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A LOGIC FOR UNKNOWN OUTCOMES

BERTRAM BRUCE

1 *Introduction* In computer question answering and problem solving programs many of the questions of modal and tense logics appear as practical design problems. One problem of particular interest appears when we allow events to have the truth value "unknown", a natural value to assign to some events which occur at other times than the present. However, allowing a third value is not as simple as it seems. Suppose that statements P and Q each have the truth value "unknown". What values should be assigned to $(P \wedge Q)$? If $(P \vee Q)$ is necessary, it should have the value "true", otherwise it has the value "unknown". The "modal" composition of truth values cannot be achieved in a three ("true", "unknown", "false") valued truth functional logic. In fact, as shown by Dugundji [1], no finite valued truth functional logic can be given the modal interpretation. Consequently, semantic analysis of most modal systems must be quasi-truth-functional or involve infinite matrices or both. For example, Kripke [2] introduces the concept of a set of "possible worlds" with a model which assigns to each well formed formula (wff) a set of truth values, one for each world. If the set of worlds is infinite then each wff will have an infinite sequence for its value. Furthermore, the composition of truth values is not strictly truth-functional since it depends on the "possibility" relation between worlds. Another example is the infinite product logic, πC_2, where C_2 is the classical two-valued propositional calculus [5]. In this logic wffs again have sequence for their values. These sequences can be viewed as the value a wff takes over time [3] and thus provide a link between modal logic and tense logic. A final example, out of many others, is the probabilistic approach as discussed by Rescher [4], [5]. He shows that assigning a probability to each wff and applying certain minimal features of a probability calculus yields a set of tautologies equivalent to the theorems of S5. Here again the logic is infinite valued and quasi-truth-functional in the compositions.

With a concern for computer applications such as question answering it seems appropriate to discuss yet another approach, which appears to have a simpler (though non-truth functional) decision procedure while requiring

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only three truth values. The system discussed (called system **C**) is a propositional logic consisting of a set of rules defining derivability and a somewhat unusual evaluation scheme.

Two kinds of implication are defined in the system: a material implication for which all the classical two valued tautologies hold, and a strict implication which satisfies all of the rules of S5, and in addition, the rule

$$\textit{If not } \text{-}\models_2 A \vee B \textit{ then } \vdash_{\mathbf{C}} (A \vee B) \supset (\Box A \vee \Box B).$$

This latter rule shows that **C** is a system of logical necessity only, unlike S5 which applies to other kinds of necessity as well. However, many of the so-called "paradoxes of material implication" do not arise in **C**. This paper presents the system **C** and shows that its set of theorems and its set of tautologies are the same.

2 *Syntax* We assume an infinite set S of statements with variables $(p, q, r, s, \ldots, A, B, C, \ldots)$ ranging over S. There are three primitive statement connectives, $\sim$ (negation), $\vee$ (disjunction), and $\Box$ (necessity). We allow abbreviations such as the following:

a) $p \wedge q = \sim(\sim p \vee \sim q)$ read "p and q"
b) $p \supset q = \sim p \vee q$ read "if p then q"
c) $p \equiv q = (p \supset q) \wedge (q \supset p)$ read "p if and only if q"
d) $\Diamond p = \sim\Box\sim p$ read "possible p"
e) $p \rightarrow q = \Box(p \supset q)$ read "p strictly implies q"
f) $p \longleftrightarrow q = p \rightarrow q \wedge q \rightarrow p$ read "p strictly equivalent to q"
g) $J_1 p = \Box p$ read "p takes the value 1"
h) $J_{\frac{1}{2}} p = \sim\Box p \wedge \sim\Box\sim p$ read "p takes the value $\frac{1}{2}$"
i) $J_0 p = \Box\sim p$ read "p takes the value 0"

The connectives J_1, $J_{\frac{1}{2}}$, and J_0 (first used for a similar reason by Rosser and Turquette [6]) are introduced here to simplify the completeness proof given in section **6**.

3 **C**-*tautologies* In order to define what it means for a statement to hold or to be a tautology, we need to define what the evaluation rules of our system are. The following paragraph presents a recursive definition of the evaluation function ($\mathrm{ev}: S \rightarrow \{0, \frac{1}{2}, 1\}$). Note that it requires the standard definition of two valued tautology.

