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A COMPLETE SYSTEM OF INDEXICAL LOGIC

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In most logical systems, the interpretation of a term or formula does not depend on a situation or context of discourse. In particular, there are no temporal contexts since all sentences are formulated in what seems to be an eternal present tense. Suppose, however, that terms and formulas with terms as situational indices as well as ordinary terms and formulas are present in a first-order logic with identity, descriptions, and no existence assumptions. Suppose also that intensional constants such as those of tense and modal logics are present as nonlogical constants so as to make possible the concise expression of certain relationships among situations. How can such a broad kind of language be formalized? What kind of semantic theory can interpret it? And what kind of logic is determined by such a semantic theory? One set of solutions to these problems is presented here.*

In the literature, the most closely related systems seem to be the "topological" logics. There, rules and interpretations for sentential formulas indexed with individual constants and first-order variables have been investigated. However, a full first-order logic and semantics of even standard type for indexing with only individual constants and variables seem to have escaped explicit formulation. For a survey of the subject, the reader is referred to Rescher and Urquhart [9]. A recent study in the area is Garson [6].

The system and semantics of the present study are also pragmatic and intensional in the sense of Montague's [7] and [8]. Although Montague has developed appropriate semantic theories in these papers and has been matched analogously by Scott in [14], no full deductive system of pragmatics or full intensional logic seems to have been published before. However,

*The main results of the present study were presented in a talk with the same title at the Royal Institute of Technology in Stockholm in May 1973. With the exception of some additions to the informal remarks, references, and introduction, the study was also presented in full at the Salzburg Colloquium in Logic and Ontology in September 1973. An abstract with completeness results was also published in the *Bulletin of the Section of Logic*, vol. 5 (1976), pp. 16-19.

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from Montague [8], Scott [14], and Gallin [5], it seems that David Kaplan and Dan Gallin have constructed some systems of this kind and proven them to be complete. Unfortunately, the author has no additional information about these matters. Another author who has developed context-dependent semantic and logical theories is Cresswell in [4]. However, there are no indexed formulas and no first-order structure.

The systems developed in the present study have evolved from those of Schock [11] and [12]. They differ from all of the above systems in the breadth of the collection of constants of odd types which are dealt with. All sorts of queer modal connectives and variable binders are present at least as nonlogical constants. Also, they are full first-order systems with only one sort of variables. The systems are free from existence assumptions and deal with existence, actuality, and non-actuality or fictiveness in addition to identity and descriptions. Only one kind of quantifiers is employed and a situationless present is provided for. This is a natural tense of mathematical and other abstract reasoning which deserves explication and is formally useful for handling situational expressions. Some additional differences from the systems of Montague or Scott are that the situational objects (indices or reference points) can be named and reasoned about in the object language, that there are two intensional logical constants for indexing rather than a strong logical constant of necessity, and that validity is more general since it does not require truth in all situations. The semantic theory also seems to be somewhat more direct and transparent than those of Montague and Scott in that fundamental reference is to truth values and objects rather than intensions. Another novelty seems to be the semantic and logical rules for the constants for indexing terms and formulas with terms. The rules are, of course, not too surprising to those with keen intuitions, but appear everywhere and are mostly new.

1 *Symbols, terms, and formulas* We presuppose a nonrepeating denumerably infinite sequence S of nameable objects called *symbols*. The symbols with even indices determine a similar sequence V of (individual) *variables*, and the remaining symbols a similar sequence K of *constants*. In the same way, nonrepeating denumerably infinite sequences T and F of *term-making* and *formula-making* constants are obtained from the even-indexed and remaining values of K, respectively. By partitioning T and F via even and noneven a few more times, we obtain the nonrepeating denumerably infinite sequences $Tklm$ and $Fklm$, where k through m are natural numbers and one of l or m is positive if k is. The values of $Tklm$ and $Fklm$ are the *k-place l-term m-formula term-making* and *formula-making constants* respectively. A constant is of *type klm* just when k through m are natural numbers with l or m positive if k is such that the constant is a value of $Tklm$ or $Fklm$. The constant is *0-place* if of type $0lm$ for some l and m. A *variable binder* is a constant which is not 0-place. A *simple constant* is one of type 000. It is an *individual constant* if term making, and a *sentential constant* otherwise. The *conceptual symbols* are the 0-place constants which are not simple. The *operation symbols* are the

term-making conceptual symbols and the *predicates* are the formula-making conceptual symbols. An operation symbol or predicate of type $0lm$ with both l and m positive is *mixed* and the remaining conceptual symbols are *pure*. A pure operation symbol or predicate of type $0l0$ for some positive l is *individual* and the remaining pure conceptual symbols are *sentential*. Notice that sentential and modal connectives are sentential predicates according to our classification of symbols.

We use the symbols $\langle, \rangle$ and $\{, \}$ in the metalanguage to mark the boundaries of finite sequences and sets, respectively. Also, given finite sequences r and s, we understand $r\,s$ to be the result of joining s to the end of r. Thus, $\langle 12\rangle^\frown\langle 34\rangle^\frown\langle\ \rangle = \langle 1234\rangle^\frown\{\ \} = \langle 1234\rangle$. Given a set x and natural number k, x^k is the set of all k-term sequences whose ranges are included in x.[1] If $s \in x^k$ and i is a positive integer $\leqslant k$, then s_i is the value which s assigns to i and s^- is s with the pair $1, s_1$ removed.

Assume now that c is a constant of type klm and that x, t, and F are in w^k, y^l, and z^m, respectively, for some w, y, and z. Then $c(xtF)$—the *application of c to x, t, and F*—is defined as follows:

1. If c is 0-place, then one of the following holds:

   a. c is simple and $c(xtF) = c$.
   b. c is an individual conceptual symbol and $c(xtF) = \langle t_1 c\rangle^\frown t^-$.
   c. c is a sentential conceptual symbol, $m = 1$, and $c(xtF) = \langle cF_1\rangle$.
   d. c is a sentential conceptual symbol, $m > 1$, and $c(xtF) = \langle F_1 c\rangle^\frown F^-$.
   e. c is a mixed conceptual symbol and $c(xtF) = t^\frown\langle c\rangle^\frown F$.

2. If c is a variable binder, then $c(xtf) = \langle c\rangle^\frown x^\frown t^\frown F$.

Observe that 1c and 1d could have been formulated together in the same sort of way as 1b: the first argument comes first, then the constant, and finally the remaining arguments. Although elegant, the result would be that 1-formula connectives come after formulas to which they are applied. Since logicians seem to always put such connectives in front of formulas, 1c and 1d have been employed for the sake of readability.

Terms and *formulas* can now be defined recursively as follows:

1. Every variable is a term.

2. If c is a constant of type klm, $x \in \text{variables}^k$ and x is nonrepeating, $t \in \text{terms}^l$, and $F \in \text{formulas}^m$, then $c(xtF)$ is a term if c is term-making and a formula otherwise.

3. Only these are terms and formulas. That is, if every variable is in K, $c(xtF)$ is in K when c is term-making and in L when c is formula-making if c is a constant of type klm, $x \in$ variables^k and x is nonrepeating, $t \in K^l$, and $F \in L^m$, then every term is in K and every formula is in L. This is the *induction principle for terms and formulas*.

This partially sequential and partially recursive definition of terms and formulas is employed here to allow proof steps with the short recursive clauses while simultaneously allowing sentential and other expressions to be written in a natural way.

Except when it is explicitly mentioned that alternative assumptions have been made, we hereafter use the letters "x" through "z", "t" through "w", and "F" through "J" as metavariables ranging over the variables, terms, and formulas, respectively. Similarly, "c" is a metavariable ranging over the constants and "T" is a metavariable ranging over both the terms and the formulas. E ("exists"), A ("is actual"), B ("is fictive"), and M ("determines a model" or "is a moment") are the first four 1-term individual predicates, I ("is identical with") is the first 2-term individual predicate, and $\ulcorner$ ("s version of" as in the name "1972s version of the king of France" or "the king of France in 1972") is the first 2-term individual operation symbol. The first 1-formula sentential predicate is $\sim$ ("not") and the first four 2-formula sentential predicates are the remaining ordinary connectives $\rightarrow$ ("only if"), $\wedge$ ("and"), $\vee$ ("or"), and $\leftrightarrow$ ("if and only if"). The first 1-term 1-formula mixed predicate is $\vdash$ ("s version of" or "yields" as in the sentence "1972 yields snow is white"). The first two 1-place 0-term 1-formula formula-making variable binders are the quantifiers $\bigwedge$ ("for any") and $\bigvee$ ("for some"). Similarly, the definite article ι ("the") is the first 1-place 0-term 1-formula term-making variable binder.

The sentential and mixed operation symbols and the mixed predicates have heretofore been neglected by logicians. In consequence, they can appear to be strange. However, if temporal units are taken into account, intuitively acceptable examples of such constants can be located. We already have $\vdash$ as a rather intuitive example of a mixed predicate if a term to which it applies denotes a definite temporal unit such as 1972. Similarly, the expressions around the variables in "the year in which F" and "the first year after t in which F" seem to function as a sentential operation symbol and mixed operation symbol, respectively.

The sequence marks in designations of terms and formulas can also be construed as parentheses of the object language. So as to simplify the reading of such designations, conventions like those for the omission of parentheses are here employed for the omission of sequence marks. In particular, sequence marks which can be reintroduced in just one way can be omitted. Also, omitted sequence marks around $\ulcorner$ and $\vdash$ dominate over those around all other constants, omitted sequence marks around 1-term or 1-formula conceptual symbols are next most dominant, and omitted sequence marks around $\rightarrow$ and $\leftrightarrow$ are least dominant. Finally, omitted sequence marks around $\wedge$ and $\vee$ associate to the left of the concerned expression.

