrb}(A) \;.\!\supset\!.\; C = A :$ $[\exists C] \,.\, C \,\varepsilon\, \mathbf{vrb}(B) \,.\, \sim(A \,\varepsilon\, \mathbf{pr}(C)) \,.\, \sim(C \,\varepsilon\, \mathbf{pr}(A))$

A3 $[AB] : A \,\varepsilon\, \mathbf{pr}(B) \;.\!\supset\!.\; A \,\varepsilon\, \mathbf{vrb}(A)$

A4 $[AB]:A\,\varepsilon\,\mathbf{pr}(B)\ .\supset.\ B\,\varepsilon\,\mathbf{vrb}(B)$

A5 $[ABC]:A\,\varepsilon\,\mathbf{pr}(B)\,.\,B\,\varepsilon\,\mathbf{pr}(C)\ .\supset.\ A\,\varepsilon\,\mathbf{pr}(C)$

A6 $[AB]\therefore A\,\varepsilon\,\mathbf{cnf}(B)\ .\equiv:\ A\,\varepsilon A:[C]:B\,\varepsilon\,\mathbf{cnf}(C)\ .\equiv.\ A\,\varepsilon\,\mathbf{cnf}(C):$
$\mathbf{vrb}(A)\infty\,\mathbf{vrb}(B):[CD]:\sim(A=B)\,.\,C\,\varepsilon\,\mathbf{vrb}(A)\,.\,D\,\varepsilon\,\mathbf{vrb}(B)\,.$
$(\mathbf{vrb}(A)\cap\mathbf{pr}(C))\infty(\mathbf{vrb}(B)\cap\mathbf{pr}(D))\ .\supset.\ C\,\varepsilon\,\mathbf{cnf}(D)$

A7 $[A]:A\,\varepsilon A\ .\supset.\ \mathbf{Fin}\{\mathbf{vrb}(A)\}$

A8 $[Aa]:A\,\varepsilon\,a\ .\supset:\ [\exists B]:B\,\varepsilon B:[C]:C\,\varepsilon\,\mathbf{vrb}(B)\ .\equiv.\ [\exists D]\,.\,D\,\varepsilon\,a\,.$
$C\,\varepsilon\,\mathbf{vrb}(D)$

The preceding axiomatic base enables explicit definitions of various terms of general syntax, i.e., terms which would be used in describing the syntax of nearly any formal language. Given below are the definitions of those terms of general syntax that we will need together with a reading and, in some cases, explanatory comments.

D1 $[Aa]\therefore A\,\varepsilon\,\mathbf{Kl}(a)\ .\equiv:\ A\,\varepsilon A:[Ba]:\ B\,\varepsilon\,\mathbf{vrb}(A)\ .\equiv.\ [\exists C]\,.\,C\,\varepsilon\,a\,.\,B\,\varepsilon\,\mathbf{vrb}(C)$

A is the Klass of a. $\mathbf{Kl}(a)$ denotes an inscription which consists of all and only those words which are themselves words in the one or more inscriptions denoted by the general name 'a'.

D2 $[A]:A\,\varepsilon\,\mathbf{vrb}\ .\equiv.\ A\,\varepsilon\,\mathbf{vrb}(A)$

A is a word. Words are the smallest possible inscriptions.

D3 $[AB]\therefore A\,\varepsilon\,\mathbf{1vrb}(B)\ .\equiv:\ A\,\varepsilon\,\mathbf{vrb}(B):[C]:C\,\varepsilon\,\mathbf{vrb}(B)\ .\supset.\ \sim(C\,\varepsilon\,\mathbf{pr}(A))$

A is the first word in (inscription) B.

D4 $[AB]::A\,\varepsilon\,\mathbf{2vrb}(B)\ .\equiv\therefore\ A\,\varepsilon\,\mathbf{vrb}(B)\therefore[C]\therefore C\,\varepsilon\,\mathbf{vrb}(B)\ .\supset:$
$C\,\varepsilon\,\mathbf{pr}(A)\ .\equiv.\ C\,\varepsilon\,\mathbf{1vrb}(B)$

A is the second word in inscription B. In a manner similar to *D4* one can define $\mathbf{3vrb}(B)$, $\mathbf{4vrb}(B)$, ... We will have occasion to use some of these terms and will presume that the definitions have been given.

D5 $[AB]\therefore A\,\varepsilon\,\mathbf{Uvrb}(B)\ .\equiv:\ A\,\varepsilon\,\mathbf{vrb}(B):[C]:C\,\varepsilon\,\mathbf{vrb}(B)\ .\supset.\ \sim(A\,\varepsilon\,\mathbf{pr}(C))$

A is the last word in (inscription) B.

D6 $[A]\therefore A\,\varepsilon\,\mathbf{expr}\ .\equiv:\ A\,\varepsilon A:[BCD]:B\,\varepsilon\,\mathbf{vrb}(A)\,.\,D\,\varepsilon\,\mathbf{vrb}(A)\,.\,B\,\varepsilon\,\mathbf{pr}(C)\ .\supset.$
$C\,\varepsilon\,\mathbf{vrb}(A)$

A is an expression. While inscriptions could be formed from scattered (i.e., non-consecutive) individuals, expressions are consequitive strings of words.

D7 $[AB]\therefore A\,\varepsilon\,\mathbf{prcd}(B)\ .\equiv:\ A\,\varepsilon\,\mathbf{expr}\,.\,B\,\varepsilon\,\mathbf{expr}:[CD]:C\,\varepsilon\,\mathbf{vrb}(A)\,.$
$D\,\varepsilon\,\mathbf{vrb}(B)\ .\supset.\ C\,\varepsilon\,\mathbf{pr}(D)$

A is an expression which precedes expression B.

D8 $[AB]:A\,\varepsilon\,\mathbf{scd}(B)\ .\equiv.\ B\,\varepsilon\,\mathbf{prcd}(A)\,.\,A\,\varepsilon A$

A is an expression which follows expression B.

D9 $[AB] \therefore A \,\varepsilon\, \mathbf{ingr}(B) \;.{\equiv}:\; A \,\varepsilon\, \mathbf{expr} \,.\, B \,\varepsilon\, \mathbf{expr} :[C]: C \,\varepsilon\, \mathbf{vrb}(A) \;.{\supset}.\; C \,\varepsilon\, \mathbf{vrb}(B)$

A is an ingredient of B.

D10 $[a] \therefore \mathbf{disj}(a) \;.{\equiv}:\; [ABC] : A \,\varepsilon\, a \,.\, B \,\varepsilon\, a \,.\, C \,\varepsilon\, \mathbf{vrb}(A) \,.\, C \,\varepsilon\, \mathbf{vrb}(B) \;.{\supset}.\; A = B$

The inscriptions a are disjoint.

D11 $[Aa] : A \,\varepsilon\, \mathbf{Cmpl}(a) \;.{\equiv}.\; A \,\varepsilon\, \mathbf{Kl}(a) \,.\, A \,\varepsilon\, \mathbf{expr} \,.\, a \subset \mathbf{expr} \,.\, \mathbf{disj}(a)$

A is a complex of a.