Let p be a wff of the form $F(p_1, p_2, \ldots, p_n)$ for some connective F. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an assignment of values 0, $\frac{1}{2}$, and 1 to $p_1, p_2, \ldots, p_n$. Then the evaluation of p, denoted $\mathrm{ev}(p)$ is defined as follows:

a) if $\models_2 p$ (p is a tautology in the classical two valued propositional logic), then $\mathrm{ev}(p) = 1$
b) if $p = p_i$ for some i, then $\mathrm{ev}(p) = \alpha_i$
c) if $p = \sim q$ then $\mathrm{ev}(p) = 1 - \mathrm{ev}(q)$
d) if $p = q \vee r$ then $\mathrm{ev}(p) = \max(\mathrm{ev}(q), \mathrm{ev}(r))$
e) if $p = \Box q$ then $\mathrm{ev}(p) = [\mathrm{ev}(q)]$ (the greatest integer less than or equal to $\mathrm{ev}(q)$)

It should be noted that the values 0, $\frac{1}{2}$, and 1 were chosen to simplify the presentation. We could have used more meaningful names such as "false" for 0, "unknown" or "indeterminate" for $\frac{1}{2}$, and "true" for 1.

In the usual way a **C**-tautology is defined to be a wff which takes only the value 1 for every assignment of values to its component statement variables. (We write $\models p$ if p is a **C**-tautology.) Clearly then (by a) above) every two valued tautology is also a **C**-tautology. Furthermore, such modal laws as $p \rightarrow (p \wedge p)$ will be seen to be **C**-tautologies. However, the so-called "paradoxes of material implication" do not arise in **C**, when we view "$\rightarrow$" as strict implication. For instance, although $p \supset (q \supset p)$ is a **C**-tautology, $p \rightarrow (q \rightarrow p)$ is not, since rule a) does not apply, and when both p and q are assigned the value $\frac{1}{2}$, $p \rightarrow (q \rightarrow p)$ takes the value $\frac{1}{2}$.

4 **C**-*theorems* In this section we present a set of rules (proper rule schemata and axiom schemata) which together with a definition of formal demonstration specify the theorems of the system **C**. None of the rules is in its most primitive form (using only $\sim$, $\vee$, and $\Box$). However, expansion of an abbreviated wff is a straightforward procedure using the definitions of section **2**. In what follows we shall use abbreviated forms for various reasons. However, it should be understood that when we speak of the length of a wff or occurrences of a symbol in a wff that we refer to the primitive form. Thus, we say that $\Box$ occurs in both $\Box A$ and in $J_1 A$.

C1 $\vdash A \supset (B \supset A)$
C2 $\vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
C3 $\vdash (\sim A \supset \sim B) \supset (B \supset A)$
C4 $\vdash A, \vdash A \supset B \Rightarrow \vdash B$
C5 $\vdash \Box A \supset A$[1]
C6 $\vdash \Box(A \supset B) \supset (\Box A \supset \Box B)$
C7 $\vdash \Diamond A \supset \Box\Diamond A$ (or $\sim\Box A \supset \Box\sim\Box A$)
C8 $\vdash A \Rightarrow \vdash \Box A$
C9 *If not* - $\models_2 (A \vee B)$ *then* $\vdash \Box(A \vee B) \supset (\Box A \vee \Box B)$

We write $\Gamma \vdash A$ to indicate that A is derivable from the set of statements Γ, that is, there is a finite sequence (called a derivation) of wffs, $\mathbf{D} = (d_1, d_2, \ldots, d_n)$ such that $A = d_n$, and each d_i is an element of Γ, or is some d_j in $\mathbf{D}$ such that $j < i$, or follows from some wffs in $d_1, d_2, \ldots, d_{i-1}$ by rules *C1-C9*. If Γ is null we say that A is a theorem, or $\vdash A$. Where there is a chance for confusion we write $\vdash_{\mathbf{C}} A$.

Rules *C1-C3* and *C4* (modus ponens) comprise a set of axioms for classical two valued logic, thus ensuring that $\vdash_2 A \Rightarrow \vdash_{\mathbf{C}} A$. The addition of rules *C5, C6,* and *C8* gives the modal system M. Adding *C7* gives the system S5. *C9* is the special axiom mentioned earlier which makes **C** a system of logical necessity.

1. Sobociński [7] has shown that in the classical axiomatization of S5 the axiom **C5** is redundant and hence it is redundant in the system **C**.