A variable x is free in T if x occurs in T without being bound by a phrase consisting of a variable binder followed by a sequence of variables one of which is x. Similarly, a term or formula t is free in T if t occurs in T without having any of its free variables bound by variable binding phrases which occur in T. That is, t is *free* in T just when t is in a term or a formula and one of the following conditions is satisfied:

1. $t = T$.

2. There are c, k, l, m, x, u, and F such that terms of type klm, x through F are in variables^k through formulas^m respectively, x is nonrepeating, $T = c(xuF)$, there is an index i of $u^\frown F$ such that t is free in $(u^\frown F)_i$, and no variable free in t is a value of x.

If f is a function and a is not a sequence, then $f\binom{a}{b}$ is the function which results from removing the pair $a,f(a)$ from f if a is an argument of f and then adding the pair a,b. If there is a natural number n such that a is nonrepeating n-term sequence, b is a finite sequence, and n is an argument of b if n is positive, then $f\binom{a}{b}$ is the function which results from removing all pairs $a_i,f(a_i)$ such that $1 \leqslant i \leqslant n$, and a_i is an argument of f from f and then adding all pairs a_i,b_i with $1 \leqslant i \leqslant n$.

If t and u are both terms or both formulas, we can replace t with u in T if t is free in T and if in T we first replace those occurrences of variables whose variable binding phrases would bind variables free in u with the first variables not occurring in either T or u. More exactly, given such t and u, ${}^t_uT$ is the U such that one of the following conditions is satisfied:

1. t is not free in T and $U = T$.

2. $t = T$ and $U = u$.

3. There are c, k, l, m, x, v, and F such that c is of type klm, x through F are in variables^k through formulas^m, respectively, x is nonrepeating, $T = c(xvF) \neq t$, and t is free in T. Let s be the sequence in order of magnitude of indices i of x such that x_i is free in u. Also, let y be the sequence with the same domain as s such that y_i = the i^{th} variable not occurring in T or u for any index i of y. Finally, let $r_0(x)$ through $r_0(F)$ be x through F, respectively, and let $r_i(x)$ through $r_i(F)$ be k, l, and m-term sequences, respectively, such that $r_i(x) = r_{i-1}(x)\binom{s_i}{y_i}$ and $(r_i(v)^\frown r_i(F))_j = {}^{x_{s_i}}_{y_i}(r_{i-1}(v)^\frown r_{i-1}(F))_j$ for indices i of y and $1 \leqslant j \leqslant l + m$. If n is a natural number such that s and y are n-term sequences, v' and F' are l and m-term sequences, respectively, and $(v'^\frown F')_j = {}^t_u(r_n(v)^\frown r_n(F))_j$ for $1 \leqslant j \leqslant l + m$, then $U = c(r_n(x)\, v'\, F')$.

Now assume that t and u are both sequences of terms or both sequences of formulas. If there is a natural number n such that t is an n-term sequence, u is a finite sequence, and n is an argument of u if n is positive, then ${}^t_uT$ (the result of simultaneously substituting u for t in T) is $s_n(u\ t\ T)$ where d is the function which assigns to any positive integer $i \leqslant n$ the i^{th} variable not occurring in u or T, $v_0(u\ t\ T) = T$, $v_i(u\ t\ T) = {}^{t_i}_{d_i}\, v_{i-1}(u\,t\,T)$ for positive $i \leqslant n$, $s_0(u\,t\,T) = v_n(u\,t\,T)$, and $s_i(u\,t\,T) = {}^{d_i}_{u_i}\, s_{i-1}(u\,t\,T)$ for positive $i \leqslant n$. That is, ${}^t_uT$ is obtained by first replacing the values of t with new variables and then the new variables with the corresponding values of u.

Assume that t and u are both terms or both formulas or both finite sequences of terms or both finite sequences of formulas. If n and k are natural numbers, t is an n-term sequence, n is an argument of u if n is positive, and R is a k-term sequence whose values are terms or formulas, then ${}_u^tR$ is the k-term sequence S such that $S_i = {}_u^tR_i$ for positive $i \leq k$. Also, if $x \in \text{variables}^k$ and q is $\wedge$ or $\vee$, then $\mathrm{C}(qxF)$—the *q-closure in x of F*—is $p_k(F)$ where $p_0(F) = F$ and $p_i(F) = qx_{k-(i-1)}\, p_{i-1}(F)$ for positive $i \leq k$. For example, $\mathrm{C}(\wedge xF)$ is obtained by successively prefixing F with universal quantifier phrases from the last variable of x out to the first.

So as to avoid the repetition of long formulations, we henceforth abbreviate the condition that k through m are natural numbers, c is a constant of type klm, $x \in \text{variables}^k$ and x is nonrepeating, $t \in \text{terms}^l$, and $F \in \text{formulas}^m$ with $\mathrm{CN}klmcxtF$.

2 *Semantic concepts* Let T and F be the sequences of term-making and formula-making constants, respectively, and let C be T or F. Also, assume that r and s are sets and that k, l, and m are natural numbers. A *C-value in s* is either $\{\ \}$ or a subset $\{e\}$ of s if $C = T$, and one of the *truth values* 0 and 1 if $C = F$. A *k-place function in s of type C* is a function defined on s^k which assigns C-values in s. A *k-place argument in s of type lm* is a sequence $q^\frown r$ where q is an l-term sequence of k-place functions in s of type T and r is an m-term sequence of k-place functions in s of type F. An *r-spread in s of type klm* is a function defined on r which assigns k-place arguments in s of type lm. Finally, an *r-intension in s of type Cklm* is a function defined on the set of all r-spreads in s of type klm which assigns C-values in s.[2] Observe that intensions evaluate not arguments, but ways of associating arguments with members of the set r. It is just this kind of span over arguments which is needed for the interpretation of intensional terminology such as that of situational indexing.

A *sentential interpreter* is a function i defined on the set of formulas which assigns truth values such that $i(\sim F) = 1 - i(F)$; $i(F \rightarrow G)$ = the smaller of 1 and $(1 - i(F)) + i(G)$; $i(F \wedge G)$ = the smaller of $i(F)$ and $i(G)$; $i(F \vee G)$ = the greater of $i(F)$ and $i(G)$; and $i(F \leftrightarrow G)$ = (1 - the greater of $i(F)$ and $i(G)$) + the smaller of $i(F)$ and $i(G)$. A *tautology* is a formula F such that $i(F) = 1$ for any sentential interpreter i.

An interpreter is here to be a function which is defined not on the constants, but rather on the set of all ordered pairs consisting of constants and situational objects in that order. Moreover, the function is to assign intensions to such pairs. The least ordinal not present in the universe of discourse is useful as a dummy object with which to determine interpretations of expressions in the nonsituational present. More exactly, i is an *interpreter* just when there are sets s, m, n, and o such that the following conditions are satisfied.

1. $m \subseteq s$.

2. o is the least ordinal $\notin s$.

3. m is the union of m and $\{o\}$.

4. i is defined on the set of all ordered pairs c,p with c a constant and $p \in n$.

5. For any pair c,p in the domain of i such that c is of type klm and a value of C, $i(c,p)$ is an n-intension in s of type $Cklm$.

As above, C is either the sequence of term-making constants or that of formula-making constants. So as to provide the *basic logical constants* E through ⅂ with their intended properties, their interpretations are fixed by the following additional clauses. It is assumed that the *r-function* is the function defined on $\{\langle\ \rangle\}$ which assigns r for any r, that $p \in n$, and that f is in the domain of $i(c,p)$ where c is the concerned logical constant.

6. $i(\mathrm{E},p)(f) = 1$ just when there is an $r \in s$ such that $f(p) = \langle$the $\{r\}$-function$\rangle$. That is, existence always represents the members of s.

7. $i(\mathrm{I},p)(f) = 1$ just when there is an $r \in s$ such that $f(p) = \langle$the $\{r\}$-function the $\{r\}$-function$\rangle$. Identity is always an identity relation in s.

8. $i(\mathrm{M},p)(f) = 1$ just when there is an $r \in m$ such that $f(p) = \langle$the $\{r\}$-function$\rangle$. M always represents the members of m—which are the existing situational objects.

9. If $i(\mathrm{A},p)(f) = 1$, then there is an $r \in s$ such that $f(p) = \langle$the-$\{r\}$-function$\rangle$. Also, if $i(\mathrm{A},p)$ is defined for f' and $f'(p) = f(p)$, then $i(\mathrm{A},p)(f') = i(\mathrm{A},p)(f)$. Actuality always represents some set of existing objects, but can represent different sets of existing objects in different situations. Nevertheless, identity always preserves actuality.

10. $i(\mathrm{B},p)(f) = 1$ just when $i(\mathrm{E},p)(f) = 1$ and $i(\mathrm{A},p)(f) = 0$. In any situation, the fictive objects are the existents which are not actual.

11. If $i(\mathrm{M},p)(f) = 1$, then $i(\mathrm{A},p)(f) = 1$ just when $f(p) = \langle$the $\{p\}$-function$\rangle$. In any situation, just that situation is actual. It follows that no situation is actual in the situationless present.

12. $i(\sim,p)(f) = 1$ just when $f(p) = \langle$the 0-function$\rangle$, $i(\rightarrow,p)(f) = 0$ just when $f(p) = \langle$the 1-function the 0-function$\rangle$, and similarly for $\wedge$, $\vee$, and $\leftrightarrow$.