D12 $[AB] : A \,\varepsilon\, \mathbf{int}(B) \;.{\equiv}.\; A \,\varepsilon\, \mathbf{vrb}(B) \,.\, B \,\varepsilon\, \mathbf{expr} \,.\, A \,\varepsilon \sim\!\left[\mathbf{1vrb}(B)\right] \,.\, A \,\varepsilon \sim\!\left[\mathbf{Uvrb}(B)\right]$

A is a word interior to expression B.

D13 $[AB] : A \,\varepsilon\, \mathbf{Int}(B) \;.{\equiv}.\; A \,\varepsilon\, \mathbf{Kl}\,(\mathbf{int}(B))$

A is the interior of expression B.

D14 $[A] : A \,\varepsilon\, \mathbf{expr\text{-}w\text{-}int} \;.{\equiv}.\; A \,\varepsilon\, A \,.\, [\exists B] \,.\, B \,\varepsilon\, \mathbf{int}(A)$

A is an expression with interior.

D15 $[ABC] : A \,\varepsilon\, \mathbf{Concat}(BC) \;.{\equiv}.\; B \,\varepsilon\, \mathbf{prcd}(C) \,.\, A \,\varepsilon\, \mathbf{Cmpl}(B \cup C)$

A is the concatenation of B and C.

D16 $[AB] : A \,\varepsilon\, \mathbf{hd}(B) \;.{\equiv}.\; A \,\varepsilon\, A \,.\, [\exists C] \,.\, B \,\varepsilon\, \mathbf{Concat}(AC)$

A is an initial segment (head) of B.

D17 $[AB] : A \,\varepsilon\, \mathbf{tl}\,(B) \;.{\equiv}.\; A \,\varepsilon\, A \,.\, [\exists C] \,.\, B \,\varepsilon\, \mathbf{Concat}(CA)$

A is a terminal segment (tail) of B.

D18 $[A] \therefore A \,\varepsilon\, \mathbf{non\text{-}rep} \;.{\equiv}:\; A \,\varepsilon\, A :[BC]: B \,\varepsilon\, \mathbf{vrb}(A) \,.\, C \,\varepsilon\, \mathbf{vrb}(A) \,.\, B \,\varepsilon\, \mathbf{cnf}(C) \;.{\supset}.\; B = C$

A is non-repeating. A is an expression no two distinct words of which are equiform.

D19 $[ABC] : A \,\varepsilon\, \mathbf{Mtch}(BC) \;.{\equiv}.\; A \,\varepsilon\, A \,.\, (\mathbf{vrb}(A) \cap \mathbf{cnf}(B)) \prec (\mathbf{vrb}(A) \cap \mathbf{cnf}(C))$

Inscription A contains more occurrences of words equiform with C than of words equiform with B.

D20 $[AB] \therefore A \,\varepsilon\, \mathbf{Uprcd}(B) \;.{\equiv}:\; A \,\varepsilon\, \mathbf{vrb} :[C]: A \,\varepsilon\, \mathbf{pr}(C) \;.{\supset}.\; \sim (C \,\varepsilon\, \mathbf{prcd}(B))$

A is the last word before B.

Along with the definitions of general syntax given above, our formulation of the directive allowing formulas similar to **AC*** to be added to Ontology will require definition of a number of terms more specifically related to the syntax of Ontology. These definitions are called "terminological explanations" (*TE*'s) and are developed within a system which we designate **MO**. **MO** is formed from system **M** by adding three new primitive terms and one new axiom.[14]

The first of these new primitives is:

$$A \ \varepsilon\, \mathrm{prntsym}(B)$$

read: A is a parenthesis symmetric to B. This term enables us to indicate matching parentheses and is needed because Ontology employs variously shaped parentheses to enclose arguments and to indicate the semantic categories of functors. The axiom that implicitly defines this term is:

$$MO1 \qquad [AB] \therefore A \ \varepsilon\, \mathrm{prntsym}(B) \ .\!\equiv\!.\ A \ \varepsilon\, \mathbf{vrb} \ .\ B \ \varepsilon\, \mathrm{prntsym}(A) \ .\sim (A \ \varepsilon\, \mathbf{cnf}(B)) : [C] : B \ \varepsilon\, \mathrm{prntsym}(C) \ .\!\supset\!.\ A \ \varepsilon\, \mathbf{cnf}(C)$$

The second primitive term we add to system **M** is:

$$A \ \varepsilon\, \mathrm{efthp}$$

read: A is an effective thesis of Prothothetic. 'efthp' is a general name for all and only those theses which actually constitute the Protothetical base on which the system of Ontology being considered is developed. As we mentioned in section **1.0**, Ontology presupposes some Protothetical base. Minimally, a single axiom for Protothetic is required, but it would also be possible to take an already extensively developed system of Protothetic. Since the semantic categories already present affect the way a system of Ontology can be developed and since the thesis or theses of a given Protothetical base will contain only certain semantic categories, some way of referring to those theses must be available. This is provided by 'efthp'.

It should be noted that 'efthp' need not be taken as primitive for in any given circumstance it would be possible to explicitly define it. For example, suppose the (finite) series

$$TP1,\ TP2,\ TP3,\ \ldots,\ TPn$$

contains the names of all those theses which form the Protothetical base for some system of Ontology. Under these circumstances, we could provide the following explicit definition of 'efthp':

$$D\alpha \qquad [A] \therefore A \ \varepsilon\, \mathrm{efthp} \ .\!\equiv\!:\ A = TP1 \ .\mathrm{v}.\ A = TP2 \ .\mathrm{v}.\ \ldots\ .\mathrm{v}.\ A = TPn$$

However, since for generality we are not presupposing any specific Protothetical base, we include 'efthp' here as a primitive term.

Finally, for reasons similar to those outlined in our comments on 'efthp', we need a term which will enable us to refer to theses which are part of the system of Ontology being developed but which are not part of that system's Protothetical base. For this we adopt:

$$A \ \varepsilon\, \mathrm{tho}$$

read: A is already a thesis of the system of Ontology under consideration. 'tho' is a general name whose extension contains Ontology's proper axiom and all theses of the system which have been added through an application of Ontology's Rule. It must be understood, however, that it is a "growing" name. At any one point in the development of the system it would be possible to explicitly indicate the extension of this term by using a technique similar to the one discussed with reference to 'efthp'. But whereas 'efthp' would have a fixed denotation that of 'tho' increases with each application of Ontology's Rule. Since the extension of 'tho' does increase, it is impossible to explicitly define it or to provide an axiomatic specification—at this stage—of its denotation.

System **MO** is to be understood as consisting of system **M** extended with the three primitive terms and axiom *MO1* given above. Besides using the primitive terms just discussed, the terminological explanations will also make explicit reference to the axioms of Protothetic and Ontology. They are given below (denoted $\mathfrak{A}\mathfrak{p}$ and $\mathfrak{A}\mathfrak{o}$ respectively).

