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Word Problems for Bidirectional, Single-Premise Post Systems

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Introduction A *bidirectional, single-premise Post system* is a Post canonical form F where, if $R_1 \rightarrow R_2$ is a rule, then $R_2 \rightarrow R_1$ is also a rule. One class of bidirectional Post systems, the Thue systems first defined in [7], have been extensively studied. Thue systems with unsolvable word problems were shown to exist by Post [5] and, more recently, Overbeek [4] demonstrated that this class of problems represents every recursively enumerable (r.e.) many-one degree of unsolvability. In this paper we extend Overbeek's result to include bidirectional extensions of Post normal systems, tag systems, and the one-letter systems introduced by Hosken [1].

Post Systems Let Σ be a finite set of symbols and let $Q_1, Q_2, \ldots, Q_n$ be new symbols called *operational variables*. A word over $\Sigma \cup \{Q_1, Q_2, \ldots, Q_n\}$, containing at least one operational variable, is called a *word form*. An *identification* of the operational variables $Q_1, Q_2, \ldots, Q_n$ is a set of pairs $\{(Q_i, W_i) | 1 \leqslant i \leqslant n\}$ where each W_i is a word over Σ. Let $Y \equiv y_1 Q_{i_1} y_2 Q_{i_2} \ldots y_m Q_{i_m} y_{m+1}$ be a word form where $y_1, y_2, \ldots, y_{m+1}$ are words over Σ and $Q_{i_1}, Q_{i_2}, \ldots, Q_{i_m}$ are operational variables. Then Y' is the result of applying the identification $\Phi = \{(Q_i, W_i) | 1 \leqslant i \leqslant n\}$ to Y, denoted Y^Φ, if $Y' \equiv y_1 W_{i_1} y_2 W_{i_2} \ldots y_m W_{i_m} y_{m+1}$.

A *single-premise Post system* $F = (\Sigma, V, P)$ is such that Σ is a finite alphabet, V is a finite set of operational variables, and P is a finite set of rules, each of the form $R_1 \rightarrow R_2$, where R_1 and R_2 are word forms. Let W_1 and W_2 be words over Σ. Then W_2 is said to be an *immediate successor* of W_1 in F, denoted $(W_1, W_2)_F$, if there exists some rule of P, $R_1 \rightarrow R_2$, and some identification Φ of V such that $R_1^\Phi \equiv W_1$ and $R_2^\Phi \equiv W_2$. W_2 is said to be *derivable* from W_1 in F, denoted $[W_1, W_2]_F$ (or $[W_1, W_2]$, whenever F is understood from context), if there exists a sequence $Y_1, Y_2, \ldots, Y_k$, where $k \geqslant 1$, of words over Σ such

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that $Y_1 \equiv W_1$, $Y_k \equiv W_2$ and for each j, $1 \leqslant j < k$, $(Y_j, Y_{j+1})_F$. The *length* of the above derivation is $k - 1$ and each Y_i is said to be *in the derivation* of W_2 from W_1.

A *Post normal system* $N = (\Sigma, \{Q\}, P)$ is a Post system where each rule is of the form $\alpha Q \to Q\beta$, for α and β words over Σ. Let $N = (\Sigma, \{Q\}, P)$ be a Post normal system where $\Sigma = \{a_1, a_2, \ldots, a_n\}$. Then N is called a *tag system* if there exist a constant positive integer d, called the *deletion number* of N, and for each i, $1 \leqslant i \leqslant n$, a word W_i, uniquely corresponding to a_i, such that

$$P = \bigcup_{i=1}^{n} \{a_i WQ \to QW_i \mid W \text{ is a word over } \Sigma \text{ and } |W| = d - 1\}.$$

A *restricted Post canonical form* (RPCF) $R = (\{1\}, \{X\}, P)$ is a system where each rule is of the form $X^a 1^b \to X^c 1^d$ (where Y^z represents z consecutive occurrences of Y). These one-letter forms are more commonly viewed as systems operating on natural numbers. In this case each rule is of the form $ax + b \to cx + d$ where a, b, c, and d are natural numbers. The natural number n_2 is said to be an immediate successor of n_1 if there is a rule $ax + b \to cx + d$ such that, for some natural number y, both $n_1 = ay + b$ and $n_2 = cy + d$. The association of this latter formulation with canonical systems is clear when we interpret a string of n consecutive 1's as representing the natural number n.