13. $i(\bigwedge,p)(f) = 1$ just when $f(p) = \langle g\rangle$ where g is the function on s^1 such that $g(\langle r\rangle) = 1$ for any $r \in s$. Similarly, $i(\bigvee,p)(f) = 0$ just when $f(p) = \langle g\rangle$ where g is the function on s^1 such that $g(\langle r\rangle) = 0$ for any $r \in s$. Universal and existential quantification are always over the existents.

14. $i(⅂,p)(f) = \{r\}$ just when $f(p) = \langle g\rangle$ where g is the function on s^1 such that $g(\langle r\rangle) = 1$ and $g(\langle q\rangle) = 0$ if $q \in s$ and $q \neq r$. Descriptions always have either empty or proper reference.

15. $i(\ulcorner,p)(f) = r$ if there are $q \in m$, g, and h such that $f(p) = \langle$the $\{q\}$-function $g\rangle$ and $f(q) = \langle h$ the r-function$\rangle$. Otherwise, $i(\ulcorner,p)(f) = \{\ \}$. In a

situation p, a term $u \ulcorner t$ refers to the object t referred to in the situation referred to by u in p. If u does not refer to a situation in p, then $u \ulcorner t$ has empty reference in p.

16. $i(\vdash, p)(f) = r$ if there are $q \in m$, g, and h such that $f(p) = \langle$the $\{q\}$-function $g\rangle$ and $f(q) = \langle h$ the r-function$\rangle$. Otherwise, $i(\vdash, p)(f) = 0$. In a situation p, a formula $u \vdash F$ has the value which F has in the situation referred to by u in p. If u does not refer to a situation in p, then $u \vdash F$ is false in p.

If i is an interpreter, then Ui, Mi, Ni, and Oi are the s, m, n, and o for which 1 through 16 above hold with respect to i respectively. An *assigner in s* is a function a such that s is a set, a is defined on the set of all variables, and either s is empty and $a(x) = \{\ \}$ or not and there is $r \in s$ such that $a(x) = \{r\}$. If i is an interpreter, a is an assigner in Ui, and $p \in$ Ni, then $\mathsf{Int}_{iap}(T)$ and $\mathsf{Int}_{ia}(T)$—the *interpretations with respect to i, a, and p* and *i and a of T*—are defined as follows:

1. If T is a variable, $\mathsf{Int}_{iap}(T) = a(T)$.

2. Assume that there are k, l, m, c, x, t, and F such that CN$klmcxtF$ and $T = c(xtF)$. Let SP($iaxtF$) be the function which assigns to any $q \in$ Ni the sequence $r^\frown s$ where r and s are l- and m-term sequences, respectively; r_j = the function g defined on Ui^k such that $g(b) = \mathsf{Int}_{ia\binom{x}{b'}g}\ t_j$ for $b \in$ Ui^k and k-term sequence b' such that $b'_n = \{b_n\}$ for $1 \leqslant n \leqslant k$ if $1 \leqslant j \leqslant l$; and s_j = the function g defined on Ui^k such that $g(b) = \mathsf{Int}_{ia\binom{x}{b'}g}\ (F_j)$ for $b \in$ Ui^k where b' is as for r if $1 \leqslant j \leqslant m$. Intuitively, SP($iaxtF$) is the *spread determined by i, a, x, t, and F*. Then $\mathsf{Int}_{iap}(T) = i(c,p)$ (SP($iaxtF$)).

3. $\mathsf{Int}_{ia}(T) = \mathsf{Int}_{ia\mathrm{O}i}(T)$.

F is *i-true* just when $\mathsf{Int}_{ia}(F) = 1$ for any assigner a in Ui and *valid* just when i-true for any interpreter i.

3 *Valid formulas*

Lemma 1 *If i is an interpreter, a is an assigner in* Ui, $p \in$ Ni, *and f is the function defined on the set of formulas such that* $f(F) = \mathsf{Int}_{iap}(F)$ *for any F, then f is a sentential interpreter.*

Proof: Assume the antecedent. By our definitions, $f(\sim G) = 1$ just when SP($ia\langle\ \rangle\langle\ \rangle\langle G\rangle$)($p$) = $\langle$the $\mathsf{Int}_{iap}(G)$-function$\rangle$ = $\langle$the 0-function$\rangle$ and so just when $\mathsf{Int}_{iap}(G) = 0$. Consequently, $f(\sim G) = 1 - f(G)$. Also, $f(G \rightarrow H) = 0$ just when $f(G) = 1$ and $f(H) = 0$ by the same sort of reasoning and so $f(G \rightarrow H)$ = the smaller of 1 and $(1 - f(G)) + f(H)$. The remaining cases follow by analogous arguments.

The next two theorems follow immediately from Lemma 1.

Theorem 1 *Every tautology is valid.*

Theorem 2 *If i is an interpreter, a is an assigner in* Ui, $p \in \mathrm{N}i$, *and* $\mathrm{Int}_{iap}(F \rightarrow G) = \mathrm{Int}_{iap}(F) = 1$, *then* $\mathrm{Int}_{iap}(G) = 1$.

Hence, *modus ponens* preserves truth and the semantics of the sentential connectives is normal in the situationless present. Obviously,

Theorem 3 $u \vdash F \rightarrow u\mathrm{M}$ *is valid.*

Lemma 2 *If i is an interpreter, a is an assigner in* Ui, $p \in \mathrm{M}i$, $q \in \mathrm{N}i$, *and* $\mathrm{Int}_{iaq}(u) = \{p\}$, *then* $\mathrm{Int}_{iap}(t) = \mathrm{Int}_{iaq}(u \ulcorner t)$ *and* $\mathrm{Int}_{iap}(F) = \mathrm{Int}_{iaq}(u \vdash F)$.

Proof: Assume the antecedent. By our definitions, $\mathrm{Int}_{iaq}(u \ulcorner t) = r$ just when $\mathrm{SP}(ia\langle\ \rangle\langle ut\rangle\langle\ \rangle)(p) = \langle$the $\mathrm{Int}_{iap}(u)$-function the $\mathrm{Int}_{iap}(t)$-function$\rangle = \langle h$ the r-function$\rangle$ for some h and so just when $\mathrm{Int}_{iap}(t) = r = \mathrm{Int}_{iaq}(u \ulcorner t)$. The argument for the second part of the consequent is analogous.

Theorem 4 $u\mathrm{M} \rightarrow \langle u \vdash \sim F \leftrightarrow \sim u \vdash F\rangle \wedge \langle u \vdash \langle F \rightarrow G\rangle \leftrightarrow \langle u \vdash F \rightarrow u \vdash G\rangle\rangle \wedge \langle u \vdash \langle F \wedge G\rangle \leftrightarrow \langle u \vdash F \wedge u \vdash G\rangle \wedge \langle u \vdash \langle F \vee G\rangle \leftrightarrow \langle u \vdash F \vee u \vdash G\rangle \wedge \langle u \vdash \langle F \leftrightarrow G\rangle \leftrightarrow \langle u \vdash F \leftrightarrow u \vdash G\rangle\rangle$ *is valid.*

Proof: Assume that i is an interpreter, a is an assigner in Ui, and $\mathrm{Int}_{ia}(u\mathrm{M}) = 1$. Hence, there is a $p \in \mathrm{M}i$ such that $\mathrm{Int}_{ia}(u) = \{p\}$. By Lemma 1 and Lemma 2, $\mathrm{Int}_{ia}(u \vdash \sim F) = \mathrm{Int}_{iap}(\sim F) = 1 - \mathrm{Int}_{iap}(F) = 1 - \mathrm{Int}_{ia}(u \vdash F) = \mathrm{Int}_{ia}(\sim u \vdash F)$. Similarly, $\mathrm{Int}_{ia}(u \vdash \langle F \rightarrow G\rangle) = \mathrm{Int}_{iap}(F \rightarrow G) =$ the smaller of 1 and $(1 - \mathrm{Int}_{iap}(F)) + \mathrm{Int}_{iap}(G) =$ the smaller of 1 and $(1 - \mathrm{Int}_{ia}(u \vdash F)) + \mathrm{Int}_{ia}(u \vdash G) = \mathrm{Int}_{ia}(u \vdash F \rightarrow u \vdash G)$. By analogous reasoning for the remaining connectives and Lemma 1, the validity of the formula of the theorem follows.

Theorem 4 asserts that the sentential connectives are absolute in that their situational truth value assignments are the situationless ones restricted to situations. For iteration of indices, we have

Theorem 5 $\langle v \ulcorner \langle u \ulcorner t\rangle \mathrm{E} \vee \langle v \ulcorner u\rangle \ulcorner t\,\mathrm{E} \rightarrow v \ulcorner \langle u \ulcorner t\rangle\, \mathrm{I}\, \langle v \ulcorner u\rangle \ulcorner t\rangle \wedge \langle v \vdash u \vdash F \leftrightarrow v \ulcorner u \vdash F\rangle$ *is valid.*

Proof: For, if i is an interpreter and a is an assigner in Ui, $\mathrm{Int}_{ia}(v \ulcorner \langle u \ulcorner t\rangle) = \mathrm{Int}_{iap}(u \ulcorner t) = \mathrm{Int}_{iaq}(t) = \mathrm{Int}_{ia}(\langle v \ulcorner u\rangle \ulcorner t)$ when p and $q \in \mathrm{M}i$, $\mathrm{Int}_{ia}(v) = \{p\}$, and $\mathrm{Int}_{iap}(u) = \{q\} = \mathrm{Int}_{ia}(v \ulcorner u)$. If there are no such p and q, then $\mathrm{Int}_{ia}(v \ulcorner \langle u \ulcorner t\rangle) = \{\ \} = \mathrm{Int}_{ia}(\langle v \ulcorner u\rangle \ulcorner t)$. By a similar argument, $\mathrm{Int}_{ia}(v \vdash u \vdash F) = \mathrm{Int}_{ia}(v \ulcorner u \vdash F)$. The theorem follows by Lemma 1.