$\mathfrak{A}\mathfrak{p}$ $\quad \llcorner pq \lrcorner \ulcorner \phi(\phi(pq) \llcorner f \lrcorner \ulcorner \phi(f(pf(p \llcorner u \lrcorner \ulcorner u \urcorner)) \llcorner r \lrcorner \ulcorner \phi(f(qr)\, \phi(qp))\urcorner)\urcorner)\urcorner$

$\mathfrak{A}\mathfrak{o}$ $\quad \llcorner Aa \lrcorner \ulcorner \phi(\varepsilon\{Aa\} \text{ϙ}(\vdash(\llcorner B \lrcorner \ulcorner \vdash(\varepsilon\{BA\})\urcorner) \text{ϙ}(\llcorner BC \lrcorner \ulcorner \phi\text{-}(\text{ϙ}(\varepsilon\{BA\} \varepsilon\{CA\})\, \varepsilon\{BC\})\urcorner$
$\quad \llcorner B \lrcorner \ulcorner \phi\text{-}(\varepsilon\{BA\} \varepsilon\{Ba\})\urcorner))\urcorner$

It should be clearly understood that $\mathfrak{A}\mathfrak{p}$ and $\mathfrak{A}\mathfrak{o}$ are the *names* of these two formulas and that the appearance in the terminological explanations of systems equiform to $\mathfrak{A}\mathfrak{p}$ and $\mathfrak{A}\mathfrak{o}$ is an explicit reference to the appropriate formula above.

We now give the necessary terminological explanations together with a reading and, in some cases, explanatory comments.[15]

TE1 $\quad [A] : A \,\varepsilon\, \mathsf{prnt} \;.\!\equiv\!:\; [\exists B] \,.\, A \,\varepsilon\, \mathsf{prntsym}(B) : A \,\varepsilon\, A$

A is a parenthesis.

TE2 $\quad [A] : A \,\varepsilon\, \mathsf{trm} \;.\!\equiv\!.\; A \,\varepsilon\, \mathbf{\mathsf{vrb}} \,.\, A \,\varepsilon \sim [\mathsf{prnt}] \,.\, A \,\varepsilon \sim [\mathbf{\mathsf{cnf(1vrb}}(\mathfrak{A}\mathfrak{p}))] \,.$
$\quad A \,\varepsilon \sim [\mathbf{\mathsf{cnf(4vrb}}(\mathfrak{A}\mathfrak{p}))] \,.\, A \,\varepsilon \sim [\mathbf{\mathsf{cnf(5vrb}}(\mathfrak{A}\mathfrak{p}))] \,.$
$\quad A \,\varepsilon \sim [\mathbf{\mathsf{cnf(Uvrb}}(\mathfrak{A}\mathfrak{p}))]$

A is a term. A term is any word other than a parenthesis or corner bracket.

TE3 $\quad [A] : A \,\varepsilon\, \mathsf{qntf} \;.\!\equiv\!.\; \mathbf{\mathsf{1vrb}}(A) \,\varepsilon\, \mathbf{\mathsf{cnf(1vrb}}(\mathfrak{A}\mathfrak{p})) \,.\, \mathbf{\mathsf{Uvrb}}(A) \,\varepsilon\, \mathbf{\mathsf{cnf(4vrb}}(\mathfrak{A}\mathfrak{p})) \,.$
$\quad \mathbf{\mathsf{Int}}(A) \,\varepsilon\, \mathbf{\mathsf{Cmpl}}(\mathsf{trm} \cap \mathbf{\mathsf{int}}(A)) \,.\, A \,\varepsilon\, \mathbf{\mathsf{non\text{-}rep}}$

A is a quantifier.

TE4 $\quad [A] \therefore A \,\varepsilon\, \mathsf{sbqntf} \;.\!\equiv\!:\; A \,\varepsilon\, \mathbf{\mathsf{expr\text{-}w\text{-}int}} : [C] : C \,\varepsilon\, \mathbf{\mathsf{hd}}(A) \;.\!\supset\!.$
$\quad C \,\varepsilon\, \mathbf{\mathsf{Mtch(Uvrb}}(\mathfrak{A}\mathfrak{p})\ \mathbf{\mathsf{5vrb}}(\mathfrak{A}\mathfrak{p})) : [C] : C \,\varepsilon\, \mathbf{\mathsf{tl}}(A) \;.\!\supset\!.$
$\quad C \,\varepsilon\, \mathbf{\mathsf{Mtch(5vrb}}(\mathfrak{A}\mathfrak{p})\ \mathbf{\mathsf{Uvrb}}(\mathfrak{A}\mathfrak{p}))$

A is a subquantifier.

TE5 $\quad [ABC] : A \,\varepsilon\, \mathsf{Gnrl}(BC) \;.\!\equiv\!.\; B \,\varepsilon\, \mathsf{qntf} \,.\, C \,\varepsilon\, \mathsf{sbqntf} \,.\, A \,\varepsilon\, \mathbf{\mathsf{Concat}}(BC)$

A is a generalization with quantifier *B* and subquantifier *C*.

TE6 $\quad [AB] : A \,\varepsilon\, \mathsf{Qntf}(B) \;.\!\equiv\!.\; A \,\varepsilon\, A \,.\, [\exists C] \,.\, B \,\varepsilon\, \mathsf{Gnrl}(AC)$

A is the quantifier of generalization B.

TE7 $[AB] : A \,\varepsilon\, \mathsf{Sbqntf}(B) \,.{\equiv}.\, A \,\varepsilon\, A \,.\, [\exists C] \,.\, B \,\varepsilon\, \mathsf{Gnrl}(CA)$

A is the subquantifier of generalization B.

TE8 $[A] : A \,\varepsilon\, \mathsf{gnrl} \,.{\equiv}.\, A \,\varepsilon\, A \,.\, [\exists BC] \,.\, A \,\varepsilon\, \mathsf{Gnrl}(BC)$

A is a generalization.

TE9 $[AB] \therefore A \,\varepsilon\, \mathsf{Essnt}(B) \,.{\equiv}{:}\, A \,\varepsilon\, A : A \,\varepsilon\, \mathbf{Int}(\mathsf{Sbqntf}(B)) \,.\mathrm{v}.\, A = B \,.\, A \,\varepsilon\, \mathbf{expr} \,.$
$A \,\varepsilon\, \sim[\mathsf{gnrl}]$

A is the essence of B.

TE10 $[AB] : A \,\varepsilon\, \mathsf{bd}(B) \,.{\equiv}.\, A \,\varepsilon\, \mathbf{int}(\mathsf{Qntf}(B))$

A is a binder in B.

TE11 $[ABC] \therefore A \,\varepsilon\, \mathsf{var}(BC) \,.{\equiv}{:}\, B \,\varepsilon\, \mathsf{bd}(C) \,.\, A \,\varepsilon\, \mathbf{int}(\mathsf{Sbqntf}(C)) \,.\, A \,\varepsilon\, \mathbf{cnf}(B) :$
$[DE] : D \,\varepsilon\, \mathbf{ingr}(C) \,.\, E \,\varepsilon\, \mathsf{bd}(D) \,.\, A \,\varepsilon\, \mathbf{cnf}(E) \,.\, A \,\varepsilon\, \mathbf{vrb}(D) \,.{\supset}.\, D = C$

A is a variable bound by B in generalization C.

TE12 $[ABC] \therefore A \,\varepsilon\, \mathsf{cnvar}(BC) \,.{\equiv}{:}\, A \,\varepsilon\, A \,.\, [\exists D] \,.\, A \,\varepsilon\, \mathsf{var}(DC) \,.\, B \,\varepsilon\, \mathsf{var}(DC)$

A is an equiform variable with B in C.

TE13 $[AB] : A \,\varepsilon\, \mathsf{var}(B) \,.{\equiv}.\, A \,\varepsilon\, \mathsf{cnvar}(AB)$

A is a variable in the generalization B.

