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GENERALISED LOGIC II

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1 This paper is a continuation of [1] in which generalised sentential logic is fully developed in a sequence of axiom systems designated *GL0* to *GL5*. In Section 2 a minor adjustment is made to the system of [1] to form the system *GL0*, and *GL1* is then formed by adding an axiom implicit in the discussion in [1]; *G.L.0* and *G.L.1* are further variants. The next two sections break new ground by adding axioms to greatly strengthen sentential generalised logic: the resulting systems are *GL2*, *G.L.2*, and *GL3*, *G.L.3*. In Section 5 it is shown that *G.L.3* captures a conventional five-valued logic, *C.L.3*, based on truth tables and that a further five-valued logic, *C.L.5*, is characteristic of an arbitrary extension of *G.L.3* designated *G.L.5*. This result is used to prove consistency and further metatheorems about the earlier systems. In Section 6 the five-valued analysis is used for further developments pointing beyond the scope of the paper.

Theorems of some system, say *GLx*, are designated "xT..." (thus the theorems of [1] become 0T...). The designation "xT..." implies that I do not believe xT... is a theorem of a weaker system of the paper than *GLx*, but not that I have proved this. Metatheorems are designated "MT...", Heyting's sentential logic "HL", Boolean sentential logic "BL", and generalised logic (any system) "GL". Expressions of the form "*x*(..*y*..)" (e.g., "*N*(..?..)", "?(..?..)",) are used to designate kinds of formula within which there are occurrences of the monadic operator *y* dominated, in a subformula or the whole formula, by the monadic operator *x*.

2 The systems discussed in this section are *GL0*, *GL1*, *G.L.0*, and *G.L.1*, and the axioms discussed are:

ECfgEKfgf . . . . A19. $\qquad$ *E$\underline{N}$fEfKsNs* . . . . A20.

[1] includes the definition D1, *Cfg* = *EKfgf* and this blocks the full development of GL. The reason is that a definition sanctions interchangeability of the definiens and the definiendum in all contexts and so, for example, D1 gives *E?Cpq?EKpqp*, which by A22 and A24 (discussed in **3**) is not a thesis of GL. Arbitrary definitions are, of course, admissible, but D1 is not arbitrary because, for example, *C* is derivable from subordinate

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proofs. D1 will be replaced by A19 and this modified system is designated *GL0*. It is emphasised that this change does not invalidate any of the theorems listed in [1], which now become the series 0T.... I must here, however, correct a slip in [1]; on p. 39, last paragraph, the last word of the first sentence should be "degenerate", not "inconsistent" (but the converse referred to cannot be proved anyway).

It was remarked in [1] that exclusion is more general and weaker than falsity; it is now pointed out that it can be interpreted as a weaker kind of negative. From 0T17 = *EKpNpKqNq* it follows that *EfKsNs* asserts no property specific to *s*, but a monadic property of *f*, viz., that it is not the case. This is designated by a new negative, "$\underline{N}$", and for the same reason that D1 is replaced by A19, $\underline{N}$ is not introduced by a definition, but by A20. *GL0* with A20 is *GL1*. It may be found helpful to verbalise "*N*" as 'false', "$\underline{N}$" as 'non', and to avoid the use of 'not'.

The nature of the segment of GL, in *g.v.s* only, is concealed by the fact that A9 to A12 and A20 are formulated in mixed *g.v.s* and *r.v.s*. One may, however, prove the following equivalent theses (in which *Eff* is used because it is the shortest thesis) and I note the equivalent axioms in parentheses: 0T66(A9), *EKEffgg*; 1T2 (A10), $EAKf\underline{N}fgg$; 1T3 (A11), $EKKf\underline{N}fgKf\underline{N}f$; 0T67 (A12), *EAEffgEff*; 1T4 (A20), $E\underline{N}fEfKg\underline{N}g$. These theses with A1 to A8, A19, and the rules, exhaust the logic of the segment of *GL1* in *g.v.s* only (in which *N* can only occur as a substitution instance).

MT1 *The segment of GL1 in g.v.s only and without N is* HL (*and without* $\underline{N}$ *also, is the Full Positive Logic*).

The proof is an easy exercise if one works from standard sets for HL and the Full Positive Logic and includes the well-known fact that subordinate proof is valid in these systems, so that it can be used both ways in the derivations. In view of MT1 a simpler basis could now, of course, be given for the segment of *GL1* in *g.v.s* only.

In [1] p. 37, a distinction was made between the logic of vagueness and the logic of truth and falsity. This was upheld by the distinction between *g.v.s* and *r.v.s* and by the restricted substitution rule. Three grounds could be cited for the distinction: firstly, that the interpretation of ??, *N*?, and more generally, formulas of the kinds ?(..?..) and *N*(..?..) is dubious; secondly, that therefore intuitive arguments for axioms in *r.v.s* would lack force if wffs in ? could be substituted under ? or *N*; and, thirdly, that freedom of substitution would probably, therefore, lead to inconsistency. The status of $\underline{N}$ is very similar. The interpretation of $?\underline{N}$ and $N\underline{N}$ is dubious, for A20 does not sanction $E?\underline{N}p?EpKsNs$ or $EN\underline{N}pENEpKsNs$ and, moreover, these could not be theses because, as will be seen later, *?EpKsNs* and *NEpKsNs* are not monadic properties of *p*—they turn out to be equivalent to *A?p?KsNs* and *KpAsNs*, respectively. More generally, of the nine kinds of wff *x*(..*y*..) in which *x*, *y*, are $\underline{N}$, *N*, or ?, five can be interpreted, but I can offer no interpretation for the four kinds ?(..?..), *N*(..?..), ?(..$\underline{N}$..), and *N*(..$\underline{N}$..). These four kinds of formulas will be called "semantically dubious".

These observations will be implemented by adding $\underline{N}$ to ? in the definition of type 2 formulas, so that formulas in both $\underline{N}$ and ? are debarred from (direct) substitution for *r.v.s*. Contrary to expectation, however, it is shown in Section 5 that the systems *GL...* remain consistent if the distinction between *g.v.s* and *r.v.s* is dropped and freedom to substitute all wffs for the variables is allowed. These generalised systems are designated *G.L.0*, *G.L.1*, etc., and in theorem lists the theorems that do not belong to the ungeneralised systems are placed at the end and marked with an asterisk. The following metatheorem shows, however, that limited substitution of formulas in $\underline{N}$ is allowable even in the ungeneralised systems.