Theorems 1 through 5 are the main principles of sentential indexical logics. Some additional principles for terms and predicates are the following:

Theorem 6 $\langle t\mathrm{M} \rightarrow t\mathrm{B}\rangle \wedge \langle t\mathrm{B} \rightarrow t\mathrm{E}\rangle$ *is valid.*

Proof: Assume that i is an interpreter and a is an assigner in Ui. If $\mathrm{Int}_{ia}(t\mathrm{M}) = 1$, there is a $p \in \mathrm{M}i$ such that $\mathrm{Int}_{ia}(t) = \{p\}$. But $\mathrm{Int}_{ia}(t\mathrm{A}) = 1$ just when $p = \mathrm{O}i \notin \mathrm{U}i$ and $\mathrm{Int}_{ia}(t\mathrm{B}) = 1$ just when $\mathrm{Int}_{ia}(t\mathrm{A}) = 0$. Hence, $p \neq \mathrm{O}i$ and $\mathrm{Int}_{ia}(t\mathrm{B}) = 1$. Also, if $\mathrm{Int}_{ia}(t\mathrm{B}) = 1$, then it follows immediately that $\mathrm{Int}_{ia}(t\mathrm{E}) = 1$. From Lemma 1, it follows that the theorem holds.

Theorem 7 *$t\mathrm{A} \leftrightarrow t\mathrm{E} \wedge \sim t\mathrm{B}$ is valid.*

Theorem 8 *$t\mathrm{I}u \rightarrow t\mathrm{E} \wedge u\mathrm{I}t \wedge \langle t\mathrm{I}v \leftrightarrow u\mathrm{I}v\rangle \wedge \langle t\mathrm{A} \leftrightarrow u\mathrm{A}\rangle \wedge \langle t\mathrm{B} \leftrightarrow u\mathrm{B}\rangle \wedge \langle t\mathrm{M} \leftrightarrow u\mathrm{M}\rangle \wedge \langle t \vdash F \leftrightarrow u \vdash F\rangle \wedge \langle t \ulcorner v\ \mathrm{E} \vee u \ulcorner v\ \mathrm{E} \rightarrow t \ulcorner v\ \mathrm{I}\ u \ulcorner v\rangle$ is valid.*

Proof: The proofs are as for that of Theorem 6. Theorem 8 is a kind of concrete indiscernibility law for predicates which is needed since general indiscernibility is not valid.

Theorem 9 *$u \vdash t\mathrm{E} \rightarrow u\mathrm{M}$ is valid.*

Theorem 10 *$\langle u \vdash t\mathrm{E} \leftrightarrow u \ulcorner t\mathrm{E}\rangle \wedge \langle u \vdash t\mathrm{M} \leftrightarrow u \ulcorner t\mathrm{M}\rangle \wedge \langle u \vdash t\mathrm{I}v \leftrightarrow u \ulcorner t\ \mathrm{I}\ u \ulcorner v\rangle$ is valid.*

Theorem 11 *$u\mathrm{M} \rightarrow u \vdash \langle\langle t\mathrm{B} \rightarrow t\mathrm{E}\rangle \wedge \langle t\mathrm{A} \leftrightarrow t\mathrm{E} \wedge \sim t\mathrm{B}\rangle\rangle \wedge \langle u \vdash \langle t\mathrm{M} \wedge t\mathrm{A}\rangle \leftrightarrow u \ulcorner t\,\mathrm{I}u\rangle \wedge u \vdash \langle t\mathrm{I}v \rightarrow \langle t\mathrm{A} \leftrightarrow v\mathrm{A}\rangle \wedge \langle t\mathrm{B} \leftrightarrow v\mathrm{B}\rangle\rangle$ is valid.*

The parts $t\mathrm{B} \leftrightarrow u\mathrm{B}$ of Theorem 8 and $t\mathrm{B} \leftrightarrow v\mathrm{B}$ of Theorem 11 follow from the respective theorems without the parts plus assumptions. The parts are here included for the sake of symmetry.

Proof: As above, the proofs are via Lemmas 1 and 2 and the analysis of the interpretations of the concerned logical constants. Theorems 9 and 10 correspond to Theorems 3 and 4, respectively. Theorem 10 expresses the absoluteness of E, M, and I, while Theorem 11 gives the usages of A and B in situations. The latter theorem is needed because A and B are not absolute although they always partition the possible object set represented by E.

Next come the quantifiers, identity, and descriptions.

Lemma 3 *If i is an interpreter, $p \in \mathrm{N}i$, both a and $a(^x_w)$ are assigners in $\mathrm{U}i$, and x is not free in T, then $\mathsf{Int}_{ia(^x_w)p}(T) = \mathsf{Int}_{iap}(T)$.*

Lemma 4 *If i is an interpreter, $p \in \mathrm{N}i$, a is an assigner in $\mathrm{U}i$, and $\mathsf{Int}_{iaq}(t) = \{w\}$ for any $q \in \mathrm{N}i$, then $\mathsf{Int}_{iap}(^x_t T) = \mathsf{Int}_{ia(^{\ x}_{\{w\}})p}(T)$.*

Lemma 5 *If t and u are both terms or both formulas, i is an interpreter, $p \in \mathrm{N}i$, a is an assigner in $\mathrm{U}i$, and $\mathsf{Int}_{iaq}(t) = \mathsf{Int}_{iaq}(u)$ for any $q \in \mathrm{N}i$, then $\mathsf{Int}_{iap}(^u_t T) = \mathsf{Int}_{iap}(T)$.*

Proof: The proofs for these three lemmas are by a straightforward induction among the terms and formulas.

Lemma 6 *If i is an interpreter, a is an assigner in $\mathrm{U}i$, and $p \in \mathrm{N}i$, then $\mathsf{Int}_{iap}(\bigwedge xF) = 1$ just when $\mathsf{Int}_{ia(^{\ x}_{\{w\}})p}(F) = 1$ for any $w \in \mathrm{U}i$, $\mathsf{Int}_{iap}(\bigvee xF) = 1$ just when there is a $w \in \mathrm{U}i$ such that $\mathsf{Int}_{ia(^{\ x}_{\{w\}})p}(F) = 1$, and $\mathsf{Int}_{iap}(\iota xF) = \{r\}$ just when $\{r\}$ is the set of all $w \in \mathrm{U}i$ such that $\mathsf{Int}_{ia(^{\ x}_{\{w\}})p}(F) = 1$.*

Proof: Assume the antecedent and let $f = \mathrm{SP}(ia\langle x\rangle\ \langle\ \rangle\ \langle F\rangle)$. Hence, $f(p) = \langle$the function g on $\mathrm{U}i^1$ such that $g(\langle w\rangle) = \mathsf{Int}_{ia(^{\ x}_{\{w\}})p}(F)$ for any $w \in \mathrm{U}i\rangle$. But $\mathsf{Int}_{iap}(\bigwedge xF) = i(\bigwedge, p)(f) = 1$ just when $g(\langle w\rangle) = 1$ for any $w \in \mathrm{U}i$. Similarly, $\mathsf{Int}_{iap}(\bigvee xF) = 0$ just when $g(\langle w\rangle) = 0$ for any $w \in \mathrm{U}i$ and $\mathsf{Int}_{iap}(\iota xF) = \{r\}$ just when $g(\langle r\rangle) = 1$ and $g(\langle w\rangle) = 0$ for any $w \in \mathrm{U}i$ such that $w \neq r$. The consequent of the lemma follows immediately.

From Lemmas 1, 3, and 6, we clearly have the following three theorems.

Theorem 12 *If x is not free in t, then $t\mathrm{E} \rightarrow \bigvee x\ t\mathrm{I}x$ is valid.*

This schema is a version of the principle of the self-identity of existents.

Theorem 13 *If x is not free in F, then $\bigwedge x\langle F \rightarrow G\rangle \leftrightarrow \langle F \rightarrow \bigwedge xG\rangle$ is valid.*

Theorem 14 *$\bigvee xF \leftrightarrow \sim\bigwedge x \sim F$ is valid.*

Lemma 7 *If i is an interpreter, a is an assigner in $\mathrm{U}i$, and x is not free in t or u, then $\mathrm{Int}_{ia}(\bigwedge x\langle x\mathrm{M} \rightarrow \langle x \ulcorner t\ \mathrm{E} \vee t\mathrm{E} \rightarrow x \ulcorner t\ \mathrm{I}\ t\rangle\rangle) = 1$ just when there is an r such that $\mathrm{Int}_{iaq}(t) = r$ for any $q \in \mathrm{N}i$. Also, if either both t and u are terms and $F = t\mathrm{E} \vee u\mathrm{E} \rightarrow t\mathrm{I}u$ or both t and u are formulas and $F = t \leftrightarrow u$, then $\mathrm{Int}_{ia}(\bigwedge x\langle x\mathrm{M} \rightarrow x \vdash F\rangle \wedge F) = 1$ just when $\mathrm{Int}_{iaq}(t) = \mathrm{Int}_{iaq}(u)$ for any $q \in \mathrm{N}i$.*

Proof: Assume the antecedent and let G and H be the formulas concerned. By Lemmas 1 through 3 and 6, $\mathrm{Int}_{ia}(G) = 1$ just when, for any $q \in \mathrm{M}i$, $\mathrm{Int}_{iaq}(t) = \mathrm{Int}_{ia}(t)$. Hence, the first part of the lemma holds. The proof that $\mathrm{Int}_{ia}(H) = 1$ just when $\mathrm{Int}_{iaq}(t) = \mathrm{Int}_{iaq}(u)$ for any $q \in \mathrm{N}i$ when t and u are both terms or both formulas is analogous.