TE14 $[A] \therefore A \,\varepsilon\, \mathsf{prntm} \,.{\equiv}{:}\, A \,\varepsilon\, \mathbf{expr\text{-}w\text{-}int} : [C] : C \,\varepsilon\, \mathbf{hd}(A) \,.{\supset}.$
$C \,\varepsilon\, \mathbf{Mtch}(\mathbf{Uvrb}(A)\ \mathbf{1vrb}(A)) : [C] : C \,\varepsilon\, \mathbf{tl}(A) \,.{\supset}.$
$C \,\varepsilon\, \mathbf{Mtch}(\mathbf{1vrb}(A)\ \mathbf{Uvrb}(A)) : \mathbf{1vrb}(A) \,\varepsilon\, \mathsf{prntsym}(\mathbf{Uvrb}(A))$

A is a parentheme.

TE15 $[Aa] \therefore A \,\varepsilon\, \mathsf{Fnct}(a) \,.{\equiv}{:}\, \mathbf{1vrb}(A) \,\varepsilon\, \mathsf{trm} : [C] : C \,\varepsilon\, a \,.{\supset}.\, C \,\varepsilon\, \mathsf{prntm} :$
$A \,\varepsilon\, \mathbf{Cmpl}(\mathbf{1vrb}(A) \cup a)$

A is a function with parenthemes a.

TE16 $[A] : A \,\varepsilon\, \mathsf{fnct} \,.{\equiv}.\, A \,\varepsilon\, A \,.\, [\exists a] \,.\, A \,\varepsilon\, \mathsf{Fnct}(a) \,.\, !\{a\}$

A is a function.

TE17 $[AB] : A \,\varepsilon\, \mathsf{prntm}(B) \,.{\equiv}.\, A \,\varepsilon\, A \,.\, [\exists a] \,.\, B \,\varepsilon\, \mathsf{Fnct}(a) \,.\, A \,\varepsilon\, a$

A is a parentheme of the function B.

TE18 $[Aa] \therefore A \,\varepsilon\, \mathsf{P\text{-}arg}(a) \,.{\equiv}{:}\, A \,\varepsilon\, \mathsf{prntm} : [C] : C \,\varepsilon\, a \,.{\supset}.$
$C \,\varepsilon\, \mathsf{trm} \cup \mathsf{gnrl} \cup \mathsf{fnct} : \mathbf{Int}(A) \,\varepsilon\, \mathbf{Cmpl}(a)$

A is a parentheme with arguments a.

TE19 $[AB] : A \,\varepsilon\, \mathsf{arg}(B) \,.{\equiv}.\, A \,\varepsilon\, A \,.\, [\exists a] \,.\, B \,\varepsilon\, \mathsf{P\text{-}arg}(a) \,.\, A \,\varepsilon\, a$

A is an argument of parentheme B.

TE20 $[AB] : A\, \varepsilon\, \mathrm{simprntm}(B)\ .\equiv.\ A\, \varepsilon\, \mathrm{prntm}\, .\, B\, \varepsilon\, \mathrm{prntm}\, .$
$\mathbf{1vrb}(A)\, \varepsilon\, \mathbf{cnf}(\mathbf{1vrb}(B))\, .\, \mathrm{arg}(A) \propto \mathrm{arg}(B)$

A is a parentheme similar to B.

TE21 $[ABCD] : A\, \varepsilon\, \mathrm{Anarg}(BCD)\ .\equiv.\ A\, \varepsilon\, \mathrm{arg}(C)\, .\, B\, \varepsilon\, \mathrm{arg}(D)\, .$
$(\mathrm{arg}(C) \cap \mathbf{prcd}(A)) \propto (\mathrm{arg}(D) \cap \mathbf{prcd}(B))\, .\, C\, \varepsilon\, \mathrm{simprntm}(D)$

A is an analogous argument to B, which are arguments of C and D respectively.

TE22 $[ABCDEF] : A\, \varepsilon\, \mathrm{Anfnct}(BCDEF)\ .\equiv.\ C\, \varepsilon\, \mathbf{Concat}(AE)\, .\, A\, \varepsilon\, A\, .$
$D\, \varepsilon\, \mathbf{Concat}(BF)\, .\, E\, \varepsilon\, \mathrm{simprntm}(F)\, .\, C\, \varepsilon\, \mathrm{fnct}\, .\, D\, \varepsilon\, \mathrm{fnct}$

TE23 $[ABCD] : A\, \varepsilon\, \mathrm{Anfnct}(BCD)\ .\equiv.\ A\, \varepsilon\, A\, .\, [\exists EF]\, .\, A\, \varepsilon\, \mathrm{Anfnct}(BCDEF)$

A in C is an analogous functor to B in D.

TE24 $[ABCD] \therefore A\, \varepsilon\, \mathrm{An}(BCD)\ .\equiv:\ A\, \varepsilon\, A : A\, \varepsilon\, \mathrm{Anarg}(BCD)\, .\mathrm{v}.\, A\, \varepsilon\, \mathrm{Anfnct}(BCD)$

A in C is analogous to B in D.

TE25 $[AB] : A\, \varepsilon\, \mathrm{1arg}(B)\ .\equiv.\ A\, \varepsilon\, \mathrm{Anarg}\, (\mathbf{10vrb}(\mathfrak{Ap})\ B\ \mathbf{Cmpl}(\mathbf{9vrb}(\mathfrak{Ap}) \cup$
$\mathbf{10vrb}(\mathfrak{Ap}) \cup \mathbf{11vrb}(\mathfrak{Ap}) \cup \mathbf{12vrb}(\mathfrak{Ap})))$

A is the first argument of parentheme B.

TE26 $[AB] : A\, \varepsilon\, \mathrm{2arg}(B)\ .\equiv.\ A\, \varepsilon\, \mathrm{Anarg}(\mathbf{11vrb}(\mathfrak{Ap})\ B\ \mathbf{Cmpl}(\mathbf{9vrb}(\mathfrak{Ap}) \cup$
$\mathbf{10vrb}(\mathfrak{Ap}) \cup \mathbf{11vrb}(\mathfrak{Ap}) \cup \mathbf{12vrb}(\mathfrak{Ap})))$

A is the second argument of parentheme B.

TE27 $[AB] \therefore A\, \varepsilon\, \mathrm{1eqvl}(B)\ .\equiv:\ \mathbf{1vrb}(B)\, \varepsilon\, \mathbf{cnf}(\mathbf{6vrb}(\mathfrak{Ap})) :$
$[\exists C]\, .\, A\, \varepsilon\, \mathrm{1arg}(C)\, .\, B\, \varepsilon\, \mathbf{Concat}(\mathbf{1vrb}(B)C) : A\, \varepsilon\, A$

A is the first equivalence of B.

TE28 $[AB] \therefore A\, \varepsilon\, \mathrm{2eqvl}(B)\ .\equiv:\ \mathbf{1vrb}(B)\, \varepsilon\, \mathbf{cnf}(\mathbf{6vrb}(\mathfrak{Ap})) :$
$[\exists C]\, .\, A\, \varepsilon\, \mathrm{2arg}(C)\, .\, B\, \varepsilon\, \mathbf{Concat}(\mathbf{1vrb}(B)C) : A\, \varepsilon\, A$

A is the second equivalence of B.