5 *Every* C-*theorem is a* C-*tautology* (*Soundness*) This section shows that any theorem of C takes the value 1 for every assignment of values to its variables. It suffices to show that each rule of C is a C-tautology.

CS1-4 $\models_{\mathsf{C}} A \supset (B \supset A)$; $\models_{\mathsf{C}} (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$; $\models_{\mathsf{C}} (\sim A \supset \sim B) \supset (B \supset A)$; $\models_{\mathsf{C}} A$, $\models_{\mathsf{C}} A \supset B \Rightarrow \models_{\mathsf{C}} B$

Proof: Obvious by step a) of the evaluation scheme.

CS5 $\models_{\mathsf{C}} \Box A \supset A$

Proof: If $\mathrm{ev}(A) \neq 1$, the $\mathrm{ev}(\Box A) = 0$.

CS6 $\models_{\mathsf{C}} \Box(A \supset B) \supset (\Box A \supset \Box B)$

Proof: If $\mathrm{ev}(\Box A \supset \Box B) \neq 1$ then $\mathrm{ev}(\Box A) = 1$ and $\mathrm{ev}(\Box B) = 0$. But then $\mathrm{ev}(A) = 1$ and $\mathrm{ev}(B) \neq 1$. Hence $\mathrm{ev}(A \supset B) \neq 1$ and $\mathrm{ev}(\Box(A \supset B)) = 0$.

CS7 $\models_{\mathsf{C}} \Diamond A \supset \Box\Diamond A$

Proof: If $\mathrm{ev}(\Box\Diamond A) \neq 1$, then $\mathrm{ev}(\Diamond A) = 0$.

CS8 $\models_{\mathsf{C}} A \Rightarrow \models_{\mathsf{C}} \Box A$

Proof: Obvious by step e).

CS9 *If not* - $\models_2 (A \vee B)$ *then* $\models_{\mathsf{C}} \Box(A \vee B) \supset (\Box A \vee \Box B)$

Proof: Assume not - $\models_2 (A \vee B)$. Then if $\mathrm{ev}(\Box A \vee \Box B) \neq 1$ then $\mathrm{ev}(\Box A) \neq 1$ and $\mathrm{ev}(\Box B) \neq 1$. Then $\mathrm{ev}(A) \neq 1$ and $\mathrm{ev}(B) \neq 1$. Then $\mathrm{ev}(A \vee B) \neq 1$ and $\mathrm{ev}(\Box(A \vee B)) = 0$.

CS10 $\vdash_{\mathsf{C}} A \Rightarrow \models_{\mathsf{C}} A$

Proof: Follows from *CS1-CS9*.

6 *Every* C-*tautology is a* C-*theorem* (*Completeness*) This section establishes some theorems of C in order to illustrate the system and then uses the theorems to prove the completeness of C. This, together with *CS10* shows that the given axiomatization of C (*C1-C9*) is complete and sound relative to the evaluation scheme given above. In the proofs below, standard results are given in an abbreviated form (for example, *C1-C4* means a theorem of ordinary two valued logic, *C1-C8* means a theorem of the Lewis system S5).

CC1 $\vdash A \supset B \Rightarrow \vdash \Box A \supset \Box B$

Proof:

1.	$\vdash A \supset B$	hypothesis
2.	$\vdash \Box(A \supset B)$	1, *C8*
3.	$\vdash \Box(A \supset B) \supset (\Box A \supset \Box B)$	*C6*
4.	$\vdash \Box A \supset \Box B$	2, 3, *C4*

CC2 $\vdash J_1 A \supset J_0 \sim A$

Proof:

1.	$\vdash A \supset \sim\sim A$	*C1-C4*
2.	$\vdash \Box A \supset \Box\sim\sim A$	1, *CC1*

CC3 $\vdash J_{\frac{1}{2}} A \supset J_{\frac{1}{2}} \sim A$

Proof:	1. $\vdash \sim\sim A \supset A$	*C1-C4*
	2. $\vdash \Box\sim\sim A \supset \Box A$	1, *CC1*
	3. $\vdash \sim\Box A \supset \sim\Box\sim\sim A$	2, *C1-C4*
	4. $\vdash \sim\Box\sim A \supset \sim\Box\sim A$	3, *C1-C4*
	5. $\vdash (\sim\Box A \wedge \sim\Box\sim A) \supset (\sim\Box\sim A \wedge \sim\Box\sim\sim A)$	4, *C1-C4*

CC4 $\vdash J_0 A \supset J_1 \sim A$

Proof: By *C1-C4*.