Bidirectional Post Systems Let $F_1 = (\Sigma, V, P_1)$ be a single-premise Post system. The *inverse* of F_1 (sometimes denoted F_1^{-1}) is the system $F_2 = (\Sigma, V, P_2)$ where P_2 contains the rule $R_2 \to R_1$ just in case P_1 contains $R_1 \to R_2$. The rules in F_2 are in effect the inverses of those in F_1. While the inverses of semi-Thue and Thue systems are themselves semi-Thue and Thue systems, respectively, the inverses of Post normal systems are not normal and the inverses of tag systems are not tag systems.

The *bidirectional extension* of any single-premise system $F_1 = (\Sigma, V, P_1)$ is the system $F_3 = (\Sigma, V, P_1 \cup P_2)$ where P_2 is the set of rules contained in the inverse of F_1. The primary research reported here concerns properties of the word problems for bidirectional extensions of Post normal systems, tag systems, and RPCF's.

Decision Problems for Post Systems If $F = (\Sigma, V, P)$ is a Post system then the *word problem* for F is the problem of determining for arbitrary W_1 and W_2 over Σ whether or not $[W_1, W_2]$. The *confluence problem* for F is the problem of determining for arbitrary W_1 and W_2 over Σ whether or not there is a W_3 over Σ such that $[W_1, W_3]$ and $[W_2, W_3]$. The *decision problem* for F *with axiom A* over Σ is the problem of determining for arbitrary W over Σ whether or not $[A, W]$. The *general word problem (general confluence problem, general decision problem with axiom)* for a class of systems, e.g., all tag systems, all Post normal systems, etc., is the family of word problems (confluence problems, decision problems with axiom) for all such systems.

Let G_1 and G_2 be a pair of *general decision problems* (that is, classes of decision problems such as the general confluence problem for tag systems) then G_1 is said to be *many-one reducible* to G_2 if there exists an effective mapping Ψ of the problems P in G_1 into the problems $\Psi(p)$ in G_2 such that, if

p is nonrecursive, then p is of the same many-one degree as $\Psi(p)$. Every nonrecursive r.e. many-one degree is said to be *represented by* G_2 if the general decision problem for r.e. sets is many-one reducible to G_2.

Deterministic Systems Let $F = (\Sigma, V, P)$ be a Post system. Then F is said to be *deterministic* if, for each word W over Σ, $(W, W_1)_F$ and $(W, W_2)_F$ implies that $W_1 \equiv W_2$. Thus, each word has at most one unique immediate successor. The confluence problem for a deterministic system is equivalent to the word problem for its bidirectional extension as is shown by the following.

Theorem 1 *Let F_1 be a deterministic Post system, let F_2 be its inverse, and let F_3 be its bidirectional extension. Then the confluence problem for F_1 is of the same one-one degree (that is, is isomorphic) to the word problem for F_3.*

Proof: Let W_1 and W_2 be any two words over F_1's alphabet. Then W_1 and W_2 conflue in F_1 just in case there is some word W_3 such that $[W_1, W_3]_{F_1}$ and $[W_2, W_3]_{F_1}$. But then $[W_1, W_3]_{F_3}$ and $[W_3, W_2]_{F_3}$, and consequently $[W_1, W_2]_{F_3}$. Hence, if W_1 and W_2 conflue in F_1, then W_1 derives W_2 in F_3.