Via Lemma 4 and Lemma 6, Lemmas 4 and 7 result in:

Theorem 15 *If y is not free in t, then $\bigwedge y\langle y\mathrm{M} \rightarrow \langle y \ulcorner t\ \mathrm{E} \vee t\mathrm{E} \rightarrow y \ulcorner t\ \mathrm{I}\ t\rangle\rangle \rightarrow \langle tE \wedge \bigwedge xF \rightarrow {}^{x}_{t}F\rangle$ is valid.*

Also, Lemmas 5 and 7 result in:

Theorem 16 *If x is not free in F and either t and u are both terms and $F = t\mathrm{E} \vee u\mathrm{E} \rightarrow t\mathrm{I}u$ or t and u are both formulas and $F = t \leftrightarrow u$, then $\bigwedge x\langle x\mathrm{M} \rightarrow x \vdash F\rangle \wedge F \rightarrow \langle {}^{u}_{t}G \leftrightarrow G\rangle$ is valid.*

Theorem 17 *$\sim\bigwedge xF \rightarrow y\mathrm{E}$ is valid.*

Proof: If i is an interpreter, a is an assigner in $\mathrm{U}i$, and $\mathrm{Int}_{ia}(\sim\bigwedge xF) = 1$, then $\mathrm{U}i$ is not empty by Lemmas 1 and 6 and so $\mathrm{Int}_{ia}(y) = a(y)$ is a subset $\{r\}$ of $\mathrm{U}i$. Hence, $\mathrm{Int}_{ia}(y\mathrm{E}) = 1$ and the theorem holds by Lemma 1.

Theorem 18 *$t\mathrm{M} \rightarrow t \ulcorner x\ \mathrm{I}\ x$ is valid.*

Proof: For $\mathrm{Int}_{iap}(x) = a(x)$ for any interpreter i, assigner a in $\mathrm{U}i$, and $p \in \mathrm{N}i$.

Lemmas 1 and 6 also give us the following two theorems:

Theorem 19 *If $y \neq x$ and y is free in neither t nor F, then $t\ \mathrm{I}\ \iota xF \leftrightarrow \bigvee y\langle\bigwedge x\langle F \leftrightarrow x\mathrm{I}y\rangle \wedge t\mathrm{I}y\rangle$ is valid.*

Theorem 20 *If i is an interpreter and F is i-true, the $\bigwedge xF$ is i-true.*

Theorems 15 and 16 provide the weak versions of universal instantiation and indiscernibility which hold in indexical logics. However, the normal versions hold for variables by Theorem 18. This is a perhaps unnatural trait of variables which is extremely useful. In particular, it lets variables reach into situational contexts at the cost of generating morning-star-type paradoxes formulated with variables instead of individual constants. The formulas of Theorem 17 express the principle that all variables denote if the universe of discourse is not empty. The description theory of indexical logics is provided by Theorem 19 and universal generalization is truth-preserving by Theorem 20. The quantifiers and ℩ are also absolute.

Theorem 21 *If* x *is not free in* u, *then* $u\mathrm{M} \rightarrow \langle u \vdash \bigwedge xF \leftrightarrow \bigwedge x\ u \vdash F\rangle \wedge \langle u \vdash \bigvee xF \leftrightarrow \bigvee x\ u \vdash F\rangle \wedge \langle u \ulcorner \text{℩} xF\ \mathrm{E} \vee \text{℩} x\ u \vdash F\ \mathrm{E} \rightarrow u \ulcorner \text{℩} xF\ \mathrm{I}\ \text{℩} x\ u \vdash F\rangle$ *is valid.*

Proof: Assume that x is not free in u, i is an interpreter, a is an assigner in $\mathrm{U}i$, and there is a $p \in \mathrm{M}i$ such that $\mathrm{Int}_{ia}(u) = \{p\}$. If $w \in \mathrm{U}i$, $a(^{x}_{\{w\}})$ is an assigner in $\mathrm{U}i$ since $\mathrm{U}i \neq \{\ \}$. Also, $\mathrm{Int}_{ia(^{x}_{\{w\}})}(u) = \mathrm{Int}_{ia}(u)$ by Lemma 3 and so $\mathrm{Int}_{ia(^{x}_{\{w\}})}(u \vdash F) = \mathrm{Int}_{ia(^{x}_{\{w\}})p}(F)$ by Lemma 2. Hence, $\mathrm{Int}_{ia}(u \vdash \bigwedge xF) = \mathrm{Int}_{iap}(\bigwedge xF) = \mathrm{Int}_{ia}(\bigwedge x\ u \vdash F)$ by Lemmas 2 and 6. Similarly, $\mathrm{Int}_{ia}(u \vdash \bigvee xF) = \mathrm{Int}_{ia}(\bigvee x\ u \vdash F)$ and $\mathrm{Int}_{ia}(u \ulcorner \text{℩} xF) = \mathrm{Int}_{ia}(\text{℩} x\ u \vdash F)$. The theorem then holds by Lemma 1.

Theorem 22 $G \wedge \langle u\mathrm{M} \rightarrow u \vdash G\rangle$ *is valid when the following conditions are satisfied*:

1. $\mathrm{C}NklmbxtF$ *and* $1 \leqslant i \leqslant k$.
2. U *is* $b(xtF)$ *and* T *is* $b(x(^{i}_{y})\ ^{xi}_{\ y}t\ ^{xi}_{\ y}F)$.
3. *There is no value of* x, t, *or* F *in which* y *is free.*
4. G *is* $T\mathrm{E} \vee U\mathrm{E} \rightarrow T\mathrm{I}U$ *if* b *is term-making and* $T \leftrightarrow U$ *if* b *is formula-making.*

Proof: Assume that 1-4 hold, that j is an interpreter and a is an assigner in $\mathrm{U}j$, and that $q \in \mathrm{N}j$. By Lemmas 1 and 2 it is sufficient to show that $\mathrm{Int}_{jaq}(T) = \mathrm{Int}_{jaq}(U)$ and so that $S = \mathrm{SP}(jaxtF) = \mathrm{SP}(jax(^{i}_{y})^{xi}_{\ y}t\ ^{xi}_{\ y}F) = S'$. If U_j is empty, $\{\ \}$ is the only function defined on $\mathrm{U}j^{k}$ and $S = S'$. So assume that $b \in \mathrm{U}j^{k}$, b' is the singleton image of b as in the definition of interpretation, and $r = t^\frown F$. By Lemmas 4 and 3, $\mathrm{Int}_{ia(^{x}_{b'}(^{i}_{y}))_q}(^{xi}_{y}r_n) = \mathrm{Int}_{ia(^{x}_{b'})_q}(r_n)$ when $1 \leqslant n \leqslant l + m$. Consequently, $S = S'$ and the theorem holds.

Theorem 23 $\mathrm{C}(\bigwedge xH) \rightarrow J \wedge \langle u\mathrm{M} \rightarrow u \vdash J\rangle$ *is valid when the following conditions are satisfied*:

1. $\mathrm{C}NklmbxtF$ *and* $1 \leqslant k$.
2. U *is* $b(xtF)$.
3. H *and* T *are* $\bigwedge y\langle y\mathrm{M} \rightarrow y \vdash \langle\langle v\mathrm{E} \vee t_i\mathrm{E} \rightarrow v\ \mathrm{I}\ t_i\rangle \wedge \langle G \leftrightarrow F_j\rangle\rangle\rangle \wedge \langle v\mathrm{E} \vee t_i\mathrm{E} \rightarrow v\ \mathrm{I}\ t_i\rangle \wedge \langle G \leftrightarrow F_j\rangle$ *and* $b(xt(^{i}_{v})\ F(^{j}_{G}))$ *respectively, if* $1 \leqslant i \leqslant l$ *and* $1 \leqslant j \leqslant m$.
4. H *and* T *are* $\bigwedge y\langle y\mathrm{M} \rightarrow y \vdash \langle G \leftrightarrow F_j\rangle\rangle \wedge \langle G \leftrightarrow F_j\rangle$ *and* $b(xtF(^{j}_{G}))$ *respectively, if* $l = 0$ *and* $1 \leqslant j \leqslant m$.
5. H *and* T *are* $\bigwedge y\langle y\mathrm{M} \rightarrow y \vdash \langle v\mathrm{E} \vee t_i\mathrm{E} \rightarrow v\ \mathrm{I}\ t_i\rangle\rangle \wedge \langle v\mathrm{E} \vee t_i\mathrm{E} \rightarrow v\ \mathrm{I}\ t_i\rangle$ *and* $b(x\ t(^{i}_{v})F)$ *respectively, if* $1 \leqslant i \leqslant l$ *and* $m = 0$.
6. *There is no value of* x *or* t *or* F *or* $\langle v\mathrm{G}\rangle$ *in which* y *is free.*
7. J *is* $T\mathrm{E} \vee U\mathrm{E} \rightarrow T\mathrm{I}U$ *if* b *is term-making and* $T \leftrightarrow U$ *if* b *is formula-making.*

Proof: Assume that 1-7 hold, h is an interpreter, a is an assigner in Ui, $q \in \mathrm{N}h$, and $\mathrm{Int}_{ha}(\mathrm{C}(\wedge xH)) = 1$. It is sufficient to show that $\mathrm{Int}_{haq}(T) = \mathrm{Int}_{haq}(U)$. If $b \in \mathrm{U}h^k$, b' is the singleton image of b, and $1 \leq i$, it follows that $\mathrm{Int}_{ha}(^x_{b'})_p(v) = \mathrm{Int}_{ha}(^x_{b'})_p(t_i)$ for any $p \in \mathrm{N}h$. Also, if $1 \leq j$, it follows that $\mathrm{Int}_{ha}(^x_{b'})_p(G) = \mathrm{Int}_{ha}(^x_{b'})_p(F_j)$ for any $p \in \mathrm{N}h$. Hence, if S is $\mathrm{SP}(haxt(^i_y)F(^j_G))$ when $1 \leq i$ and $1 \leq j$, $\mathrm{SP}(haxtF(^j_G))$ when $i = 0$ and $1 \leq j$, and $\mathrm{SP}(haxt(^i_v)F)$ when $1 \leq i$ and $j = 0$, then $S = \mathrm{SP}(haxtF)$. But then $\mathrm{Int}_{haq}(T) = \mathrm{Int}_{haq}(U)$ holds in each of the three possible cases.