MT2 *In the GL... series of systems, wffs containing occurrences of* $\underline{N}$ *can be substituted for r.v.s, provided that the wff to be substituted is not semantically dubious and that no occurrences of the r.v.s in question are dominated by N or ? in the wffs in which they occur.*

Proof: Let the arguments of the disputed occurrences of $\underline{N}$ be $F_1, F_2, \ldots F_x, \ldots$. As the wff to be substituted is not semantically dubious, one may put EF_xKsNs in every position where the substitution of $\underline{N}F_x$ is disputed, for every F_x. But provided the conditions of the theorem are satisfied, A20 can by an obvious recursion be used to obtain an equivalent formula in which EF_xKsNs is replaced by $\underline{N}F_x$.

Setting aside mere substitution instances of HL, the following list of theorems of *GL0* and *GL1* continues the list in [1].

0T63.	*EK?Apq?KpqK?p?q.*	0T64.	*EA?Apq?KpqA?p?q.*
0T65.	*CpCqENpNq.*	0T66.	*EKEffgg.*
0T67.	*EAEffgEff.*	0T68.	*CKEpNpEqNqKKEpqE?p?qENpNq.*
1T1.	$EKf\underline{N}fKsNs.$	1T2.	$EAKf\underline{N}fgg.$
1T3.	$EKKf\underline{N}fgKf\underline{N}f.$	1T4.	$E\underline{N}fEfKg\underline{N}g.$
1T5.	$CNp\underline{N}p.$	1T6.	$Cp\underline{N}Np.$
1T7.	$\underline{N}KpNp.$	1T8.	$CpE\underline{N}pNp.$
1T9.	$C\underline{N}?pE\underline{N}pNp.$	1T10.	$CpCqE\underline{N}pNq.$
1T11.	$CK\underline{N}?p\underline{N}?qK\underline{N}?Apq\underline{N}?Kpq.$	1T12.	$EK\underline{N}p\underline{N}NpKK\underline{N}p?p\underline{N}Np.$
1T13.	$EK\underline{N}p\underline{N}NpEpNp.$	*1T14.	$CpENpN\underline{N}\underline{N}p.$
*1T15.	$CE\underline{N}pN\underline{N}pENpN\underline{N}\underline{N}p.$	*1T16.	$\underline{N}KpN\underline{N}\underline{N}p.$
*1T17.	$CE\underline{N}pN\underline{N}p\underline{N}EpNp.$		

3 The systems discussed in this section are *GL2* and *G.L.2*, and the axioms discussed are:

ENEpqAKNpqKpNq A21	*E?EpqA?p?q* A22
ENCpqKpNq A23	*E?Cpq?q* A24

The intuitive basis of A1 to A20 was strong, that of A21 to A23 continues fairly strong, but that of A24 is weak and complicated. However, one may reasonably point out that these are the only axioms in *NE*, *?E*, *NC*, and *?C* and are therefore the sole explication of the meaning of these functions, which confers a measure of liberty in their choice, provided the choice is intelligible. This will be shown for A24 and also that it satisfies a number of desirable requirements; moreover, it proves satisfactory in the later development of GL.

A21 and A23 call for little comment. Both are theses of BL and intuitively persuasive, but it is worth mentioning that among such candidates there are a number of nonstarters, because in GL $EpNq$ does not entail either $ENpq$ or $NEpq$. A22 is also clear; for firstly, if Epq is vague it would be odd not to hold that so must be one of its arguments, and, secondly, if say p is vague, the axiom formulates the intuition that Epq must be vague because one cannot have a nonvague equivalence to a vague sentence.

As to $?Cpq$ one may reasonably demand satisfaction of the following requirements: (a) If $?Cpq$ then at least one of its arguments is vague and a sufficient condition for $?Cpq$ must include for this. (b) There are two functions of p and/or q, F_1, F_2, such that $C?CpqF_1$ and $CF_2?Cpq$ are the only axioms in $?Cpq$, i.e., the system includes a necessary and a sufficient condition for $?Cpq$. The following are also desirable: (c) $F_1 = F_2$, i.e., $E?CpqF$ is an axiom. (d) A nonvague conclusion may not be drawn from premises all of which are vague. (e) A vague conclusion may not be drawn from premises all of which are nonvague. (f) $?Cpq$ should be stronger than $?Epq$, indeed $E?EpqA?Cpq?Cqp$ is outstandingly plausible. (g) The axioms for $?Cpq$ should be simple and intelligible. (h) It is shown later, independently of A24, that there is a system in which the segment in *g.v.s* is strengthened from HL to BL: (d) and (e) should hold in that system.

One may reason as follows: 1. By (a), $C?CpqC\underline{N}?q?p$. 2. By (d), $p, Cpq, \rightarrow q$, is debarred when $?p$, $?Cpq$, and $\underline{N}?q$, therefore by (1), $C?CpqC\underline{N}?qK?pA\underline{N}p\underline{N}q$, and by (h), $C?CpqA?qK?pA\underline{N}p\underline{N}q$. 3. By (b) and (2), F_1 must be strong enough to secure $CF_1A?qK?pA\underline{N}p\underline{N}q$. 4. Similarly, by (e), $p, Cpq, \rightarrow q$, is debarred when $\underline{N}?p$, $\underline{N}?Cpq$ and $?q$, therefore by (h), $CKKKp\underline{N}?pq?q?Cpq$ and by (b) F_2 must be weak enough to secure $CKKKp\underline{N}?pq?qF_2$. 5. By (b), CF_2F_1, so (3) gives a necessary and (4) a sufficient condition for both F_1 and F_2, moreover, by (a) both F_1 and F_2 must contain $?p$ and/or $?q$ categorically, therefore: 6. Both F_1 and F_2 must take the form $AK?qF_3KK?pA\underline{N}p\underline{N}qF_4$, where F_3, F_4, are arbitrary functions with certain constraints on F_3. F_4 can be inconsistent (cancelling the second disjunct) but not F_3.

The obvious choice is $F_1 = F_2 = ?q$; indeed, it would be difficult to make any other choice simple and intelligible. The choice makes the antecedent condition for $?Cpq$ as weak and therefore as inclusive as possible, without unduly weakening the consequent and it gives 2T6 = $E?EpqA?Cpq?Cqp$. Intuitively, $E?Cpq?q$ emphasises the interpretation of Cpq as hypothetical inference. Thus, if q is vague the hypothetical inference to q is vague, irrespective of whether p is vague or not and, similarly, if q is nonvague, so is the inference to it. This is a simple and intelligible interpretation of $?Cpq$.