Going in the other direction, assume $[W_1, W_2]_{F_3}$. Let $W_1 \equiv U_1$, $W_2 \equiv U_n$, and $(U_1, U_2)_{F_3}, \ldots, (U_{n-1}, U_n)_{F_3}$ represent a derivation in F_3 such that there is no shorter length path from W_1 to W_2. We claim that there must exist some j, $0 \leqslant j < n$, such that $(U_m, U_{m+1})_{F_1}$, $1 \leqslant m \leqslant j$, and $(U_m, U_{m+1})_{F_2}$, $j < m < n$. This, in effect, says that once a rule from F_2 is used we can never again choose one from F_1. If this claim were false then for some k, $(U_k, U_{k+1})_{F_1}$ and $(U_{k-1}, U_k)_{F_2}$. But, since F_1 is deterministic $U_{k+1} \equiv U_{k-1}$ and there is a derivation of length $n - 3$, contradicting the fact that $n - 1$ is minimal. Thus our claim is verified. But then $[W_1, U_{j+1}]_{F_1}$ and $[U_{j+1}, W_2]_{F_2}$ which implies that $[W_1, U_{j+1}]_{F_1}$ and $[W_2, U_{j+1}]_{F_1}$ and hence W_1 and W_2 conflue in F_1.

Word Problems for Bidirectional Extensions The general word problems for bidirectional extensions of tag and, consequently, Post normal systems may be trivially shown to represent every r.e. many-one degree. This is accomplished as follows.

Lemma 1 *Every nonrecursive r.e. many-one degree is represented by the general confluence problem for tag systems.*

Proof: While not explicitly claimed there, this result follows from the construction in Hughes [2] and the fact that every r.e. many-one degree is represented by the general confluence problem for register machines [3].

Theorem 2 *Every nonrecursive r.e. many-one degree is represented by each of the general word problems for the bidirectional extensions of tag and Post normal systems.*

Proof: Tag systems are clearly deterministic and thus, by Theorem 1, the confluence problem for a tag system is of the same many-one degree as the word problem for its bidirectional extension. Thus the degree result of Lemma 1 may be carried over to the general word problem for bidirectional extensions of tag systems. The result for Post normal systems is a consequence of the fact that every tag system is also a Post normal system.

The case for the bidirectional extensions of restricted Post canonical forms is not demonstrated as easily as was done for tag systems. Our basis for the result is in the work of Overbeek [3] in which he established that every nonrecursive r.e. many-one degree is represented by the confluence problem for n-register machines. What will be shown here is an effective procedure which when given an arbitrary n-register machine R will produce a bidirectional RPCF F such that the confluence problem for R is of the same many-one degree as the word problem for F.

An *n-register machine* R is a system having n registers each capable of storing any nonnegative integer. R is defined by an ordered set of m rules, each having one of the forms:

$\mathrm{ADD}_i(j)$, where $0 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m+1$; or
$\mathrm{SUB}_i(j,k)$, where $0 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m+1$ and $1 \leqslant k \leqslant m+1$.

A *configuration* of R is an $(n+1)$-tuple $Z = (h, r_0, r_1, \ldots, r_{n-1})$ where $1 \leqslant h \leqslant m+1$ and each r_i is a natural number. If $Z = (h, r_0, r_1, \ldots, r_{n-1})$ and $Z' = (j, s_0, s_1, \ldots, s_{n-1})$ are configurations of R, then Z' is the *immediate successor* of Z in R if either

a. rule h is $\mathrm{ADD}_i(j)$, $s_i = r_i + 1$, and $s_t = r_t$ for $t \neq i$; or
b. rule h is $\mathrm{SUB}_i(j,k)$, $r_i > 0$, $s_i = r_i - 1$, and $s_t = r_t$ for $t \neq i$; or
c. rule h is $\mathrm{SUB}_i(k,j)$, $r_i = 0$, and $s_t = r_t$ for $0 \leqslant t < n$.

If $h = m + 1$, none of the rules apply; this case can be thought of as a *terminal* configuration.

We shall now demonstrate an effective procedure which when applied to an arbitrary n-register machine R, produces a bidirectional RPCF F such that the confluence problem for R is of the same many-one degree as the word problem for F.