CC5 $\vdash J_i A \supset J_{1-i} \sim A$

Proof: By definition from *CC2-CC4*.

CC6 $\vdash J_1(A \vee B) \supset J_1(B \vee A)$

Proof:	1. $\vdash A \vee B \supset B \vee A$	*C1-C4*
	2. $\vdash \Box(A \vee B) \supset \Box(B \vee A)$	1, *CC1*

CC7 $\vdash J_0(A \vee B) \supset J_0(B \vee A)$

Proof:	1. $\vdash \sim(A \vee B) \supset \sim(B \vee A)$	*C1-C4*
	2. $\vdash \Box\sim(A \vee B) \supset \Box\sim(B \vee A)$	1, *CC1*

CC8 $\vdash J_{\frac{1}{2}}(A \vee B) \supset J_{\frac{1}{2}}(B \vee A)$

Proof:	1. $\vdash \Box(B \vee A) \supset \Box(A \vee B)$	*CC6*
	2. $\vdash \sim\Box(A \vee B) \supset \sim\Box(B \vee A)$	1, *C1-C4*
	3. $\vdash \Box\sim(B \vee A) \supset \Box\sim(A \vee B)$	*CC7*
	4. $\vdash \sim\Box\sim(A \vee B) \supset \sim\Box\sim(B \vee A)$	3, *C1-C4*
	5. $\vdash (\sim\Box(A \vee B) \wedge \sim\Box\sim(A \vee B)) \supset (\sim\Box(B \vee A) \wedge \sim\Box\sim(B \vee A))$	2, 4, *C1-C4*

CC9 $\vdash \Box A \supset \Box(A \vee B)$

Proof:	1. $\vdash A \supset (A \vee B)$	*C1-C4*
	2. $\vdash \Box A \supset \Box(A \vee B)$	1, *C1-C4*

CC10 $\vdash \Box A \supset (J_i B \supset \Box(A \vee B))$

Proof:	1. $\vdash \Box A \supset \Box(A \vee B)$	*CC9*
	2. $\vdash \Box A \supset (J_i B \supset \Box(A \vee B))$	1, *C1-C4*

CC11 $\vdash J_i A \supset (\Box B \supset \Box(A \vee B))$

Proof:	1. $\vdash \Box B \supset (J_i A \supset \Box(B \vee A))$	*CC10*
	2. $\vdash J_i A \supset (\Box B \supset \Box(B \vee A))$	1, *C1-C4*
	3. $\vdash J_i A \supset (\Box B \supset \Box(A \vee B))$	2, *CC6*, *C1-C4*

CC12 $\vdash J_0 A \supset (J_0 B \supset J_0(A \vee B))$

Proof:	1. $\vdash \sim A \supset (\sim B \supset \sim(A \vee B))$	*C1-C4*
	2. $\vdash \Box\sim A \supset \Box(\sim B \supset \sim(A \vee B))$	1, *CC1*
	3. $\vdash \Box\sim A \supset (\Box\sim B \supset \Box\sim(A \vee B))$	2, *CC1*

CC13 $\vdash \Diamond B \supset \Diamond(A \vee B)$

Proof:
1. $\vdash \sim(A \vee B) \supset \sim B$ — *C1-C4*
2. $\vdash \Box \sim(A \vee B) \supset \Box \sim B$ — 1, *CC1*
3. $\vdash \sim \Box \sim B \supset \sim \Box \sim(A \vee B)$ — 2, *C1-C4*

CC14 $\vdash(\Box \sim A \sim \sim \Box B) \supset \sim \Box(A \vee B)$

Proof:
1. $\vdash(A \vee B) \supset (\sim A \supset B)$ — *C1-C4*
2. $\vdash \Box(A \vee B) \supset \Box(\sim A \supset B)$ — 1, *CC1*
3. $\vdash \Box(A \vee B) \supset (\Box \sim A \supset \Box B)$ — 2, *CC1*
4. $\vdash \sim(\Box \sim A \supset \Box B) \supset \sim \Box(A \vee B)$ — 3, *C1-C4*

CC15 $\vdash J_0 A \supset (J_{\frac{1}{2}} B \supset J_{\frac{1}{2}}(A \vee B))$