TE29 $[AB] : A\, \varepsilon\, \mathrm{1trm}(B)\ .\equiv.\ A\, \varepsilon\, \mathrm{1eqvl}(\mathbf{Essnt}(B))$

A is the first term of B.

TE30 $[AB] : A\, \varepsilon\, \mathrm{2trm}(B)\ .\equiv.\ A\, \varepsilon\, \mathrm{2eqvl}(\mathbf{Essnt}(B))$

A is the second term of B.

TE31 $[AB] :: A\, \varepsilon\, \mathrm{tho}(B)\ .\equiv\therefore\ A\, \varepsilon\, \mathrm{efthp}\, .\mathrm{v}.\, A\, \varepsilon\, \mathrm{tho} : A\, \varepsilon\, A : B\, \varepsilon\, \mathrm{tho} :$
$A\, \varepsilon\, \mathbf{prcd}(B)\, .\mathrm{v}.\ A = B$

A is a thesis of Ontology with respect to theses B. '$\mathrm{tho}(B)$' is a general name denoting all theses present in the system before the addition of B together with B itself.

TE32 $[AB] : A\, \varepsilon\, \mathrm{ingrtho}(B)\ .\equiv.\ A\, \varepsilon\, A\, .\, [\exists C]\, .\, C\, \varepsilon\, \mathrm{tho}(B)\, .\, A\, \varepsilon\, \mathbf{ingr}(C)$

A is an ingredient of a thesis of Ontology with respect to B.

TE33 $[AB] \therefore A \varepsilon \text{fro}(B) .\equiv: A \varepsilon A : A \varepsilon \text{tho}(B) .\text{v}. [\exists C] . C \varepsilon \text{ingrtho}(B) . C \varepsilon \text{sbqntf} . A \varepsilon \textbf{Int}(C) .\text{v}. [\exists C] . C \varepsilon \text{ingrtho}(B) . A \varepsilon \text{1arg}(C) .\text{v}. [\exists C] . C \varepsilon \text{ingrtho}(B) . A \varepsilon \text{2arg}(C)$

A is a propositional phrase with respect to B.

TE34 $[ABC] \therefore A \varepsilon \text{1homosemo}(BC) .\equiv: A \varepsilon A : A \varepsilon \text{fro}(C) . B \varepsilon \text{fro}(C) .\text{v}. [\exists D] . D \varepsilon \text{ingrtho}(C) . A \varepsilon \text{cnvar}(BD) .\text{v}. [\exists DE] . D \varepsilon \text{ingrtho}(C) . E \varepsilon \text{ingrtho}(C) . A \varepsilon \text{An}(BDE)$

A is a direct homoseme of B with respect to thesis C. This identifies A and B as being of the same semantic category.

TE35 $[ABC] :: A \varepsilon \text{homosemo}(BC) .\equiv\therefore A \varepsilon A \therefore B \varepsilon \text{1homosemo}(BC) \therefore [a] \therefore B \varepsilon a : [D] : D \varepsilon a .\supset. D \varepsilon \text{1homosemo}(DC) : [DE] : D \varepsilon a . E \varepsilon \text{1homosemo}(DC) .\supset. E \varepsilon a :\supset. A \varepsilon a$

A is of the same semantic category as B with respect to thesis C.

TE36 $[AB] \therefore A \varepsilon \text{fnctgen}(B) .\equiv: A \varepsilon \text{fnct} . \text{prntm}(A) \precsim \text{prntm}(B) : [CD] : C \varepsilon \text{prntm}(A) . D \varepsilon \text{prntm}(B) . (\text{prntm}(A) \cap \textbf{scd}(C)) \infty (\text{prntm}(B) \cap \textbf{scd}(D)) .\supset. C \varepsilon \text{simprntm}(D)$

A is a function generated by function B.

TE37 $[ABCDE] :: C \varepsilon \text{consto}(BADE) .\equiv\therefore D \varepsilon \text{homosemo}(EB) . C \varepsilon \textbf{cnf}(D) : [F] : F \varepsilon \text{ingrtho}(B) .\supset. D \varepsilon \sim\{\text{var}(F)\} : [\exists FG] . F \varepsilon \textbf{ingr}(A) . G \varepsilon \text{ingrtho}(B) . C \varepsilon \text{An}(EFG)$

TE38 $[ABC] : C \varepsilon \text{consto}(BA) .\equiv. C \varepsilon C . [\exists DE] . C \varepsilon \text{consto}(BADE)$

C in A is a constant of Ontology with respect to thesis B.

TE39 $[ABCDEF] : A \varepsilon \text{q-homosemo}(BCDEF) .\equiv. E \varepsilon \text{homosemo}(FC) . A \varepsilon A . [\exists GI] . G \varepsilon \textbf{ingr}(D) . I \varepsilon \text{ingrtho}(B) . A \varepsilon \text{An}(EGI) . [\exists GI] . G \varepsilon \textbf{ingr}(D) . I \varepsilon \text{ingrtho}(C) . B \varepsilon \text{An}(FGI)$

TE40 $[ABDF] : F \varepsilon \text{q-homosemo}(DBA) .\equiv. F \varepsilon F . [\exists GH] . F \varepsilon \text{q-homosemo}(DBAGH)$

F and D *would be* of the same semantic category if they were parts of theses of Ontology.

TE41 $[ABCDE] \therefore C \varepsilon \text{fncto}(BADE) .\equiv: D \varepsilon \text{homosemo}(EB) . C \varepsilon \text{fnctgen}(D) : [\exists FG] . F \varepsilon \textbf{ingr}(A) . G \varepsilon \text{ingrtho}(B) . C \varepsilon \text{An}(EFG)$

C is a function of Ontology.

TE42 $[ABFGHI] \therefore F \varepsilon \text{varo}(GBAHI) .\equiv: H \varepsilon \text{homosemo}(GB) : [\exists JK] : K \varepsilon \textbf{ingr}(A) . J \varepsilon \text{ingrtho}(B) . H \varepsilon \text{An}(IJK) : F \varepsilon \text{cnvar}(IA) . I \varepsilon \textbf{ingr}(\text{1trm}(A))$

F is a variable of Ontology.

TE43 $[ABCDE] \therefore C\ \varepsilon$ pprntmo$(BADE)$.≡: $D\ \varepsilon$ homosemo(BB) . $E\ \varepsilon$ prntm(D) $C\ \varepsilon$ prntm(2trm(A)) . arg$(C) \infty$ arg(E) : $[FG]$: $F\ \varepsilon$ arg(C) . $G\ \varepsilon$ arg(E) . (arg$(C) \cap$ **prcd**(F)) ∞ (arg$(E) \cap$ **prcd**(G)) . $[\exists HI]$. $F\ \varepsilon$ varo$(GBAHI)$

TE44 $[ABCDE]$: $C\ \varepsilon$ 1pprntmo$(BADE)$.≡. $C\ \varepsilon$ pprntmo$(BADE)$. **Uvrb**$(D)\ \varepsilon$ **vrb**(E)

TE45 $[ABCDEFG]$: $C\ \varepsilon$ 2pprntmo$(BADEFG)$.≡. $C\ \varepsilon$ pprntmo$(BADE)$. $F\ \varepsilon$ prntm(D) . **Uprcd**$(F)\ \varepsilon$ **ingr**(E) . $G\ \varepsilon$ smprntm(F)

The three terminological explanations immediately above are used in *TE46* to identify certain parenthemes and to insure appropriate choice of enclosing parentheses.