The systems *GL2* and *G.L.2* are formed by adding A21 to A24 to the systems *GL1* and *G.L.1*. A select list of theorems is given below; proofs are not difficult to find. To avoid cumbersome formulations, the following abbreviations are used in 2T29 to 2T34: "$(1)p$" for "$Kp\underline{N}?p$", "$(2)p$" for "$Kp?p$", "$(3)p$" for "$EpNp$", "$(4)p$" for "$K?pNp$", "$(5)p$" for "$K\underline{N}?pNp$". By 1T12, 1T13, $(3)p$ is equivalent to $KK\underline{N}p?p\underline{N}Np$ and for obvious reasons p will be said to be completely true iff $(1)p$, completely vague iff $(3)p$, and completely false iff $(5)p$. According to 2T29 to 2T33, $(1)p, (2)p, (3)p, (4)p, (5)p$, are mutually exclusive. According to 2T34 they are jointly exhaustive iff both p and $?p$ conform to the excluded middle law.

The truth tables given in Section 5 for N, A, K, C, and E can be interpreted as a summary of a further 105 theorems of *GL2*. For in those tables, if an entry for $|p| = i$, $|q| = j$, is $|Fpq| = k$, then $CK(i)p(j)q(k)Fpq$ is a theorem of *GL2*, e.g., $|p| = 4$, $|q| = 3$, $|Apq| = 3$, can be interpreted as $CKK?pNpEqNqEApqNApq$. Of course, many of these theorems are weak and not very interesting. The tables can also be used to form theorems that are functions of functions.

2T1. $E?p?Epp$.
2T2. $E?p?Cpp$.
2T3. $E?p?EpNp$.
2T4. $E?r?CpCqr$.
2T5. $E?Cpr?Cqr$.
2T6. $E?EpqA?Cpq?Cqp$.
2T7. $E?EpqA?Crp?Csq$.
2T8. $C?EpqA?Epr?Eqs$.
2T9. $E?Epq?EpNq$.
2T10. $E?EpqA?Apq?Kpq$.
2T11. $AA?EpqEpqEp\underline{N}q$.
2T12. $AA?CpqCpqCqp$.
2T13. $C?CpqC?CqrE?q?r$.
2T14. $E?Cpr?CqNr$.
2T15. $ENENpqNEpNq$.
2T16. $ENEpqNENpNq$.
2T17. $ENEpNpApNp$.
2T18. $ENEpqKApqNKpq$.
2T19. $CNCpqCqp$.
2T20. $CNCpqCNpq$.
2T21. $ENCpCqrKKpqNr$.
2T22. $ENEpqANCpqNCqp$.
2T23. $CKpqKNENpqNEpNq$.
2T24. $CKEpNpqEEpqNEpq$.
2T25. $EAKpqKNpNqNEpNq$.
2T26. $C\underline{N}CpqA?EpqNEpq$.
2T27. $CEp?pANpKCqp?Cqp$.
2T28. $CK\underline{N}?Crp\underline{N}?CsqAAEpqNCpqNCqp$.
2T29. $C(1)pKKK\underline{N}(2)p\underline{N}(3)p\underline{N}(4)p\underline{N}(5)p$.
2T30. $C(2)pKKK\underline{N}(1)p\underline{N}(3)p\underline{N}(4)p\underline{N}(5)p$.
2T31. $C(3)pKKK\underline{N}(1)p\underline{N}(2)p\underline{N}(4)p\underline{N}(5)p$.
2T32. $C(4)pKKK\underline{N}(1)p\underline{N}(2)p\underline{N}(3)p\underline{N}(5)p$.
2T33. $C(5)pKKK\underline{N}(1)p\underline{N}(2)p\underline{N}(3)p\underline{N}(4)p$.
2T34. $EAAAA(1)p(2)p(3)p(4)p(5)pKKAp\underline{N}pA?p\underline{N}?pANp\underline{N}Np$.
*2T35. $ENpNC\underline{N}pp$.
*2T36. $A\underline{N}EpNp\underline{N}E\underline{N}pN\underline{N}p$.

4 The systems discussed in this section are *GL3*, *G.L.3*, and the axiom discussed is:

$Af\underline{N}f$ . . . . A25.

The intuitive basis of A25 is best studied indirectly, by first studying 3T1 = $AAAA(1)p(2)p(3)p(4)p(5)p$. Spelled out, the theorem asserts that either p is completely true, or it is true and vague, or it is completely vague, or it is vague and false, or it is completely false. Intuitively, this seems all inclusive and therefore to impose no constraint on p, but this is not so (by 2T34). It might next be supposed that 3T1 were invalid, because incompatible with the vagueness of $(1)p, (2)p, (3)p, (4)p$, and $(5)p$, but in the system *G.L.5*, considered below, all these functions can be vague and 3T1 is a thesis; moreover, all except $(3)p$ can be completely vague and by 1T12, 1T13, $(3)p$ is equivalent to $KK\underline{N}p?p\underline{N}Np$, which can be completely vague and by Ru5 can be substituted for $(3)p$ in 3T1. So there would seem to be good intuitive grounds for adopting 3T1 and the resulting system would still be a system of *GL*.

Further insight is to hand, for by 2T34 the adoption of 3T1 is equivalent to the adoption of $Ap\underline{N}p$ and $A?p\underline{N}?p$ as theses. Moreover, in the generalised sequence *G.L.1*, *G.L.2*, ... this is equivalent to the adoption of A25. Finally, the adoption of A25 is equivalent to the replacement of HL by BL as the system of *g.v.s*. Designating the new systems (*GL2*, *G.L.2*, with A25 added) "*GL3*", "*G.L.3*", the present and the earlier finding can be combined by remarking that *GL3* stands in the same relation to BL as *GL2* to HL: *GL2* utilises HL for its system of *g.v.s* and *GL3* utilises BL. HL and BL are not, of course, the only systems that could be used.

MT3 *The segment of GL3 in g.v.s only and without N is* BL.

It should be mentioned that given a sufficiently rich (perhaps over-rich) language, the liar-type paradoxes can be restored in *GL3*.