Proof:
1. $\vdash \Diamond B \supset \Diamond(A \vee B)$ — *CC13*
2. $\vdash(\Box \sim A \wedge \sim \Box B) \supset \sim(A \vee B)$ — *CC14*
3. $\vdash(\Box \sim A \wedge \sim \Box B \wedge \Diamond B) \supset (\sim \Box(A \vee B) \wedge \sim \Box \sim(A \vee B))$ — 1, 2, *C1-C4*
4. $\vdash \Box \sim A \supset ((\sim \Box B \wedge \Diamond B) \supset (\sim \Box(A \vee B) \wedge \Diamond(A \vee B)))$ — 3, *C1-C4*

CC16 $\vdash J_{\frac{1}{2}} A \supset (J_0 B \supset J_{\frac{1}{2}}(A \vee B))$

Proof:
1. $\vdash J_0 B \supset (J_{\frac{1}{2}} A \supset J_{\frac{1}{2}}(B \vee A))$ — *CC15*
2. $\vdash J_{\frac{1}{2}} A \supset (J_0 B \supset J_{\frac{1}{2}}(B \vee A))$ — 1, *C1-C4*
3. $\vdash J_{\frac{1}{2}} A \supset (J_0 B \supset J_{\frac{1}{2}}(A \vee B))$ — 2, *CC8*

CC17 If not $(i = j = \frac{1}{2})$ *then* $\vdash J_i A \supset (J_j B \supset J_{\max(i,j)}(A \vee B))$

Proof: For $i = j = 0$, *CC12*. For $i = 1$, *CC10*. For $j = 1$, *CC11*. For $i = 0$, $j = \frac{1}{2}$, *CC15*. For $i = \frac{1}{2}$, $j = 0$, *CC16*.

CC18 $\vdash \Diamond(A \wedge B) \supset \Diamond(A \vee B)$

Proof:
1. $\vdash \sim(A \vee B) \supset \sim(A \wedge B)$ — *C1-C4*
2. $\vdash \Box \sim(A \vee B) \supset \Box \sim(A \wedge B)$ — 1, *CC1*
3. $\vdash \sim \Box \sim(A \wedge B) \supset \sim \Box \sim(A \vee B)$ — 2, *C1-C4*

CC19 If not $-\models_2 (A \vee B)$ *then* $\vdash J_{\frac{1}{2}} A \supset (J_{\frac{1}{2}} B \supset J_{\frac{1}{2}}(A \vee B))$

Proof: Assume not $-\models_2 (A \vee B)$
1. $\vdash \Box(A \vee B) \supset (\Box A \vee \Box B)$ — *C9*
2. $\vdash(\sim \Box A \sim \sim \Box B) \supset \sim \Box(A \vee B)$ — 1, *C1-C4*
3. $\vdash(\sim \Box \sim A \wedge \sim \Box \sim B) \supset \sim \Box \sim(A \wedge B)$ — 2, *C1-C4*
4. $\vdash(\sim \Box \sim A \wedge \sim \Box \sim B) \supset \sim \Box \sim(A \vee B)$ — 3, *CC18*, *C1-C4*
5. $\vdash(\sim \Box A \wedge \sim \Box \sim A \wedge \sim \Box B \wedge \sim \Box \sim B) \supset (\sim \Box(A \vee B) \wedge \sim \Box \sim(A \vee B))$ — 2, 4, *C1-C4*
6. $\vdash(\sim \Box A \wedge A) \supset ((\sim \Box B \wedge B) \supset (\sim \Diamond(A \vee B) \wedge \Diamond(A \vee B)))$ — 5, *C1-C4*

CC20 $\vdash \Box A \Rightarrow \vdash A$

Proof:
1. $\vdash \Box A$ — hypothesis
2. $\vdash A$ — 1, *C5*, *C4*

CC21 $\vdash J_1 A \supset J_1 J_1 A$

Proof: By *C1-C8*.

CC22 $\vdash J_{\frac{1}{2}} A \supset J_0 J_1 A$

Proof: 1. $\vdash \sim\Box A \supset \Box\sim\Box A$ *C7*
2. $\vdash J_{\frac{1}{2}} A \supset J_0 J_1 A$ 1, *C1-C4*

CC23 $\vdash J_0 A \supset J_0 J_1 A$

Proof: 1. $\vdash \Box A \supset A$ *C5*
2. $\vdash \sim A \supset \sim\Box A$ 1, *C1-C4*
3. $\vdash \Box\sim A \supset \Box\sim\Box A$ 2, *CC1*

CC24 $\vdash J_i A \supset J_{[i]} J_1 A$

Proof: By *CC21-CC23*.