Let R be an n-register machine with m rules. Let p_i denote the i^{th} prime number, with $p_0 = 2$. The bidirectional RPCF will have the rules:

Set 1. $p_r p_s X \longleftrightarrow p_r p_t X$ for $n \leqslant r \leqslant m+n$, $n \leqslant s \leqslant m+n$, and $n \leqslant t \leqslant m+n$.
$p_r p_s p_i X \longleftrightarrow p_r p_s X$ for $n \leqslant r \leqslant m+n$, $n \leqslant s \leqslant m+n$, and $0 \leqslant i < n$.
$p_r p_s X \longleftrightarrow p_r p_s p_i X$ for $n \leqslant r \leqslant m+n$, $n \leqslant s \leqslant m+n$, and $0 \leqslant i < n$.

Set 2. If rule h of R is $\mathrm{ADD}_i(j)$ then include the rule

$$p_{h+n-1} X \longleftrightarrow p_{j+n-1} p_i X.$$

If rule h of R is $\mathrm{SUB}_i(j,k)$ then include the rule

$$p_{h+n-1} p_i X \longleftrightarrow p_{j+n-1} X$$

and the rules

$$p_{h+n-1} p_i X + p_{h+n-1} t \longleftrightarrow p_{k+n-1} p_i X + p_{k+n-1} t$$

for each t, $1 \leqslant t < p_i$.

We now show that the word problem for F is of the same many-one degree as the confluence problem for R. Let $Z = (h, r_0, r_1, \ldots, r_{n-1})$ be an arbitrary configuration of R. We define $G(Z)$ as the natural number $p_{h+n-1} p_0^{r_0} p_1^{r_1} \ldots p_{n-1}^{r_{n-1}}$. A natural number α is *normal* if there exists a configuration Z of R such that $G(Z) = \alpha$.

Let Z be an arbitrary configuration of R. We wish to show the following:

1. When applied to normal numbers, the rules of F are deterministic in a forward ($\rightarrow$) or left-to-right direction; $G(Z)$ has at most one forward immediate successor in F, and

2. Z' is the immediate successor of Z in R implies $G(Z')$ is the forward immediate successor of $G(Z)$ in F.

To show these results, we note that no member of Rule Set 1 can apply to a normal number α, since a normal number has only one prime factor p_w, where $n \leqslant w \leqslant m + n$. p_w in this case will be p_{h+n-1}, with $1 \leqslant h \leqslant m + 1$, and $\alpha = p_{h+n-1}y$. Thus, exactly one of the following will be true:

a. $h = m + 1$, and α will have no forward successor in F.

b. Rule h of R is $\text{ADD}_i(j)$. In this case α has the forward successor $p_{j+n-1}p_iy$.

c. Rule h of R is $\text{SUB}_i(j,k)$ and $y = p_ir$ for some r. The forward successor of α will be $p_{j+n-1}r$.

d. Rule h of R is $\text{SUB}_i(j,k)$ and y is not divisible by p_i. Then $\alpha = p_{h+n-1}$ $(p_ir + t)$ for some natural number r and $1 \leqslant t < p_i$, and α has the forward successor $p_{k+n-1}y$.

This establishes determinism for F in a forward direction for normal numbers. Let F' be the restriction of F to the forward or left-to-right rules only. We have then shown the following.

Lemma 2 *There is a one-to-one relationship between the confluence problem for R and the confluence problem for F′ restricted to normal numbers.*

In addition, Theorem 1 establishes that the confluence problem for F' is of the same one-one degree as the word problem for F, where each is restricted to normal numbers. Thus we may conclude

Lemma 3 *The word problem for F restricted to normal numbers is of the same one-one degree as the confluence problem for R.*

We now show that questions about abnormal numbers in F (and F') are either trivially decidable or are reducible to questions about normal numbers.

Let α be an abnormal number. Then α is abnormal due to one of the following disjoint set of reasons.

1. α is not divisible by any p_i, for $n \leqslant i \leqslant m + n$.

2. α is divisible by p_rp_s, where $n \leqslant r \leqslant m + n$ and $n \leqslant s \leqslant m + n$.

3. α is the product of some y and r where y is normal and r is not divisible by any p_i with $0 \leqslant i \leqslant m + n$.