TE46 $[AB]$ ∴: $A\ \varepsilon$ 1defo(B) .≡:: $A\ \varepsilon A$. **1vrb**(Essnt(A)) $\varepsilon \sim$ ⟦var(A)⟧ .
1vrb(2trm (A)) $\varepsilon \sim$ ⟦var (A)⟧ .
1vrb(2trm (A)) $\varepsilon \sim$ ⟦consto(BA)⟧ ∴ $[C]$ ∴ $C\ \varepsilon$ trm .
$C\ \varepsilon$ **vrb**(1trm(A)) .⊃: $[\exists G]$. $C\ \varepsilon$ bd(G) .v. $[\exists D]$.
$D\ \varepsilon$ **ingr**(A) . $C\ \varepsilon$ var(D) .v. $C\ \varepsilon$ contso(BA) ∴ $[CD]$:
$D\ \varepsilon$ **ingr**(A) . $C\ \varepsilon$ bd(D) .⊃. $[\exists E]$. $E\ \varepsilon$ var(CD) : $[C]$:
$C\ \varepsilon$ bd(A) .⊃. $[\exists D]$. $D\ \varepsilon$ **vrb**(1trm(A)) . $D\ \varepsilon$ var(CA) :
$[C]$: $C\ \varepsilon$ bd(A) .⊃. $[\exists D]$. $D\ \varepsilon$ **vrb**(2trm(A)) . $D\ \varepsilon$ var(CA) ∴
$[DEF]$ ∴ $F\ \varepsilon$ **vrb**(1trm(A)) . $D\ \varepsilon$ **vrb**(1trm(A)) .
$F\ \varepsilon$ cnvar(DE) .⊃: $D = F$ .v. $F\ \varepsilon$ q-homosemo (DBA) ∴ $[C]$:
$C\ \varepsilon$ gnrl . $C\ \varepsilon$ **ingr** (**Int**(A)) .⊃. $[\exists DEF]$. $D\ \varepsilon$ homosemo(BB) .
$E\ \varepsilon$ ingrtho(B) . $F\ \varepsilon$ **ingr** (A) . $D\ \varepsilon$ Anagr(CEF) ∴ $[CD]$:
$C\ \varepsilon$ gnrl . $C\ \varepsilon$ **ingr**(A) . $D\ \varepsilon$ Essnt(C) .⊃: $D\ \varepsilon$ **vrb** .v. $[\exists E]$.
$E\ \varepsilon$ fro(B) . $D\ \varepsilon$ fnctgen(E) ∴ $[C]$: $C\ \varepsilon$ fnct .
$C\ \varepsilon$ **ingr**(1trm(A)) .⊃: $[\exists D]$. $D\ \varepsilon$ gnrl . $C\ \varepsilon$ Essnt(D) .v.
$[\exists DE]$. $C\ \varepsilon$ fncto$(BADE)$: $[CD]$: $D\ \varepsilon$ **int**(C) .
$C\ \varepsilon$ prntm(2trm(A)) .⊃. $D\ \varepsilon$ var(A) : 2trm$(A)\ \varepsilon$ **non-rep** :
$[CDE]$: $C\ \varepsilon$ 1pprntmo$(BADE)$. **Uvrb**(2trm(A)) ε **ingr** (C) .⊃.
$C\ \varepsilon$ simprntm(E) ∴ $[CDEFG]$: $C\ \varepsilon$ 2pprntmo$(BADEFG)$.
$G\ \varepsilon$ **ingr** (A) . **Uprcd**$(G)\ \varepsilon$ **ingr** (C) .⊃. $C\ \varepsilon$ simprntm(E) ∴
$[CE]$: $C\ \varepsilon$ prntm(2trm(A)) . **Uvrb**(2trm(A)) ε **ingr** (C) .
$E\ \varepsilon$ ingrtho(B) . $C\ \varepsilon$ simprntm(E) .⊃. $[\exists D]$. $C\ \varepsilon$ 1pprntmo$(BADE)$.
$[CEG]$: $C\ \varepsilon$ prntm(2trm(A)) . $G\ \varepsilon$ prntm . $G\ \varepsilon$ **ingr** (A) .
Uprcd$(G)\ \varepsilon$ **ingr** (C) . $E\ \varepsilon$ ingrtho(B) . $C\ \varepsilon$ simprntm(E) .⊃.
$[\exists DF]$. $C\ \varepsilon$ 2pprntmo$(BADEFG)$

A is a definition of the first kind with respect to B (or) A is a protothetical type definition with respect to thesis B. *TE46* formulates the conditions a formula which is to be added as a protothetical type definition must meet. This *TE* can be used to state one of the seven original directives of Ontology's Rule. We give it here because it must be used in formulating one of the succeeding *TE*'s.

TE47 $[AB]$: $A\ \varepsilon$ Sgnfnct(B) .≡. $A\ \varepsilon$ Anfnct (ABB) . $A\ \varepsilon A$

A is the significant functor of B. Given a functor B, Sgnfnct(B) is the functor formed by dropping the last parentheme of B.

TE48 $[AB] : A \varepsilon \mathrm{Neg}(B) \;.\!\equiv\!.\; A \varepsilon A \,.\, \mathbf{1vrb}(B) \varepsilon \mathbf{cnf}(\mathbf{15vrb}(\mathfrak{A}_0)) \,.$
$\mathbf{2vrb}(B) \varepsilon \mathbf{cnf}(\mathbf{7vrb}(\mathfrak{A}_0)) \,.\, \mathrm{arg}(\mathbf{Cmpl}(\mathbf{vrb}(B) \cap \sim[\mathbf{1vrb}(B)])) \varepsilon A$

A is the negate of *B*. *A* is the propositional expression being negated.

TE49 $[ABCD] : A \varepsilon \mathrm{Imp}(BCD) \;.\!\equiv\!.\; A \varepsilon \mathbf{Concat}(\mathbf{1vrb}(A)D) \,.$
$\mathbf{1vrb}(A) \varepsilon \mathbf{cnf}(\mathbf{38vrb}(\mathfrak{A}_0)) \,.\, B \varepsilon \mathrm{1arg}(D) \,.\, C \varepsilon \mathrm{2arg}(D)$

TE50 $[ABC] \therefore A \varepsilon \mathrm{Imp}(BC) \;.\!\equiv\!: [\exists D] \,.\, A \varepsilon \mathrm{Imp}(BCD) : A \varepsilon A$

A is an implication whose first argument is *B* and second argument is *C*.

TE51 $[AB] \therefore A \varepsilon \mathrm{Antcdnt}(B) \;.\!\equiv\!: [\exists C] \,.\, B \varepsilon \mathrm{Imp}(AC) : A \varepsilon A$

A is the antecedent of implication *B*.

TE52 $[AB] \therefore A \varepsilon \mathrm{Cnsqnt}(B) \;.\!\equiv\!: [\exists C] \,.\, B \varepsilon \mathrm{Imp}(CA) : A \varepsilon A$

A is the consequent of implication *B*.