CC25 $\vdash (J_1 A \supset J_1 B) \supset ((J_{\frac{1}{2}} A \supset J_1 B) \supset ((J_0 A \supset J_1 B) \supset J_1 B))$

Proof: 1. $\vdash (A \supset B) \supset (((\sim A \wedge \sim C) \supset B) \supset ((C \supset B) \supset B))$ *C1-C4*
2. $\vdash (J_1 A \supset J_1 B) \supset (((\sim J_1 A \wedge \sim J_0 A) \supset J_1 B) \supset ((J_0 A \supset J_1 B) \supset J_1 B))$ 1

CC26 Let A be a wff composed in the usual way from the statement variables $p_1, p_2, \ldots, p_k$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be a given evaluation (assignment of values) of $p_1, p_2, \ldots, p_k$. Let $p_1', p_2', \ldots, p_k'$ be $J_{\alpha_1} p_1, J_{\alpha_2} p_2, \ldots, J_{\alpha_k} p_k$ and A' be $J_{\mathrm{ev}(A)} A$. Then $p_1', p_2', \ldots, p_k' \vdash A'$.

Proof: The proof is by induction on the rules of formation. There are five cases.

a) A is atomic. Then $A = p_i$, $A' = p_i'$, and $p_i' \vdash p_i' \Rightarrow p_i' \vdash A'$.
b) $\models_2 A$. Then $A' = J_1 A$. By the completeness of two valued logic $\models_2 A \Rightarrow \vdash_2 A$. By *C1-C4*, $\vdash_2 A \Rightarrow \vdash_{\mathsf{C}} A$. By *C8*, $\vdash_{\mathsf{C}} J_1 A \Rightarrow \vdash_{\mathsf{C}} A'$. Thus $p_1', p_2', \ldots, p_k' \vdash_{\mathsf{C}} A'$.
c) $A = \sim B$. By hypothesis of induction, we have $p_1', p_2', \ldots, p_k' \vdash B'$. Let $i = \mathrm{ev}(B)$. Then $B' = J_i B$, $\mathrm{ev}(A) = 1 - \mathrm{ev}(B) = 1 - i$, and $A' = J_{1-i} A = J_{1-i}(\sim B)$. By *CC5* we get $p_1', p_2', \ldots, p_k' \vdash A'$.
d) $A = B \vee C$. By hypothesis of induction, we have $p_1', p_2', \ldots, p_k' \vdash B'$ and $p_1', p_2', \ldots, p_k' \vdash C'$. Let $i = \mathrm{ev}(B)$ and $j = \mathrm{ev}(C)$. Then $B' = J_i(B)$ and $C' = J_j(C)$, $\mathrm{ev}(A) = \max(i, j)$ (since case a does not occur), and $A' = J_{\max(i,j)} A = J_{\max(i,j)}(B \vee C)$. By *CC17* or *CC19*, we get $p_1', p_2', \ldots, p_k' \vdash A'$.
e) $A = \Box B$. By the hypothesis of induction we have $p_1', p_2', \ldots, p_k' \vdash B'$. Let $i = \mathrm{ev}(B)$. Then $B' = J_i B$, $\mathrm{ev}(A) = [i]$, and $A' = J_{[i]} A = J_{[i]} J_1 A$. By *CC24* we get $p_1', p_2', \ldots, p_k' \vdash A'$.

CC27 If Γ is a set of wffs and A and B are wffs, then $\Gamma, A \vdash_{\mathsf{C}} B$ implies $\Gamma \vdash_{\mathsf{C}} (A \supset B)$.

Proof: Let $\mathsf{D} = (d_1, d_2, \ldots, d_n)$ be a derivation of B from $\Gamma \cup \{A\}$. For each $i = 1, 2, \ldots, n$, define a sequence Y_i as follows:

a) If d_i is an instance of *C1-C9* or an element of Γ, let Y_i be $(d_i, d_i \supset (A \supset d_i), A \supset d_i)$.

b) If $d_i = A$ let Y_i be a derivation of $A \supset A$ from the null set. (Since $\vdash_2 A \supset A$ we have $\vdash_{\mathbf{C}} A \supset A$.)
c) If d_i follows from previous wffs of **D**, say d_j and $d_k = d_j \supset d_i$, by *C4*, then let Y_i be (select the least j and k) $(Y_j, Y_k, (A \supset (d_j \supset d_i)) \supset ((A \supset d_j) \supset (A \supset d_i)), (A \supset d_j) \supset (A \supset d_i), (A \supset d_i))$.