A conjecture of [1] can now be reformulated and proved:

MT4 *If $F(N)$ is a thesis of* BL *in N and in the variables* $p_1, p_2, \ldots . p_n, \ldots$ *then* $CKK\ldots\underline{N}?p_1\underline{N}?p_2\ldots\underline{N}?p_nF(N)$ *is a theorem of GL3.*

Outline proof: 1. $F(N)$ contains neither ? nor $\underline{N}$. Therefore, by repeated applications of A13, A14, A21, A23, and finally A17, it can be shown equivalent to $F'(N)$, in which all occurrences of "N" are single occurrences standing against variables. 2. Given $KK\ldots\underline{N}?p_1\ldots\underline{N}?p_n$, by repeated application of 1T9 = $C\underline{N}?pE\underline{N}pNp$, $F'(N)$ can be shown equivalent to $F'(\underline{N})$, in which $\underline{N}$ is substituted for N. 3. $F(\underline{N})$, in which $\underline{N}$ is substituted for N in $F(N)$, is a theorem of *GL3* by BL. Also, all the theses appealed to in (1) are valid in $\underline{N}$. Therefore $F'(\underline{N})$ is a theorem of *GL3* equivalent to $F(N)$.

MT5 *If $\underline{N}?p$ is added as an axiom to any of the systems GL1 to GL3 or G.L.1 to G.L.3, the system degenerates into* BL *with twin negatives.*

Discussion: It is easy to check that with the proposed addition, all axioms collapse into theses of BL and that the distinction between *g.v.s* and *r.v.s* becomes redundant: also, the axioms include a complete set for BL. $E\underline{N}pNp$ is a thesis, but both $\underline{N}$ and N remain in the system, though either can be treated as redundant: this situation is described as BL with twin negatives.

Plainly, A25 is primarily a supplement to the system of *g.v.s* and so introduces an important class of theorems that it is unnecessary to list, as they are merely substitution instances of BL. There is also, however, an enrichment of the logic of a variable, displayed below in 3T1 to 3T11, and some supplementation of the logic of two or more variables, displayed in 3T12 to 3T20. Finally, there is much enrichment of the formal relations between the two negatives, displayed in 3T21 to 3T31.

The difference between the *GL*... and the *G.L*.... series of systems can now be assessed. The theorems in $N(..\underline{N}..)$ collected in *1T14 to *1T17, *2T35, *2T36, and *3T21 to *3T31 are of theoretical interest, but I suggest that they leave this kind of formula semantically dubious. There is no comparable range of theorems in the other semantically dubious forms, $?(..\underline{N}..)$, $N(..?..)$, and $?(..?..)$, partly because there are no axioms in these forms and partly because all axioms in N and ? except A20 are equivalences carrying the operator in both members. Summarising, in the *G.L*... series the uninterpretable formal relation between the two negatives is more developed than in the *GL*... series and substitution of semantically dubious formulas for the variables is allowed.

3T1. $AAAA(1)p(2)p(3)p(4)p(5)p$.
3T2. $EpA(1)p(2)p$.
3T3. $E?pAA(2)p(3)p(4)p$.
3T4. $ENpA(4)p(5)p$.
3T5. $A\underline{N}p\underline{N}Np$.
3T6. $AEpNpE\underline{N}pNp$.
3T7. $AApNpEpNp$.
3T8. $EApNp\underline{N}EpNp$.
3T9. $E\underline{N}EpNpNEpNp$.
3T10. $AAEp?pEpNpE?pNp$.
3T11. $CApNpAEp?pE?pNp$.
3T12. $EEEpqNEpqAKEpNpqKEqNqp$.
3T13. $EEEpqNEpqAECpqNCpqECqpNCqp$.
3T14. $EECpqNCpqKpEqNq$.
3T15. $AEpNpKE\underline{N}pNpEp\underline{N}Np$.
3T16. $E?Epq\underline{N}K\underline{N}?p\underline{N}?q$.
3T17. $EK\underline{N}?p\underline{N}?q\underline{N}?Epq$.
3T18. $E\underline{N}?EpqK\underline{N}?Apq\underline{N}?Kpq$.
3T19. $E\underline{N}?EpqK\underline{N}?Crp\underline{N}?Csq$.
3T20. $C\underline{N}?EpqKACrpKrNpACsqKsNq$.
*3T21. $CN\underline{N}pp$.
*3T22. $AE\underline{N}pNpEpN\underline{N}p$.
*3T23. $ANEpNpNE\underline{N}pN\underline{N}p$.
*3T24. $CE\underline{N}pN\underline{N}pp$.
*3T25. $AE\underline{N}pN\underline{N}pEpN\underline{N}p$.
*3T26. $AEpN\underline{N}pENpN\underline{N}\underline{N}p$.
*3T27. $AEp\underline{N}NpENpN\underline{N}\underline{N}p$.
*3T28. $CKp\underline{N}?\underline{N}pN\underline{N}p$.
*3T29. $ENCp\underline{N}pN\underline{N}p$.
*3T30. $E\underline{N}EpNpNKp\underline{N}p$.
*3T31. $ENKp\underline{N}pNKpNp$.

5 *GL* will now be related to conventional many-valued logic. Without defining the latter, it is noted that it has conventional, i.e., discrete values, in contrast to the 'generalised' values of *GL* that are not all discrete, that 'quasi-values' as described in [2] are allowed, and that the logic is constructed by truth tables, presupposing therefore, freedom of substitution, as in the *G.L*.... series. A standpoint similar to [2] is assumed and some symbolism is borrowed from that source with cursory explanation.

$(1)p$, $(2)p$, $(3)p$, $(4)p$, $(5)p$, being mutually exclusive and jointly exhaustive, they will serve as specifications of the values of a five-valued conventional logic, provided suitable theses are available. For example, $CKKp?pK?qNqK?KpqNKpq$, which is a thesis of *G.L.3*, specifies that if $|p| = 2$ and $|q| = 4$ then $|Kpq| = 4$ (where $|p|$ is the value of p). In this way the following system, designated "*C.L.3*", may readily be derived from *G.L.3*.

				q	Apq	Kpq	Cpq	Epq
p	Np	$\underline{N}p$	$(?)p$	p	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5
+1	5	345	345	1	1 1 1 1 1	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5
+2	4	345	12	2	1 2 2 2 2	2 2 3 4 5	1 2 3 4 5	2 2 3 4 4
3	3	12	12	3	1 2 3 3 3	3 3 3 4 5	1 2 2 2 1	3 3 2 2 2
4	2	12	12	4	1 2 3 4 4	4 4 4 4 5	1 2 2 2 1	4 4 2 2 2
5	1	12	345	5	1 2 3 4 5	5 5 5 5 5	1 2 2 2 1	5 4 2 2 1