Theorems 22 and 23 provide the special principles of the indexical logic of variable binders. Bound variables can be rewritten even within situational formulas, but coextensional terms or formulas can only be interchanged if they are coextensional in all situations. However, such terms or formulas are even interchangeable within situational formulas.

The formulas of the above theorems are those fundamental to indexical logics. For the sake of subsequent applications, some consequences of these formulas will now be derived.

Corollary 1 *If F is a tautology, then $u\mathrm{M} \rightarrow u \vdash F$ is valid.*

Proof: Let K be the intersection of all sets of formulas closed under *modus ponens* which have as members $F \rightarrow \langle G \rightarrow F\rangle$, $\langle \sim F \rightarrow \sim G\rangle \rightarrow \langle G \rightarrow F\rangle$, $\langle F \rightarrow \langle G \rightarrow H\rangle\rangle \rightarrow \langle\langle F \rightarrow G\rangle \rightarrow \langle F \rightarrow H\rangle\rangle$, $F \wedge G \rightarrow \sim\langle F \rightarrow \sim G\rangle$, $\sim\langle F \rightarrow \sim G\rangle \rightarrow F \wedge G$, $F \vee G \rightarrow \sim\langle \sim F \wedge \sim G\rangle$, $\sim\langle \sim F \wedge \sim G\rangle \rightarrow F \vee G$, $\langle F \leftrightarrow G\rangle \rightarrow \langle F \vee G \rightarrow F \wedge G\rangle$, and $\langle F \vee G \rightarrow F \wedge G\rangle \rightarrow \langle F \leftrightarrow G\rangle$ for any F, G, and H. Clearly, K is the set of tautologies. Also, if L is the set of all F such that $u\mathrm{M} \rightarrow u \vdash F$ is valid, each of the formulas above is in L by Theorems 1, 2, and 4 and L is closed under *modus ponens* by the same theorems. Hence, K is included in L and the corollary holds.

Corollary 2 $\langle v \ulcorner u\ \mathrm{E} \vee u\mathrm{E} \rightarrow v \ulcorner u\ \mathrm{I}\ u\rangle \rightarrow \langle v \ulcorner \langle u \ulcorner t\rangle\ \mathrm{E} \vee u \ulcorner t\ \mathrm{E} \rightarrow v \ulcorner \langle u \ulcorner t\rangle\ \mathrm{I}\ u \ulcorner t\rangle \wedge \langle v \vdash u \vdash F \leftrightarrow u \vdash F\rangle$ *is valid.*

Proof: Let G be the antecedent of the formula, let $H \wedge H'$ be the consequent, and $J \rightarrow J'$ be H. By Theorems 9, 6, 1, and 2, $G \rightarrow \langle u \ulcorner t\ \mathrm{E} \rightarrow v \ulcorner u\ \mathrm{I}\ u\rangle$ is valid. Hence, by Theorems 8, 1, 2, and 5, $G \rightarrow \langle u \ulcorner t\ \mathrm{E} \rightarrow J'\rangle$ is valid. Also, by Theorems 5, 1, 2, 9, and 6, $G \rightarrow \langle v \ulcorner \langle u \ulcorner t\rangle \mathrm{E} \rightarrow v \ulcorner \langle u \ulcorner t\rangle\ \mathrm{I}\ \langle v \ulcorner u\rangle \ulcorner t \wedge v \ulcorner u\ \mathrm{I}\ u\rangle$ is valid. Consequently, by Theorems 8, 1, and 2, $G \rightarrow \langle v \ulcorner \langle u \ulcorner t\rangle\ \mathrm{E} \rightarrow J'\rangle$ and so $G \rightarrow \langle J \rightarrow J'\rangle$ is valid. Finally, by Theorems 5, 9, 6, 3, 1, and 2, $G \rightarrow \langle v \vdash u \vdash F \rightarrow v \ulcorner u\ \mathrm{I}\ u\rangle$ and $G \rightarrow \langle u \vdash F \rightarrow v \ulcorner u\ \mathrm{I}\ u\rangle$ are valid and so $G \rightarrow H'$ is valid via Theorem 8. Hence, $G \rightarrow H \wedge H'$ is valid by Theorems 1 and 2.

Corollary 3 *If y is not free in t, then* $\wedge y\langle y\mathrm{M} \rightarrow \langle y \ulcorner t\ \mathrm{E} \vee t\mathrm{E} \rightarrow y \ulcorner t\ \mathrm{I}\ t\rangle\rangle \rightarrow \langle t\mathrm{E} \wedge {}^x_tF \rightarrow \vee xF\rangle$ *is valid.*

Proof: The proof is by Theorems 15, 1, 2, and 14.

Corollary 4 $\wedge y\langle y\mathrm{M} \rightarrow \langle y \ulcorner x\,\mathrm{E} \vee x\mathrm{E} \rightarrow y \ulcorner x\ \mathrm{I}\ x\rangle\rangle$ *is valid.*

Proof: The proof is by Theorems 18, 1, 2, and 20.

In the proofs of the remaining corollaries, reference to Theorems 1 and 2 is usually omitted.

Corollary 5 *The following formulas are valid*:

1. $\bigwedge xx\mathrm{E}$
2. $\bigvee xF \rightarrow y\mathrm{E}$
3. $\bigvee xx\mathrm{E} \leftrightarrow y\mathrm{E}$
4. $\bigvee yy\mathrm{E} \wedge \bigwedge x\mathrm{F} \rightarrow {}^{x}_{z}\mathrm{F}$
5. $\bigvee yy\mathrm{E} \wedge {}^{x}_{z}F \rightarrow \bigvee xF$
6. $\langle \bigvee yy\mathrm{E} \rightarrow \bigwedge x\mathrm{F}\rangle \rightarrow \bigwedge x\mathrm{F}$.

Proof: 1 holds via Theorems 17, 20, and 13; 2 via Theorems 14 and 17; 3 via Corollaries 3 and 4 with 2; 4 via Theorem 15 and Corollary 4 with 3; and 5 via 4 and Theorem 14. 6 is a consequence of 3 and Theorem 17.

Corollary 6 *If x is not free in F, then $F \rightarrow G$ is valid only if $F \rightarrow \bigwedge xG$ is and $G \rightarrow F$ is valid only if $\bigvee xG \rightarrow F$ is.*

Proof: This is a consequence of Theorems 20, 13, and 14.

Corollary 7 *If x is not free in F, then $F \rightarrow \bigwedge xF$ and $\bigvee xF \rightarrow F$ are valid.*

Proof: The proof is by Corollary 6 and the validity of $F \rightarrow F$.

Corollary 8 *The following formulas are valid*:

1. $\bigwedge x\langle F \rightarrow G\rangle \rightarrow \langle \bigwedge xF \rightarrow \bigwedge xG\rangle$
2. $\bigwedge x\langle F \leftrightarrow G\rangle \rightarrow \langle \bigwedge xF \leftrightarrow \bigwedge xG\rangle$
3. $\bigwedge x\langle F \rightarrow G\rangle \rightarrow \langle \bigvee xF \rightarrow \bigvee xG\rangle$
4. $\bigwedge x\langle F \leftrightarrow G\rangle \rightarrow \langle \bigvee xF \leftrightarrow \bigvee xG\rangle$
5. $\bigwedge x\langle F \wedge G\rangle \leftrightarrow \bigwedge xF \wedge \bigwedge xG$
6. $\bigvee x\langle F \vee G\rangle \leftrightarrow \bigvee xF \vee \bigvee xG$.

Proof: Via Theorem 17 and 3 and 4 of Corollary 5, $\bigwedge x\langle F \rightarrow G\rangle \wedge \bigwedge xF \wedge \sim\bigwedge xG \rightarrow \langle F \rightarrow G\rangle \wedge F$ is valid. Consequently, 1 holds via Corollary 6 and Theorem 1. 2 through 6 follow from 1 in the usual manner.

Corollary 9 $\bigwedge xF \leftrightarrow \bigwedge x\langle x\mathrm{E} \rightarrow F\rangle$ *and* $\bigvee xF \leftrightarrow \bigvee x\langle x\mathrm{E} \wedge F\rangle$ *are valid.*

Proof: By 1 of Corollary 8, $\bigwedge x\langle x\mathrm{E} \rightarrow F\rangle \rightarrow \langle \bigwedge xx\mathrm{E} \rightarrow \bigwedge x\mathrm{F}\rangle$ is valid. Also, by 3 and 4 of Corollary 5, $\bigwedge xF \rightarrow \langle x\mathrm{E} \rightarrow F\rangle$ is valid. From 1 of Corollary 5 and Corollary 6, it follows that $\bigwedge xF \leftrightarrow \bigwedge x\langle x\mathrm{E} \rightarrow F\rangle$ is valid. Also, since $\bigwedge x\langle x\mathrm{E} \rightarrow \langle F \leftrightarrow x\mathrm{E} \wedge F\rangle\rangle$ is valid, $\bigwedge x\langle F \leftrightarrow x\mathrm{E} \wedge F\rangle$ is valid by 1 of Corollary 8 and 1 of Corollary 5 and $\bigvee xF \leftrightarrow \bigvee x\langle x\mathrm{E} \wedge F\rangle$ is valid by 4 of Corollary 8.