Case 1: If α is not divisible by any p_i, with $n \leqslant i \leqslant m + n$, then none of the rules can be applied, and so α can derive only itself.

Case 2: If α is divisible by $p_r p_s$, where $n \leqslant r \leqslant m+n$ and $n \leqslant s \leqslant m+n$, then α is of the form xyz, where x is not divisible by any prime p_i where $i \geqslant n$, y is not divisible by any prime p_j when $j < n$ or $j > m+n$, and z is not divisible by any prime p_i where $i \leqslant m+n$. Possibly $z = 1$. Let K be the number of prime divisors of y greater than 1 (K must be at least 2). K is the number of factors representing rules. α then derives by Rule Set 1 any β, where $\beta = uvz$; u is not divisible by any prime p_i where $i \geqslant n$, v is not divisible by any prime p_i where $j < n$ or $j > m+n$, and v has k prime divisors.

Case 3: If $\alpha = yr$, where y is normal and r is not divisible by any prime p_i, where $i \leqslant m+n$, then by Rule Set 2 α derives any $\beta = xr$ where y derives x. Case 3 thus reduces to questions about normal numbers.

As a result, derivability questions about abnormal numbers are either trivial or they reduce to questions about normal numbers. Combining this with Lemma 3 and the fact that every nonrecursive r.e. many-one degree is represented by the general confluence problem for n-register machines, we have proven the following.

Theorem 3 *Every nonrecursive r.e. many-one degree is represented by the general word problem for bidirectional RPCF's.*

Bidirectional Systems with Axiom We will show in this section that our results also hold for bidirectional extensions of systems with axiom. We start by showing that every nonrecursive r.e. many-one degree is represented by the decision problem for bidirectional RCPF's with axiom. To do this, we will first demonstrate that a slightly nonstandard version of the n-register machine yields the same many-one degree results.

We will use as a basis the halting problem for register machines. This was shown by Shepherdson [6] to represent every nonrecursive r.e. many-one degree. Given R, an arbitrary n-register machine with k rules, we construct R' by adding the rules $k+1, k+2, \ldots, k+n$, where rule $k+i+1$ is $\mathrm{SUB}_i(k+i+1, k+i+2)$ for $0 \leqslant i < n$.

Rule $k+n+1$ can be regarded as the terminal rule. Thus if the terminal state is reached in R', all the registers will have been zeroed out.

Lemma 4 *Every nonrecursive r.e. many-one degree is represented by the halting problem for the revised n-register machines (that is, by n-register machines that zero all registers before halting).*

Given a revised n-register machine R, we present a method of constructing an RPCF F and axiom A such that the halting problem for R is of the same many-one degree as the decision problem for F with axiom A. We will then show how the inverse rules for F can be added so that the same result will hold for the bidirectional RCPF.

Let R be an arbitrary revised n-register machine with m rules. Let p_i denote the i^{th} prime, with $p_0 = 2$. The axiom for the desired RPCF is p_{m+n+1} and the rules are the following:

a. If rule h of R is $\mathrm{ADD}_i(j)$ then add the rule

$$p_{j+n-1} p_i X \rightarrow p_{h+n-1}$$

b. If rule h of R is $\mathrm{SUB}_i(j,k)$ then add the rule

$$p_{j+n-1}X \to p_{h+n-1}p_iX$$

and the rules

$$p_{k+n-1}p_iX + p_{k+n-1}t \to p_{h+n-1}p_iX + p_{h+n-1}t, \text{ for each } t,\ 1 \leqslant t < p_i.$$

Define configurations of R, normal numbers, and $G(Z)$ as before. The axiom will be the word $p_{m+n+1}p_0^0p_1^0 \ldots p_{n-1}^0 = p_{m+n+1}$. Thus F will simulate R backwards from the configuration $(m + n + 1, 0, 0, \ldots, 0)$.