MT6 *G.L.3 captures C.L.3.*

Proof: 1. The above tables are derived from *G.L.3*. 2. We validate the derivation of further tables in *C.L.3*. Only dyadic functions are discussed, leaving monadic functions as a subsequent easy exercise. Case I: All inputs and outputs strict valued. Let $x_1', x_2', \ldots x_n', \ldots$ $(1 \leqslant x_n' \leqslant 5)$ be values, $x_1, x_2, \ldots x_n, \ldots$ the corresponding functions in *G.L.3* that specify these values and F_1, $F_2, \ldots$ any dyadic functions. In *C.L.3* a derivation takes the form: (a) if $|p| = x_1'$, $|q| = x_2'$, then $|F_1pq| = x_3'$ and $|F_2pq| = x_4'$; (b) if $|p| = x_3'$, $|q| = x_4'$, then $|F_3pq| = x_5'$; therefore (c) if $|p| = x_1'$, $|q| = x_2'$, then $|F_4pq| = |F_3F_1pqF_2pq| = x_5'$. In *G.L.3* the corresponding premise-theses are: (A) $CKx_1px_2qx_3F_1pq$ and $CKx_1px_2qx_4F_2pq$; and (B) $CKx_3px_4qx_5F_3pq$. Substituting in (B), $CKx_3F_1pqx_4F_2pqx_5F_3F_1pqF_2pq$ and therefore by (A), (C) $CKx_1px_2qx_5F_4pq$, validating (c). Case II: Quasi values. Initially these are outputs of monadic functions, but by recursion they become outputs of dyadic functions. Then in the derivation (a) to (c), x_1', x_2', are strict values, but x_3', x_4', x_5', may be quasi values. If x_n' is a quasi value, it is effectively a set of values and x_n the corresponding disjunction of functions. Then e.g., in (b) the set of pairs $x_3' \times x_4'$ is used as inputs to F_3, to obtain the output set x_5'. Correspondingly, in (B) the logical product of the disjunctions x_3p and x_4p (which in normal form corresponds to $x_3' \times x_4'$) implies the corresponding disjunction x_5F_3pq. Therefore (a) and (b) are the case iff (A) and (B) are the case and the argument under Case I is still valid. 3. By (1) and (2) a function always gives outputs of 1, 2, or 12, in *C.L.3* only if the corresponding thesis can be derived in *G.L.3*.

Let all strict and quasi values be classified into (a) designated, (b) undesignated, and (c) mixed, this last consisting of quasi values such as 25 or 234. Then it can be seen that *C.L.3* is not characteristic of *G.L.3*, because whereas it verifies all axioms for designated and for undesignated inputs, some axioms (e.g., A15, A20) give mixed outputs for mixed inputs and are therefore not tautologies of *C.L.3*. The position is clarified by noting that mixed values can only be obtained as outputs of formulas of the forms $?(\ldots\underline{N}\ldots)$, $N(\ldots\underline{N}\ldots)$, $?(\ldots?\ldots)$, and $N(\ldots?\ldots)$, (e.g., if $|p| = 3$, $|?\underline{N}p| = 12345$): for these are the semantically dubious forms which lack axioms in *G.L.3*. The obvious step is to formally complete $\underline{N}$ and ? (relative to five-valued conventional logic) by adding axioms to eliminate the quasi values. Now in *G.L.3* and *C.L.3*,? is obviously incomplete, but $\underline{N}$ is another matter, for $\underline{N}p$ merely asserts that p is not the case, i.e., that

$|p| = 345$ and it does not mean more than this; thus, working through the possible cases, $\underline{N}$ means as much and no more than is displayed in the table in *C.L.3*. $\underline{N}$ might be called "semantically complete" in *G.L.3* and *C.L.3* and any formal completion "artificial"; formal completion is nevertheless valuable.

In serving to specify values, the functions $(1)p$, $(2)p$, $(3)p$, $(4)p$, and $(5)p$, also specify a parametric operator, as shown on the left in the tables below. It is easy to show that Vvp with C and K will describe any truth table in strict values. Also, by experimenting with permutations of monadic operators and then combining these with dyadic operators, one can show that every quasi value is the output of some function of *C.L.3* for some combination of inputs, the full set being too lengthy to list here. A few examples are cumbersome, e.g., the shortest form that I can find for 135 is $|KAN?\underline{N}ppCN?\underline{N}qq|$ when $|p| = 3$ and $|q| = 5$. It follows that Vvp with C, K, $\underline{N}$, N and $(?)$ will describe all strict and quasi valued tables. These properties of *G.L.3* and *C.L.3* differ from autodescriptivity in [2] in that, firstly, they include for an axiom system and for quasi values and to achieve the latter, they include $\underline{N}$, N and $(?)$ and, secondly, the truth criterion in [2] is $EpV1p$, whereas in *GL* it is the weaker requirement $EpAV1pV2p$. We shall use the words "self-descriptive" and put:

MT7 *G.L.3 and C.L.3 are self-descriptive.*

	$v \backslash p$	Vvp 1	2	3	4	5
$V1p = Kp\underline{N}(?)p$	1	12	345	345	45	5
$V2p = Kp(?)p$	2	5	2	3	4	5
$V3p = EpNp$	3	5	4	2	4	5
$V4p = K(?)pNp$	4	5	4	3	2	5
$V5p = K\underline{N}(?)pNp$	5	5	45	345	345	12

p	$?p$	$\underline{N}_1p$	$\underline{N}_2p$	$\underline{N}_3p$	$\underline{N}_4p$
1	5	5	4	3	4
2	2	3	4	3	3
3	1	2	2	2	2
4	2	1	2	2	1
5	5	1	2	2	2

Formally, there are 72 completions of $(?)p$, but the choice can be narrowed. Firstly, it is desirable to retain MT5, because the logic should be applicable to a domain of nonvague sentences, but this entails making $?p$ normal, because otherwise $\underline{N}p$ and/or $\underline{N}Np$ can be derived from $\underline{N}?p$ by contraposition of the first and last entries of the table. This condition is captured by adding $C\underline{N}?p\underline{N}??p$, or by contraposition, 4T1, to the axioms. Secondly, it is desirable that $|?p|$ should be the same when $|p| = 2$ and $|p| = 4$. This leaves four alternatives and the most plausible is that shown above, which is captured by adding 4T2 = $EEpNpK?p\underline{N}??p$ to the axioms.