Before presenting the last two terminological explanations that we will need, it will be useful to comment on some of the factors which motivate them. For reasons which will be explained in the next section, it is necessary to be able to identify the semantic categories of those variables and functors of a system of Ontology which could be a part of or introduced into a system of Protothetic. Of course these functors and variables can be introduced into Ontology on account of Ontology's Protothetical base, but the point to be made is that they are not specifically Ontological. Let us call the semantic categories of all such variables and functors "purely protothetical" categories. In terms of their Ajdukiewicz symbols such categories are easily characterized for their Ajdukiewicz symbols would contain only combinations of the symbols 'S' and '/'. We must, however, characterize these categories syntactically.

Intuitively, purely protothetical categories are those already present in the theses of Ontology's Protothetical base or those introduced by protothetical type definitions (i.e., those satisfying *TE46*) which themselves contain only purely protothetical categories. *TE53* syntactically characterizes these "purely protothetical" definitions and *TE54* syntactically characterizes the purely protothetical semantic categories.

TE53 $[BC] :: C \varepsilon \mathrm{ppldefo}(B) \;.\!\equiv\!:: C \varepsilon \mathrm{1defo}(B) :: [a] :: [D] \,.$
$D \varepsilon \mathrm{efthp} \;.\!\supset\!.\; D \varepsilon a \therefore [EF] \therefore E \varepsilon \mathrm{1defo}(B) \,.\, F \varepsilon \mathrm{var}(E) \;.\!\supset\!.$
$[\exists GH] \,.\, G \varepsilon a \,.\, H \varepsilon \mathbf{ingr}(G) \,.\, F \varepsilon \mathrm{homosemo}(HB) \;:\!\supset\!.\; E \varepsilon a \therefore$
$[D] : D \varepsilon a \,.\, D \varepsilon \sim[\mathrm{efthp}] \;.\!\supset\!.\; D \varepsilon \mathrm{1defo}(B) \;\therefore\!\supset\!.\; C \varepsilon a$

C is a *purely protothetical* definition of the protothetical type with respect to thesis *B*.

TE54 $[AB] \therefore A \varepsilon \mathrm{ppsemo}(B) \;.\!\equiv\!: A \varepsilon A : [\exists CD] : C \varepsilon \mathrm{efthp} \,.\, D \varepsilon \mathbf{ingr}(C) \,.$
$A \varepsilon \text{q-homosemo}(DB) :\!\mathrm{v}\!: [\exists CD] : C \varepsilon \mathrm{ppldefo}(B) \,.\, D \varepsilon \mathbf{ingr}(C) \,.$
$A \varepsilon \text{q-homosemo}(DB)$

A is of a purely protothetical semantic category with respect to thesis *B*.

3.2 *The Choice Directive* Using the previous terminological explanations we can now define the exact conditions a formula must meet in order to be suitable to be added as a choice thesis to Ontology. These are given in *TE55*. The conjuncts of the definiens of *TE55* are labeled separately so they may be identified in later comments.

TE55 $[AB] \therefore A \,\varepsilon\, \mathrm{axcho}(B) \;.{\equiv}::$

C1 $\mathbf{Int}(\mathrm{Qntf}(\mathrm{Neg}(A))) \,\varepsilon\, \mathbf{vrb} \,.\, A \,\varepsilon\, A :$

C2 $[CD] : D \,\varepsilon\, \mathbf{ingr}(A) \,.\, D \,\varepsilon\, \mathrm{qntf} \,.\, C \,\varepsilon\, \mathbf{int}(D) \;.{\supset}.\; [\exists EF] \,.\, E \,\varepsilon\, \mathrm{var}(CF) :$

C3 $[\exists C] : C \,\varepsilon\, \mathrm{prntm}(\mathrm{Antcdnt}(\mathbf{Int}(\mathrm{Sbqntf}(\mathrm{Neg}(\mathrm{Essnt}(\mathrm{Neg}(A))))))) \,.$
$\mathbf{Int}(C) \,\varepsilon\, \mathrm{var}(\mathrm{Neg}(\mathrm{Essnt}(\mathrm{Neg}(A)))) :$

C4 $[\exists CE] : C \,\varepsilon\, \mathrm{Cnsqnt}(\mathrm{Essnt}(\mathrm{Neg}(\mathrm{Essnt}(\mathrm{Neg}(A))))) \,.\, E \,\varepsilon\, \mathbf{Int}(\mathrm{prntm}(C)) \,.$
$\mathrm{Sgnfnct}(E) \,\varepsilon\, \mathrm{var}(\mathrm{Neg}(A)) \,.\, \mathbf{Int}(\mathrm{prntm}(E)) \,\varepsilon\, \mathbf{cnf}(\mathrm{Sgnfnct}(C)) :$

C5 $[CDF] : C \,\varepsilon\, \mathbf{ingr}(A) \,.\, F \,\varepsilon\, \mathrm{cnvar}(DC) \;.{\supset}.\; F \,\varepsilon\, \text{q-homosemo}(DBA) \therefore$

C6 $[C] \therefore C \,\varepsilon\, \mathrm{fnct} \,.\, C \,\varepsilon\, \mathbf{ingr}(A) \;.{\supset}:\; C = A \,.\mathrm{v}.\, [\exists D] \,.\, D \,\varepsilon\, \mathrm{gnrl} \,.$
$C \,\varepsilon\, \mathrm{Essnt}(D) \,.\mathrm{v}.\, [\exists DE] \,.\, C \,\varepsilon\, \mathrm{fncto}(BADE) \therefore$

C7 $[CDEFG] : C \,\varepsilon\, \mathrm{prntm}(\mathrm{Essnt}(\mathrm{Neg}(\mathrm{Essnt}(\mathrm{Neg}(A))))) \,.\, D \,\varepsilon\, \mathrm{arg}(C) \,.$
$E \,\varepsilon\, \mathrm{arg}(C) \,.\, F \,\varepsilon\, \mathrm{Sgnfnct}(D) \,.\, G \,\varepsilon\, \mathrm{Sgnfnct}(E) \;.{\supset}.$
$F \,\varepsilon\, \mathrm{cnvar}(G\ \mathrm{Neg}(\mathrm{Essnt}(\mathrm{Neg}(A)))) :$

C8 $[C] : C \,\varepsilon\, \mathrm{var}(\mathrm{Neg}(\mathrm{Essnt}(\mathrm{Neg}(A)))) \;.{\supset}.\; C \,\varepsilon\, \sim\![\mathrm{ppsemo}(B)]$

A is a choice thesis for Ontology with respect to thesis *B*.

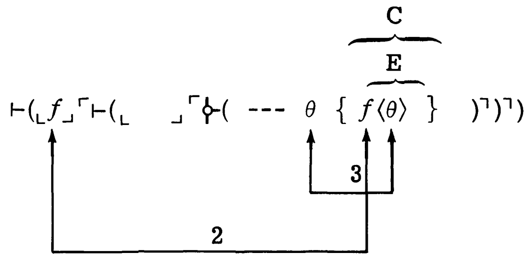
Regarding the various conjuncts of *TE55*, note that C2, C5, and C6 specify requirements that must be satisfied by any thesis of a system of Ontology. C2 stipulates that variables in quantifiers actually bind something and thus it eliminates as prospective theses formulas with vacuous quantifiers. C5 stipulates that any two equiform variables in the same generalization in *A* must be of the same semantic category. C6 stipulates that the semantic category of any function which appears in a prospective choice thesis must have been previously introduced into the system.