It follows (by induction) that for each $i = 1, 2, \ldots, n$, that Y_i is a derivation of $A \supset d_i$ from Γ. Since d_n is B, Y_n is a derivation of $A \supset B$ from Γ.

The completeness theorem follows directly from *CC26* and *CC27*.

CC28 $\models_{\mathbf{C}} A \Rightarrow \vdash_{\mathbf{C}} A$

Proof: Let A be a wff composed in the usual way from the variables $p_1, p_2, \ldots, p_k$. Define $p_1', p_2', \ldots, p_k', A'$ as in *CC26*. Then $\models_{\mathbf{C}} A$ implies for every evaluation, $\mathrm{ev}(A) = 1$, hence $A' = J_1 A$. By *CC26*, $p_1', p_2', \ldots, p_k' \vdash_{\mathbf{C}} J_1 A$ for every evaluation. In particular,

$$p_1', p_2', \ldots, p_{k-1}', J_1 p_k \vdash_{\mathbf{C}} J_1 A$$
$$p_1', p_2', \ldots, p_{k-1}', J_{\frac{1}{2}} p_k \vdash_{\mathbf{C}} J_1 A$$
$$p_1', p_2', \ldots, p_{k-1}', J_0 p_k \vdash_{\mathbf{C}} J_1 A$$

By *CC27*,

$$p_1', p_2', \ldots, p_{k-1}' \vdash_{\mathbf{C}} J_1 p_k \supset J_1 A$$
$$p_1', p_2', \ldots, p_{k-1}' \vdash_{\mathbf{C}} J_{\frac{1}{2}} p_k \supset J_1 A$$
$$p_1', p_2', \ldots, p_{k-1}' \vdash_{\mathbf{C}} J_0 p_k \supset J_1 A$$

By *CC25*, $p_1', p_2', \ldots, p_{k-1}' \vdash_{\mathbf{C}} J_1 A$. By induction we can get $\vdash_{\mathbf{C}} J_1 A$, and by *CC20* $\vdash_{\mathbf{C}} A$.

7 *Conclusion*

CCS1 $\models_{\mathbf{C}} A \Leftrightarrow \vdash_{\mathbf{C}} A$

Proof: Follows immediately from *CC28* and *CS10*.

Theorem *CCS1* shows that the evaluation scheme given above assigns the value 1, for all assignments of values to the component variables, to those and only those wffs which are derivable from the axioms *C1-C9*. Thus, the system **C** is complete and sound relative to the axioms and evaluation scheme presented.

C is an interesting system in several ways. As mentioned, its strict implication operator ($\rightarrow$) avoids many of the implication paradoxes which are also avoided by the Lewis systems. Furthermore, no Gödel sequence (see [1] or [5]) holds for the strict equivalence; that is, wffs of the form

$$(p_1 \leftrightarrow p_2) \vee (p_1 \leftrightarrow p_3) \vee (p_2 \leftrightarrow p_3) \vee (p_1 \leftrightarrow p_4) \vee (p_2 \leftrightarrow p_4) \vee$$
$$(p_3 \leftrightarrow p_4) \vee \ldots \vee (p_{n-1} \leftrightarrow p_n)$$

are not theorems. The reason that **C** is not a counter example to Dugundji's theorem is that it is not a strictly truth functional logic; that is, no finite truth table could be constructed to replace its evaluation scheme.

It seems reasonable (although it has not been shown here) to expect that changing rule a) of the evaluation scheme so that "necessities" other than logical necessity are captured would make **C** equivalent to S5. As it stands, **C** is strictly stronger than S5 because of the axiom *C9*. Aside from any intrinsic interest **C** may hold as a system for purely logical necessity, it appears that its main value may be in computational areas. There are many efficient methods for checking for two valued tautologies, so the evaluation scheme should be easy to implement in a question answering system. The "unknown" truth value could then be assigned to certain future (or past) statements or events. Questions involving several events might then take the value "unknown", provided they were not tautologies.

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