G.L.3 with 4T1, 4T2, added as axioms, will be designated "*G.L.4*" and the conventional system that it captures "*C.L.4*". $\underline{N}$ being semantically complete, *G.L.4* might be called a "semantical completion" of *G.L.3* relative to five-valued conventional logic and is, I suggest, the most satisfactory one, but from the standpoint of *GL* it is semantically dubious. Insofar as one is not committed to analysis in conventional logic, the problem is to find acceptable axioms in $?(..?..)$ and $N(..?..)$ and here one might start again with *GL2* or *GL3* and utilise their restricted substitution rule to sanction the addition of axioms inconsistent with the generalisation of the axioms in ? of these systems. This is a main reason for retaining both sequences of systems.

To choose an artificial completion for $\underline{N}$ we again retain MT5 and therefore a normal table, captured by adding $C\underline{N}?p\underline{N}?\underline{N}p$, or contrapositioning, 5T1 below, to the axioms. Probably the most plausible further addition that completes the table and one having interesting consequences discussed in Section 6, is 5T2 = $ENpN\underline{N}\underline{N}p$. When $|p| = 3$, $|Np| = 3 = |N\underline{N}\underline{N}p|$, therefore $|\underline{N}\underline{N}p| = 3$, while $|\underline{N}p| = 12$: but $|\underline{N}p|$ cannot be 1 as by 5T1 this gives $|\underline{N}\underline{N}p| = 5$, so $|\underline{N}p| = 2$ and therefore when $|\underline{N}p| = 2$, $|\underline{N}\underline{N}p| = 3$. The table is now 5, 3, 2, x, 1, with $x = 12$ and by again appealing to 5T2, x cannot be 2, so finally one obtains $\underline{N}_1$ above, captured by adding 5T1, 5T2, to *G.L.4* to form the system *G.L.5*, which captures *C.L.5*, the system formed by replacing $\underline{N}$ with $\underline{N}_1$ (in discussing *G.L.5*, either "$\underline{N}$" or "$\underline{N}_1$" may be used, according to context).

MT8 *C.L.5 is characteristic of G.L.5.*

Outline proof: 1. It has been shown that *G.L.5* captures *C.L.5*. 2. Use the $\underline{N}$-*A*-*K*-*C*-*E* segment of *C.L.5* to validate any convenient set for BL. This at once validates several axioms of *G.L.5*, all the rules and, in particular, subordinate proof. 3. The remaining axioms check against the tables of *C.L.5* (some are complicated and best tabulated). It is worth remarking that all the axioms of *G.L.5* give outputs of 2 for some inputs.

MT9 *The systems GL0 to GL5 and G.L.0 to G.L.5 are consistent.*

MT10 *The systems GL0 to GL5 and G.L.0 to G.L.5 do not degenerate into* BL (*nor, e.g., GL2, G.L.2 into* HL).

Proof: For degeneration, $\underline{N}?p$ would have to be a thesis, which it is not in *C.L.5* and *C.L.5* is adequate for the systems listed.

The following metatheorem is about those wffs that are not semantically dubious. These will be called Type 3 formulas and consist of all wffs except the kinds $N(..\underline{N}..)$, $?(..\underline{N}..)$, $N(..?..)$, and $?(..?..)$: a rigorous definition can be given, but is not helpful here.

Lemma *If a Type 3 formula is a tautology or quasi tautology of C.L.3, it is a thesis of GL3.*

Proof: By MT6 the formula is a thesis of *G.L.3*: we show that an evaluation in *C.L.3* can be arranged to correspond to a derivation in *GL3*. No mixed inputs need be considered, as these require substitution of non-Type 3 formulas, and as quasi inputs require substitution of formulas in ? and/or $\underline{N}$, they are limited to subformulas neither containing nor dominated by ? or N. Let $F_1, F_2, \ldots$ be subformulas. 1. For each subformula $?F_m$ calculate $|F_m|$ in *C.L.3* for each set of inputs. As F_m contains neither ? nor $\underline{N}$, this corresponds to a derivation in *GL3*. For each $|F_m|$, either $?F$ or $\underline{N}?F$ in *GL3* and therefore either $|?F_m| = 12$ or $|?F_m| = 345$ without further calculation, though the table in *C.L.3* gives the same result. 2. For each subformula NF_n proceed similarly and weaken $|NF_n|$ to 12 or 345 if NF_n is dominated by neither *N* nor ?, otherwise proceed to the highest occurrence of *N* or ?. 3. In subformulas neither containing nor dominated by *N* or ? weaken any strict inputs to 12 or 345. 4. Procedures (1) to (3) correspond to derivations in *GL3* and the resulting assignments of 12 and 345 are equivalent to the assignments in *C.L.3* and are assigned to every smallest subformula not dominated by ? or *N*. But by the tables for *C.L.3*, this suffices to assign 12 or 345 to the whole formula, this further calculation corresponding to a derivation in BL and therefore in *GL3*.

MT11 *For Type 3 formulas, GL3 is decidable and complete relative to five-valued conventional logic.*

Proof: Let x_1, x_2, be a pair of equivalent values or quasi values and y_1, y_2, another pair. 1. If x_1 is an input to both (?) and ? the two outputs are equivalent. 2. If x_1 is an input to $\underline{N}$ and x_2 to $\underline{N}_1$ the two outputs are equivalent. 3. If x_1, y_1, is an input to *A*, *K*, *C*, or *E* and x_2, y_2, another input to the same function, the two outputs are equivalent. 4. In evaluating the same Type 3 formula in *C.L.3* and *C.L.5* the input values to *N*, (?), and ? are strict and the input values to (?) and ? are the same, therefore: 5. By (1), (2), (3), the formula is a tautology of *C.L.5* iff it is a tautology or quasi tautology of *C.L.3*. 6. By (5) MT8 and the lemma, *GL3* is complete for Type 3 formulas and the truth tables of *C.L.3* or *C.L.5* provide a decision procedure.

Corollary *A Type 3 formula is a thesis of G.L.3 iff it is a thesis of GL3.*

MT12 *If either $\underline{N}?p$ or $\underline{N}Kp?p$ is added as an axiom to any system among GL3 to GL5 or G.L.3 to G.L.5, the resulting system is* BL *with twin negatives and the corresponding system among C.L.3 to C.L.5 degenerates correspondingly.*

Outline proof: 1. $\underline{N}?p$ has been discussed *passim* and presents no problems. 2. In the *C.L....* series, addition of $\underline{N}Kp?p$ requires immediate deletion of rows and columns 2 and 4 and as, if $|p| = 3$, then $|Epp| = 2$, row and column 3 must go as well. 3. As *C.L.3* is not characteristic of *G.L.3* and *C.L.4* is stronger, it is necessary to exhibit the corresponding proof for the *GL* and *G.L.* series; it is: [a. *EpNp*. b. *?p* (a). c. *?EpNp* (b, A22). d. *EEpNpEpNp* (0T1). e. *?EEpNpEpNp* (c, A22). f. *KE...?E...* (d, e). g. $\underline{N}KE...?E...$ (by $\underline{N}Kp?p$). h. *KsNs* (f, g).] i. $\underline{N}EpNp$.