Let W derive $(m + n + 1, 0, 0, \ldots, 0)$ in R. The derivation proceeds by a sequence of rule applications $g_1, g_2, \ldots, g_n$. The rules of F that will be applied correspond to the rules of R in the order $g_n, \ldots, g_2, g_1$. Thus if the configuration W derives the configuration $(m + n + 1, 0, 0, \ldots, 0)$ in R, then p_{m+n+1} derives $G(W)$ in F.

Let p_{m+n+1} derive $G(W)$ in F. The sequence of steps $h_1, h_2, \ldots, h_j$ in this derivation corresponds to the rules $h_1', h_2', \ldots, h_j'$ in R, and applied to W in the reverse order, starting with h_j' in R, yields the configuration $(m + n + 1, 0, 0, \ldots, 0)$. Thus the following has been shown.

Lemma 5 *p_{m+n+1} derives $G(W)$ in F iff W derives $(m + n + 1, 0, 0, \ldots, 0)$ in R.*

We now show that adding the inverses of rules F yields nothing additional, so that $[p_{m+n+1}, G(W)]_{F\cup F^{-1}}$ iff W derives $(m + n + 1, 0, 0, \ldots, 0)$ in R.

As shown before, the inverses of the rule set are deterministic. Thus if $[p_{n+m+1}, G(W)]_F$, $[G(W), G(W')]_F$ and $(G(W'), G(W''))_{F^{-1}}$ then $W \equiv W''$. Since it is impossible to go back past the axiom, then $[p_{n+m+1}, G(W)]_F$ iff $[p_{n+m+1}, G(W)]_{F\cup F^{-1}}$.

Theorem 4 *Every nonrecursive r.e. many-one degree is represented by the general decision problem for bidirectional RCPF's with axiom.*

Degree results as shown above can be easily obtained for tag systems and, consequently, for Post normal systems with axiom. The basis for our proof lies in the constructions presented in Hughes [2]. There, revised n-register machines were used to prove the following lemma.

Lemma 6 *Let m be an arbitrary nonrecursive r.e. many-one degree. Then there exists a tag system T and a fixed word A such that the problem to decide of an arbitrary word A whether or not W derives A is of degree m. Furthermore, T and A may be chosen so that A is a terminal word.*

Using the above we can now prove our final result.

Theorem 5 *Every nonrecursive r.e. many-one degree is represented by each of the general decision problems for the bidirectional extensions of tag and Post normal systems with axiom.*

Proof: For an arbitrary r.e. many-one degree m, let T and A be chosen as in Lemma 6 and let S be the bidirectional extension of T. With T^{-1} denoting the inverse rules of T, the deterministic nature of T's rules ensures that $[A,W]_{T^{-1}}$,

$(W,W')_{T^{-1}}$ and $(W',W'')_T$ implies $W \equiv W''$. Combining this with the fact that A is terminal in T, we get that $[A,W]_{T^{-1}}$ iff $[A,W]_S$. But then $[A,W]_S$ iff $[W,A]_T$ and our proof is complete.

REFERENCES

[1] Hosken, W. H., "Some Post canonical systems in one letter," *BIT*, vol. 12 (1972), pp. 509-515.

[2] Hughes, C. E., "Many-one degrees associated with problems of tag," *The Journal of Symbolic Logic*, vol. 38 (1973), pp. 1-17.

[3] Overbeek, R., "The representation of many-one degrees by decision problems of Turing machines," *Proceedings of the London Mathematical Society*, vol. 26 (1973)., pp. 167-183.

[4] Overbeek, R., "The representation of many-one degrees by the word problem for Thue systems," *Proceedings of the London Mathematical Society*, vol. 26 (1973), 184-192.

[5] Post, E. L., "Recursive unsolvability of a problem of Thue," *The Journal of Symbolic Logic*, vol. 12 (1947), pp. 1-11.

[6] Shepherdson, J. C., "Machine configuration and word problems of given degree of unsolvability," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik,* vol. 11 (1965), pp. 149-175.

[7] Thue, A., "Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln," *Skrifter utgit av Videnskapsselskapet i Kristiania, I. Matematisk-naturvidenskabelig klasse 1914*, no. 10, 34 pages.

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