Corollary 10 $\bigwedge xF \leftrightarrow \sim\bigvee x \sim F$ *is valid.*

Proof: For $\bigwedge xF \leftrightarrow \bigwedge x \sim\sim F$ and $\bigwedge x \sim\sim F \leftrightarrow \sim\bigvee x \sim F$ are valid by 2 of Corollary 8 and Theorem 14.

Corollary 11 *If x is not free in t, then $t\mathrm{E} \leftrightarrow \bigvee xt\mathrm{I}x$ and $\bigvee xt\mathrm{I}x \leftrightarrow t\mathrm{I}t$ are valid.*

Proof: By Theorem 8, $t\mathrm{I}x \vee t\mathrm{I}t \rightarrow t\mathrm{E}$ and $t\mathrm{I}x \rightarrow x\mathrm{I}t \wedge \langle t\mathrm{I}t \leftrightarrow x\mathrm{I}t\rangle$ are valid. Hence, $t\mathrm{E} \leftrightarrow \bigvee x t\mathrm{I}x$ is valid by Corollary 6 and Theorem 12 while $\bigvee x t\mathrm{I}x \leftrightarrow t\mathrm{I}t$ is valid by Theorem 12.

Corollary 12 *If y is free in neither F nor G nor H and both $F \rightarrow \bigvee xG$ and $F \wedge {}^{x}_{y}G \rightarrow H$ are valid, then $F \rightarrow H$ is valid.*

Proof: Assume the antecedent. By Corollary 6 and Theorem 22, $\bigvee xG \rightarrow \langle F \rightarrow H\rangle$ and so $F \rightarrow H$ are valid.

Corollary 13 *$t\mathrm{I}u \leftrightarrow u\mathrm{I}t$ and $t\mathrm{I}u \wedge u\mathrm{I}v \rightarrow t\mathrm{I}v$ are valid.*

Proof: The corollary follows from Theorem 8.

Corollary 14 *If z is not free in t or u, then the following formulas are valid*:

1. $\bigwedge z\langle z\mathrm{M} \rightarrow z \ulcorner t \mathrm{I}\, t\rangle \wedge \bigwedge z\langle z\mathrm{M} \rightarrow z \ulcorner u \mathrm{I}\, u\rangle \wedge t\mathrm{I}u \rightarrow \bigwedge z\langle z\mathrm{M} \rightarrow z \vdash \langle t\mathrm{E} \vee u\mathrm{E} \rightarrow t\mathrm{I}u\rangle\rangle \wedge \langle t\mathrm{E} \vee u\mathrm{E} \rightarrow t\mathrm{I}u\rangle$
2. $x\mathrm{I}y \rightarrow \langle {}^{y}_{x}F \leftrightarrow F\rangle$
3. $\sim\bigvee xx\mathrm{E} \rightarrow \langle {}^{u}_{t}F \leftrightarrow F\rangle$.

Proof: Assume the antecedent. By 5 and 1 of Corollary 8 and Theorem 10, Corollary 13, and Theorem 4, 1 is valid. Hence, 2 is valid by Theorems 18, 20, and 16. Finally, $\sim\bigvee xx\mathrm{E} \rightarrow \bigwedge z\langle z\mathrm{M} \rightarrow z \vdash \langle t\mathrm{E} \vee u\mathrm{E} \rightarrow t\mathrm{I}u\rangle\rangle \wedge \langle t\mathrm{E} \vee u\mathrm{E} \rightarrow t\mathrm{I}u\rangle$ is valid by Theorem 17 and 3 of Corollary 5, and so 3 is valid by Theorem 16.

Corollary 15 *If x is the sequence in standard order of variables free in F, y is the first variable, t is the sequence defined on the domain of x whose only value is $\iota y \sim y\mathrm{E}$, and $F' = \langle \bigvee yy\mathrm{E} \rightarrow \mathrm{C}(\bigwedge xF)\rangle \wedge \langle \sim\bigvee yy\mathrm{E} \rightarrow {}^{x}_{t}F\rangle$, then $F' \rightarrow F$ is valid. Also, F' is valid just when F is.*

Proof: Assume the antecedent. $F' \rightarrow \langle \bigvee yy\mathrm{E} \rightarrow F\rangle \wedge \langle \sim\bigvee yy\mathrm{E} \rightarrow F\rangle$ is valid by 4 of Corollary 5 and 3 of Corollary 14 and so $F' \rightarrow F$ is valid. Hence, F is valid when F' is. Also, if F is valid, F' is valid by Corollary 6 and 3 of Corollary 14.

Corollary 16 *If $y \neq x$ and y is not free in F, then the following formulas are valid*:

1. $\iota xF\mathrm{E} \leftrightarrow \bigvee y \bigwedge x\langle F \leftrightarrow x\mathrm{I}y\rangle$
2. $\bigvee y \bigwedge x\langle F \leftrightarrow x\mathrm{I}y\rangle \leftrightarrow \bigvee x\langle F \wedge \iota xF \mathrm{I}\, x\rangle$
3. $\bigwedge y\langle y\mathrm{M} \rightarrow y \ulcorner \iota xF \mathrm{I}\, \iota xF\rangle \rightarrow \langle \bigvee y \bigwedge x\langle F \leftrightarrow x\mathrm{I}y\rangle \rightarrow {}^{x}_{\iota xF} F\rangle$
4. $\iota xF\mathrm{E} \rightarrow \bigwedge y\langle y\mathrm{M} \rightarrow \langle \bigwedge x\langle y \vdash F \leftrightarrow F\rangle \leftrightarrow y \ulcorner \iota xF \mathrm{I}\, \iota xF\rangle\rangle$.

Proof: Assume the antecedent and let z be a variable not occurring in x, y, or F. By Corollary 11 and Theorem 19, $\iota xF\mathrm{E} \leftrightarrow \bigvee y\langle \bigwedge x\langle F \leftrightarrow x\mathrm{I}y\rangle \wedge \iota xF \mathrm{I}\, y\rangle$ is valid. Also, $\bigwedge x\langle F \leftrightarrow x\mathrm{I}z\rangle \wedge \iota xF \mathrm{I}\, z \rightarrow \bigvee y \bigwedge x\langle F \leftrightarrow x\mathrm{I}y\rangle \wedge \bigvee x\langle F \wedge \iota xF \mathrm{I}\, x\rangle$ is valid by Theorem 8 and 3 through 5 of Corollary 5. By Corollary 12, it follows that $\iota xF\mathrm{E} \rightarrow \bigvee y \bigwedge x\langle F \leftrightarrow x\mathrm{I}y\rangle \wedge \bigvee x\langle F \wedge \iota xF \mathrm{I}\, x\rangle$ is valid. On the other hand, $\bigvee y \bigwedge x\langle F \leftrightarrow x\mathrm{I}y\rangle \rightarrow \bigvee y\langle y\mathrm{E} \wedge \bigwedge x\langle F \leftrightarrow x\mathrm{I}y\rangle\rangle$ is valid by Corollary 9 and

$z\mathrm{E} \wedge \bigwedge x\langle F \leftrightarrow x\mathrm{I}z\rangle \to \bigvee z \bigvee y\langle \bigwedge x\langle F \leftrightarrow x\mathrm{I}y\rangle \wedge z\mathrm{I}y\rangle$ is valid by Corollary 11 and 3 and 5 of Corollary 5. Hence, $\bigvee y \bigwedge x\langle F \leftrightarrow x\mathrm{I}y\rangle \to \iota xF\mathrm{E}$ is valid by Corollary 12, Theorems 19 and 20, 4 of Corollary 8, and Corollary 11. Finally, since ${}^{x}_{z}F \wedge \iota xF \;\mathrm{I}\; z \to \iota xF\mathrm{E}$ is valid by Theorem 8, $\bigvee x\langle F \wedge \iota xF \;\mathrm{I}\; x\rangle \to \iota xF\mathrm{E}$ is valid by Corollary 6 and Theorem 22. It follows that 1 and 2 are valid.

By 1 of Corollary 8, Theorem 1, and Theorem 15, $\bigwedge y\langle y\mathrm{M} \to y \sqsubset \iota xF \;\mathrm{I}\; \iota xF\rangle \wedge \iota xF\mathrm{E} \wedge \bigwedge x\langle F \leftrightarrow x\mathrm{I}z\rangle \to \langle {}^{x}_{\iota xF} F \leftrightarrow \iota xF \;\mathrm{I}\; z\rangle$ is valid. But $\langle \bigwedge x\langle F \leftrightarrow x\mathrm{I}z\rangle \wedge \iota xF \;\mathrm{I}\; z \to {}^{x}_{\iota xF} F\rangle \to \langle \iota xF\mathrm{E} \to {}^{x}_{\iota xF} F\rangle$ is valid by Corollary 6, Theorems 19 and 20, 4 of Corollary 8, and Corollary 11. It follows from 1 that 3 is valid.