MT12 foils any expectation that one might derive a three-valued conventional logic from *GL* by making $\underline{N}Kp?p$ an axiom. The affinities of *GL* are with two- and five-valued conventional logic, not with three-valued conventional logic.

4T1.	*C??p?p*. (Axiom of *G.L.4*)	4T2.	$EEpNpK?p\underline{N}??p$. (Axiom of *G.L.4*)
*4T3.	$E\underline{N}?pN?p$.	*4T4.	$C\underline{N}N?p?p$.
*4T5.	*E??p??Np*.	*4T6.	*E???p??p*.

*4T7. $E?Kp?p?p$.	*4T8. $E?Ep?p?p$.
*4T9. $\underline{N}E?pN?p$.	*4T10. $E??pAKp?pK?pNp$.
*4T11. $EEKp?pNKp?pEpNp$.	*4T12. $E?Ap?pAKp?pK?pNp$.
*4T13. $E?Ap?p?Cp?p$.	
5T1. $C?\underline{N}p?p$. (Axiom of *G.L.5*)	5T2. $ENpN\underline{N}\underline{N}p$. (Axiom of *G.L.5*)

6 In this section points are made about modality, truth functional completeness, and the generalisation of the number of values.

The monadic operators of *GL* are quasi modal. In *G.L.3* put L for $N\underline{N}$ and M for $\underline{N}N$, then $CLpp$, $CpMp$, $ELp\underline{N}M\underline{N}p$ and $EM\underline{N}\underline{N}p\underline{N}L\underline{N}p$ are theorems. To satisfy Łukasiewicz's criteria for modality, the last of these should entail $EMp\underline{N}L\underline{N}p$, but it is easy to show that this is so only if $ENpN\underline{N}\underline{N}p$ is a thesis, which requires (with the normality criterion) the replacement of $\underline{N}$ by $\underline{N}_1$, as noted above. Working in *C.L.5* one obtains:

p	$\underline{N}p$	$\underline{N}\underline{N}p$	Lp	LLp	$LLLp$	Mp	MMp	$MMMp$
1	5	1	1	1	1	1	1	1
2	3	2	3	4	5	1	1	1
3	2	3	4	5	5	2	1	1
4	1	5	5	5	5	3	2	1
5	1	5	5	5	5	5	5	5

As a modalisation of *G.L.5* this is inordinately strong, as the axioms of *G.L.5* all give outputs of 2 for some inputs. Stronger input functions are needed, calling attention to the point that because *C.L.5* is normal it is truth functionally incomplete. Probably the simplest addition to secure completeness is $\underline{N}_2$, specifiable by adding $E\underline{N}_2p\underline{N}_1p$ and $??\underline{N}_2p$ as axioms to *G.L.5*. As the tables for A and K are suitably behaved, to prove completeness we require (see [2]) the set of functions $\textcircled{i}$ (where $|\textcircled{i}p| = i$ regardless of input) and the set $\boxed{i}$ (where if $|p| = i$ then $|\boxed{i}p| = 1$ and if $|p| \neq i$ then $|\boxed{i}p| = 5$); they are as follows. $\boxed{1}p = LLLp$, $\boxed{2}p = ?KNLpN\boxed{3}p$, $\boxed{3}p = ?\underline{N}_1?\underline{N}_1p$, $\boxed{4}p = K?MpN\boxed{3}p$, $\boxed{5}p = \boxed{1}Np$, $\textcircled{1}p = N\textcircled{5}p$, $\textcircled{2}p = ?\underline{N}_2p$, $\textcircled{3}p = \underline{N}_1\textcircled{2}p$, $\textcircled{4}p = N\textcircled{2}p$, $\textcircled{5}p = ?\boxed{1}p$. The resulting system offers scope for constructing functions to provide suitable inputs to L and M, while preserving a meaningful relation to *C.L.5* and *G.L.5* and therefore to GL.

By abandoning normality and intuitive plausibility, truth functional completeness can be obtained with a single weak negative. $\underline{N}_3$ is axiomatically simple, it is obtained by adding $?\underline{N}p$ and $CpE\underline{N}pN\underline{N}p$ to *G.L.4*. $\boxed{1}p = KKKpN\boxed{2}pN\boxed{3}pN\boxed{4}p$, $\boxed{2}p = ?KK\underline{N}_3p??p??\underline{N}_3\underline{N}_3p$, $\boxed{3}p = ?Kp??K\underline{N}_3p\underline{N}_3Np$, $\boxed{4}p = ?K\underline{N}_3NpK??Np??\underline{N}_3\underline{N}_3Np$, $\boxed{5}p = KKKNpN\boxed{2}pN\boxed{3}pN\boxed{4}p$, $\textcircled{1}p = ?\textcircled{3}p$, $\textcircled{2}p = A\underline{N}_3p\underline{N}_3\underline{N}_3p$, $\textcircled{3}p = K\underline{N}_3p\underline{N}_3\underline{N}_3p$, $\textcircled{4}p = N\textcircled{2}p$, $\textcircled{5}p = N\textcircled{1}p$. $\underline{N}_4$ gives a relatively simple Post negative, it is $AN\underline{N}_4p?KYYp?AYYpYYYp$, where $Y = \underline{N}_4Np$.

The formal structure of the tables for C and E is unfamiliar and merits study. The following instructions for constructing the tables are given for $n = 2k + 1 > 4$ values, of which j are designated, where $n > 2j$. Below, the instructions are exemplified for $n = 5$: entries in accordance with instruction (3) are placed in single parentheses (. . .), those in accordance with (4) in double parentheses ((. . .)), and for quasi values the final choice is underlined. The notion of 'degree of vagueness' will be used, which loosely expressed is the number of places of the value from the nearest end of the sequence 1, 2, ... n and expressed precisely is $(\frac{1}{2}(n - 1) - |\frac{1}{2}(n + 1) - |p||)$.