C1, C3, C4, and C7 specify the external structure and internal relationships of prospective choice theses. C1 necessitates that *A* be the negation of a generalization and that the interior of the quantifier of this generalization be a single variable. C3 together with C1 require *A* to have the following form:

$$\vdash(\llcorner f \lrcorner \ulcorner \vdash(\llcorner \quad \lrcorner \ulcorner \phi(\overbrace{(\quad)}^{\mathrm{C}}) \urcorner) \urcorner)$$

1

In the skeletal formula above, 'C' represents the parentheme whose existence is required by C3. The line marked '1' illustrates the requirement imposed by C3 that the interior of C be bound by and therefore conformal with a term appearing in the interior quantifier at the left terminus of the line. C4 further specifies that *A* have parenthemes with definite characteristics. C4 together with C1 and C3 require that *A* have the form sketched on the following page:



In the diagram above, 'C' and 'E' represent the components identified in C4. The line marked '2' illustrates the requirement imposed by C4 that the significant functor of E be a variable in the negate of A. The line marked '3' illustrates the requirement that the interior of the parentheme of E be conformal with the significant functor of E. C7 requires that the "words" represented by asterisks in the diagram below be conformal.

We come finally to C8. C8 has the effect of eliminating as candidates for choice theses any formulas which contain variables and functors which are of purely protothetical semantic categories. This restriction is imposed because it can be shown that formulas similar to **AC*** containing only purely protothetical semantic categories are not independent of Ontology.[16] For that matter neither are they independent of Protothetic. This dependence is a consequence of the fact that there are only a finite number of distinguishable functors in each purely protothetical semantic category. Hence a "choice function" can be constructed for each of these categories and its existence need not be assumed. Although no logical catastrophe would result if C8 were deleted, we include it in order that our choice directive not be needlessly redundant.

TE55 enables us to succinctly state the modification to the Rule of Ontology, *cf.* [16], which will insure that the Axiom of Choice is available for each semantic category. One merely adds to the original directives composing Ontology's Rule the additional directive

$$A \ \varepsilon \, \mathrm{axcho}(B)$$

Our goal of extending Ontology so that it includes the Axiom of Choice is thus achieved.

NOTES

1. The use of upper and lower case letters in the sentence form '$A \ \varepsilon \ a$' has no formal import. If it is known that a formula in which a name variable appears is true only if that particular name variable designates a single individual, then as a matter of informal convention that name variable is written in upper case letters. It is a consequence of Ontology's axiom that '$A \ \varepsilon \ a$' is true only if the subject term designates exactly one individual.

2. For a discussion of various proper axioms for Ontology, see [29].

3. Axiom *AO* and the rest of the formulas of Chapter One and Chapter Two employ a modified Peano-Russell system of notation in which: square brackets are used for quantifiers, binary functors are often written between their arguments, and dots are used to indicate parenthetical groupings and conjunctions in accord with well known conventions.

4. The Peano-Russell notation differs significantly from the system of notation devised by Leśniewski for use with his systems. In Leśniewski's notation functors *always* precede their arguments which are enclosed in parentheses; lower corner brackets, '⌞' and '⌟', enclose quantifiers; upper corner brackets, '⌜' and '⌝', are used to indicate the scope of quantifiers; and the propositional functors '~', '.', 'v', '⊃', and '≡' are symbolized as '⊢', 'ϙ', '-ϙ-', 'ϕ-', and 'ϕ' respectively. Furthermore, Ontology does not contain the particular quantifier, the effect of it being obtained through the use of the negation of a universal generalization of a negation. The use in Chapters One and Two of Peano-Russell notation, including the particular quantifier, should be considered a concession to the demands of convenience and familiarity. The work of Chapter Three presupposes the use of Leśniewski's symbolism.

5. A system of Protothetic and hence any Prothetical base for Ontology contains only semantic categories generated from sentences, that is, the Ajdukiewicz symbol for such categories contains only the symbols 'S' and '/'.

6. Though Leśniewski's theory of *semantic* categories determines the *syntax* of Ontology, the qualifier 'semantic' was used in order to emphasize that the motivation behind them was semantical rather than purely formal.

7. A formal statement of the Rule of Ontology can be found in [16] which itself refers to [15]. A formal presentation of the Rule of Protothetic which is incorporated into the Rule of Ontology can be found in [21] and [23].

8. Our informal presentation of the Rule of Ontology follows that presented in [5] to which the reader is referred for further details.

9. To my knowledge an explicit proof of this has not yet been published. Sobociński has remarked privately that such a proof was known to Leśniewski, and Canty in [5] remarks that Thomas Scharle had developed a proof. Canty also points out that the proof involves showing that: 1) for each semantic category contained in Ontology, one can define a "higher level" 'ε' as a functor analogous to the primitive 'ε' of Ontology and 2) given this definition and the Rule of Ontology one can always obtain, as a derived thesis, a formula equiform to *AO* except for the shape of the parentheses which enclose the arguments of the defined "higher ε". The second point guarantees that for any derived thesis employing the primitive 'ε' one can derive an analogous thesis employing a "higher-level" 'ε' (though not necessarily the converse). This together with the ability to introduce analogous functors would guarantee the repeatability at a "higher level" of any derivation on a "lower level".

10. This thesis, as well as the one immediately following, cannot be properly derived without the use of an auxiliary definition. The definition required for *T1.1.36* could be, for example:

D1.1.011.1 $[ab\phi]:\phi\{ab\}.\equiv.*\underset{5}{\langle}\phi a\underset{5}{\rangle}\{b\}$

The derivation could then proceed as follows:

T1.1.36	$[ab\phi]:\phi\{ab\}.b \circ c .\supset. \phi\{ac\}$	
PR	$[ab\phi]:Hp(2).\supset.$	
3.	$*\langle \underset{5}{\phi} \underset{5}{a}\rangle\{b\}$	[1; *D1.1.011.1*]
4.	$*\langle \underset{5}{\phi} \underset{5}{a}\rangle\{c\}$	[2; 3; *E1.1.01*]
	$\phi\{ac\}$	[4; *D1.1.011.1*]

Auxiliary definitions are required in situations like this because Ontology prohibits the use of incomplete symbols—the functor substituted must be of the same semantic category as that for which it is substituted. However, since auxiliary definitions are always definable for use in situations such as this, we will not explicitly give them.

11. The proof of a similar "fixed point" theorem in the field of logical type theory can be found in [4].

12. Using a system of Ontology as the logical base of a system which will ultimately be used to define the syntactical terms necessary for formulating a modification to the Rule of Ontology may bring up visions of vicious circles. It should be noted, however, that the development of Ontology which serves as a base for system **M** must be considered as independent of any system which would employ the modification of the Rule presented in this Chapter. The two are related as different tokens of the same type.

13. In presenting definitions in this chapter, we reverse previous practice and write definiendums on the left.

14. System **MO** is essentially Rickey's system **MP** (*cf.*, [21] or [23]) with some minor modifications and adjustments chief among which are the addition of the primitives 'efthp' and 'tho'.

15. Most of the "Terminological Explanations" (*TE*'s) presented in Chapter Three are identical with or analogous to those originally formulated by Rickey in [21].

16. V. F. Rickey pointed this out to me in a letter in which he sketched a proof. Rickey's argument hinges on the fact that there are only a finite number of distinguishable elements in each purely prothetical semantic category. Hence a choice function for these can be defined. The situation is analogous to that of the Axiom of Choice *vis à vis* finite sets.

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