1. Form the three pairs of values and quasi values: (a) $1, n$ (e.g., 1, 5); (b) $12 \ldots j, (n - j + 1)(n - j + 2) \ldots n$ (e.g., 12, 45); (c) $12 \ldots j, (j + 1)(j + 2) \ldots n$ (e.g., 12, 345). The first member of each pair is called its 'T-analogue' and the second its 'F-analogue'. Any strict value occurring in either member of a pair will be called an 'element' of that pair. 2. In the tables for both C and E and for all inputs such that both $|p|$ and $|q|$ are elements of pair (a), use pair (a) for the entries in accordance with the normality criterion, i.e., use the T and F analogues as in BL. 3. Repeat for pair (b), the entries being quasi values. For $|Cpq|$ choose the element of each entry whose degree of vagueness is nearest that of $|q|$ and for $|Epq|$ the nearest to whichever of $|p|$ and $|q|$ is most vague. 4. Repeat (2) for pair (c). For designated entries choose j. Undesignated entries will include $|p|$ or $|q|$ but not both; choose this.

$p \backslash q$	Cpq 1	2	3	4	5	Epq 1	2	3	4	5
1	1	$(1\underline{2})$	$((\underline{3}45))$	$(\underline{4}5)$	5	1	$(1\underline{2})$	$((\underline{3}45))$	$(\underline{4}5)$	5
2	$(\underline{1}2)$	$(1\underline{2})$	$((\underline{3}45))$	$(\underline{4}5)$	$(4\underline{5})$	$(1\underline{2})$	$(1\underline{2})$	$((\underline{3}45))$	$(\underline{4}5)$	$(\underline{4}5)$
3	$((\underline{1}2))$	$((1\underline{2}))$	$((1\underline{2}))$	$((1\underline{2}))$	$((\underline{1}2))$	$((\underline{3}45))$	$((\underline{3}45))$	$((1\underline{2}))$	$((1\underline{2}))$	$((1\underline{2}))$
4	$(\underline{1}2)$	$(1\underline{2})$	$((1\underline{2}))$	$(1\underline{2})$	$(\underline{1}2)$	$(\underline{4}5)$	$(\underline{4}5)$	$((1\underline{2}))$	$(1\underline{2})$	$(1\underline{2})$
5	1	$(1\underline{2})$	$((1\underline{2}))$	$(1\underline{2})$	1	5	$(\underline{4}5)$	$((1\underline{2}))$	$(1\underline{2})$	1

MT13 *C.L.3 to C.L.5 can all be generalised to analogues having the same tautologies and quasi tautologies and having any number of values n satisfying* $n = 2k + 1 > 4$ *and j designated values satisfying* $j > 1$ *and* $n > 2j$.

Proof: The analogues are to include the original systems. Call value 1 a type 1 value, values 2 to j type 2 values, values $j + 1$ to $n - j$ type 3 values, values $n - j + 1$ to $n - 1$ type 4 and value n type 5. Case I: *C.L.5*. 1. Use the construction given above for C and E and the usual generalisations of A and K. Consider any entry of any analogue, suppose the inputs are of types g and h and the output of type i; then the output is of type i for every other pair of inputs of types g and h, both in that and every other analogue. 2. Similarly for the usual generalisation of N, if one type h input gives a type i output, so does every other. For ? and $\underline{N}$, the simplest output assignments having the same property for inputs of types 1, 2, 3, 4, 5, are for $|?|$, $n, j, 1, j, n$, respectively, and for $|\underline{N}|$, n, $\frac{1}{2}(n + 1), j, 1, 1$, respectively. 3. By (1) and (2) a function gives outputs of types 1 and/or 2 for all inputs in one analogue iff it does so in every other. Case II: *C.L.3* and *C.L.4*. The position is not essentially altered, because the quasi values of $\underline{N}$ and (?) are simply the sequence of all types 1 and 2 values and the sequence of all types 3, 4, and 5 values. In $\underline{N}$ the latter is the output for the first j values and the former for the remainder, in (?) the latter is the output for values 1 and n and the former for the remainder.

Returning to *GL* itself, with its three generalised values, one might study the generalisation of *GL* to $n > 3$ values and the question of the constraints to be placed upon the permutations of values of a single sentence to secure consistency. This raises too many problems to treat here.

APPENDIX

Two alternatives to *G.L.5/C.L.5* are less plausible, but merit brief mention for their technical interest. They are called *SyA* and *SyB* and in both cases the designation refers ambiguously to the truth tables or to the axiom system, as the one is characteristic of the other. In what follows, strings of digits are used for value tables, not for quasi values.

In both systems ? is 52225: in *SyA* $\underline{N}$ is 54221 and in *SyB* $\underline{N}$ is 53221. To capture either system by addition to *G.L.3*, axioms 4T2 and 5T2 must (unplausibly) be rejected and axioms 4T1 and 5T1 strengthened to $E??p?p$ and $E?\underline{N}p?p$. Then to obtain *SyA* add $\vdash \underline{N}E\underline{N}pN\underline{N}p$ and to obtain *SyB* add $\vdash CKp?p\underline{N}N\underline{N}p$. Both systems are adequate for *G.L.3*, although ? only takes two values and in *SyA* $\underline{N}$ only differs from N in one value. The addition of $\vdash \underline{N}?p$ to either system captures the familiar BL in 1, 5.

The addition of $\vdash \underline{N}EpNp$ to *SyA* captures one of many systems in 1, 2, 4, 5, derivable from *C.L.3* (there are 256 of them). The addition of $\vdash ?p$ to *SyA* captures one of two systems in 2, 3, 4, derivable from *C.L.3*: in it, ? is not specific to the middle value, but is the tautology function 222. The addition of both $\vdash \underline{N}EpNp$ and $\vdash ?p$ to *SyA* or to *G.L.3* captures BL with twin negatives in 2, 4 (the only two-valued system derivable from *C.L.3*, except BL in 1, 5). To that extent, *GL* for a domain of vague but not completely vague sentences is BL: but this result cannot be obtained from *G.L.5/C.L.5* or from *SyB*.

The addition of $\vdash ?p$ to *SyB* captures the remaining system in 2, 3, 4, and again ? is 222. This system is truth functionally complete and is a varient, quoted in [2], p. 341, of Słupecki's system S_3'', see [3]. An alternative derivation is the addition of $\vdash ?p$ and $\vdash \underline{N}N\underline{N}p$ to *G.L.3*. But $\underline{N}N\underline{N}p$ is the exclusion of a semantically dubious function and it is easy to show that by this with $\vdash ?p$ all semantically dubious forms are either excluded or rendered nugatory. To that extent, *GL* for a domain of vague and non-dubious sentences is S_3'': but this result cannot be obtained from *G.L.5/C.L.5* or from *SyA*.

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