By 2 and 4 of Corollary 8, $y\mathrm{M} \wedge \bigwedge x\langle y \vdash F \leftrightarrow F\rangle \to \langle\langle \bigvee z \bigwedge x\langle y \vdash F \leftrightarrow x\mathrm{I}z\rangle \wedge \iota xF \;\mathrm{I}\; z\rangle\rangle \leftrightarrow \bigvee z\langle \bigwedge x\langle F \leftrightarrow x\mathrm{I}z\rangle \wedge \iota xF \;\mathrm{I}\; z\rangle\rangle\rangle$ is valid. Hence, $y\mathrm{M} \wedge \bigwedge x\langle y \vdash F \leftrightarrow F\rangle \to \langle y \sqsubset \iota xF \;\mathrm{I}\; \iota xF \leftrightarrow \iota xF\mathrm{E}\rangle$ is valid by Theorems 19 and 21 and Corollaries 11 and 13. On the other hand, $y\mathrm{M} \wedge y \sqsubset \iota xF \;\mathrm{I}\; \iota xF \to \bigvee x\langle y \vdash F \wedge \iota x\; y \sqsubset F \;\mathrm{I}\; x\rangle \wedge \bigvee x\langle F \wedge \iota xF \;\mathrm{I}\; x\rangle$ is valid by Theorem 21, Theorem 8, and both 1 and 2. But then $y\mathrm{M} \wedge y \sqsubset \iota xF \;\mathrm{I}\; \iota xF \to \bigwedge x\langle y \vdash F \leftrightarrow F\rangle$ is valid by Corollary 12, 1 and 2, 2 through 5 of Corollary 5, Corollary 13, 2 of Corollary 14, and Corollary 6. Hence, 4 is valid by Corollary 6.

Corollary 17 *$w\mathrm{M} \to w \vdash H$ is valid if one of the following conditions is satisfied*:

1. *H is a formula of Theorems* 3 *through* 5, 8 *through* 14, 17 *through* 19, *or* 21
2. *H is a formula of Corollaries* 1 *and* 2, *Corollaries* 4 *and* 5, *Corollaries* 7 *through* 11, *or Corollaries* 13 *and* 14, *or one of* 1, 2, *and* 4 *of Corollary* 16.

Proof: The proofs are by means of the original theorems and corollaries together with the absoluteness Theorems 4, 10, and 21, the association of iterated indices by Theorem 5, and the rewriting of bound variables by Theorem 22. In the same way:

Corollary 18 *$G \wedge w\mathrm{M} \to w \vdash H$ is valid when $G \to H$ is a formula of Theorems* 15 *and* 16, *a formula of Corollary* 3, *or* 3 *of Corollary* 16.

Notice that, in spite of Theorem 6,

Corollary 19 *$u\mathrm{M} \to u \vdash \langle \iota x\; x\mathrm{M} \wedge x\mathrm{A}\; \mathrm{M} \wedge \iota x\; x\mathrm{M} \wedge x\mathrm{A}\; \mathrm{A}\rangle$ is valid.*

Let y and z be distinct variables not occurring in u. By Theorems 11, 18, and 8 and Corollaries 6 and 11, $u\mathrm{M} \wedge z\mathrm{I}u \to \bigwedge y\langle u \vdash \langle y\mathrm{M} \wedge y\mathrm{A}\rangle \leftrightarrow y\mathrm{I}z\rangle \wedge z\mathrm{I}z$ is valid and so $u\mathrm{M} \wedge z\mathrm{I}u \to z \;\mathrm{I}\; \iota y\; u \vdash \langle y\mathrm{M} \wedge y\mathrm{A}\rangle$ is as well by Theorem 8, 3 and 5 of Corollary 5, and Theorem 19. By Theorems 6, 21, and 22 and Corollaries 13, 11, and 12, it follows that $u\mathrm{M} \to u \sqsubset \iota x\; x\mathrm{M} \wedge x\mathrm{A} \;\mathrm{I}\; u$ is valid. The corollary then holds by Theorem 11.

Now let $t = \iota x\; x\mathrm{M} \wedge x\mathrm{A}^3$ and let $F = \mathord{\sim}\langle t\mathrm{M} \wedge t\mathrm{A}\rangle$. By Theorem 4 and Corollary 19, $u\mathrm{M} \to \mathord{\sim} u \vdash F$ is valid and so $u\mathrm{M} \to u \vdash F$ is not valid although

F is valid by Theorems 6 and 7. In other words, there is a valid formula which is situationally invalid.

4 *The system* **L** An *axiom* of **L** is any of the formulas of Theorems 1, 3 through 19, and 21 through 23. A formula is **L**-*provable* just when it is in every set K such that every axiom of **L** is in K, G is in K when F and $F \to G$ are for any F and G, and $\wedge xF$ is in K when F is for any x and F. If K is a set of formulas, then K *implies* F just in case either F is **L**-provable or there is conjunction k of members of K such that $k \to F$ is **L**-provable. K is *consistent* just in case there is an F such that K does not **L**-imply F. Finally, K is *satisfiable* just when there are an interpreter i and assigner in Ui a such that, for any $F \in K$, $\mathrm{Int}_{ia}(F) = 1$.

Theorem 24 *F is* **L**-*provable just when $\{\sim F\}$ is not consistent and F is valid just when $\{\sim F\}$ is not satisfiable.*

Proof: F is **L**-provable just when $\sim F \wedge \sim F \to G$ is **L**-provable for any formula G via tautologies and *modus ponens*. Also, F is valid just when $\mathrm{Int}_{ia}(\sim F) = 0$ for any interpreter i and assigner in Ui a.

Theorem 25 *Corollaries 1 through 19 hold when "valid" is replaced with "***L**-*provable".*

Proof: The formulas of these corollaries were shown to be valid by derivations from the axioms of **L** via *modus ponens* and universal generalization.

Theorem 26 *If F is* **L**-*provable, k is an individual constant, and x does not occur in F, then ${}^{k}_{x}F$ is* **L**-*provable.*

Proof: Let K be the set of all formulas F such that, if k is an individual constant and x does not occur in F, then both F and ${}^{k}_{x}F$ are **L**-provable. Since every axiom of **L** is in K and K is closed under *modus ponens* and universal generalization, the theorem follows.

The following theorem is a strong assertion of the soundness and semantic completeness of **L**.

Theorem 27 *If K is a set of formulas, then K is consistent just when K is satisfiable.*

Proof: Assume the antecedent. Since the individual constants can be mapped unto the individual constants with even indices, it can be assumed without loss of generality that S is some nonrepeating denumerably infinite sequence of individual constants none of which occurs in members of K. Some correlation of the formulas with the positive integers is also taken for granted.

Assume first that K is satisfiable. If K is empty, then, since $F = \sim\wedge xx\mathrm{E}$ is not valid and so not **L**-provable by Corollary 5 and Theorems 1 through 23, K does not imply F. Similarly, if k is a conjunction of members of K and $k \to F$ is **L**-provable, $k \to F$ and so $\sim k$ are valid by Theorems 1 through

23. This contradicts the assumption that K is satisfiable and $k \to F$ is again not **L**-provable. Hence, K is consistent.

The proof of the converse is more complex. Let f, g, and K' be such that f and g are functions on the natural numbers, $f(0) = K$, $g(0) =$ the set of members of members of the range of f, and $K' =$ the set of members of members of the range of g. Also, if n is a natural number, then $f(n + 1) =$ the union of $f(n)$ and $\{\vee xF \to \wedge x\langle x\mathrm{M} \to x \ulcorner k \mathrm{~I~} k\rangle \wedge k\mathrm{E} \wedge {}^{x}_{k}F\}$ where $\vee xF$ is the $n + 1^{\text{th}}$ existential generalization and $k = S(n + 1)$. On the other hand, $g(n + 1) =$ the union of $g(n)$ and $\{F\}$ if this set is consistent, and $g(n + 1) = g(n)$ otherwise where F is the $n + 1^{\text{th}}$ formula.

Assume now that K is consistent. Clearly, K' is not empty. If K' is not consistent, there is a conjunction k' of members of K' such that $k' \to \sim \wedge xx\mathrm{E}$ is **L**-provable. Since k' has finitely many conjuncts, there then exists a j such that j is the least natural number j for which k' is a conjunction of members of $g(j)$. However, since $g(n + 1)$ is consistent if $g(n)$ is for any natural number n, it follows via mathematical induction that $j = 0$. Consequently, there is an i such that i is the least natural number i for which k' is a conjunction of members of $f(i)$. But, if n is a natural number and $f(n + 1)$ is not consistent, $f(n)$ implies $\vee xF \wedge \langle {}^{x}_{k}F \to \sim \langle \wedge x\langle x\mathrm{M} \to x \ulcorner k \mathrm{~I~} k\rangle \wedge k\mathrm{E}\rangle\rangle$ where k is an individual constant occurring in neither F nor some member of $f(n)$ by tautologies and *modus ponens*. Since $\vee xF$ is not valid and so not **L**-provable by Theorems 1 through 23, there is a conjunction k'' of members of $f(n)$ such that $k'' \to \vee xF$ and $k'' \wedge {}^{x}_{k}F \to \sim \langle \wedge x\langle x\mathrm{M} \to x \ulcorner k \mathrm{~I~} k\rangle \wedge k\mathrm{E}\rangle$ are **L**-provable by tautologies and *modus ponens*. Let y be a variable not occurring in k'' or $\vee xF$. Since ${}^{kx}_{yk}F = {}^{x}_{y}F$, it follows from Theorem 26 that $k'' \wedge {}^{x}_{y}F \to \sim\langle \wedge x\langle x\mathrm{M} \to x \ulcorner y \mathrm{~I~} y\rangle \wedge y\mathrm{E}\rangle$ is **L**-provable. Since $\wedge x\langle x\mathrm{M} \to x \ulcorner y \mathrm{~I~} y\rangle$ is the universal generalization of an axiom of **L**, it follows from the part of Theorem 25 corresponding to existential instantiation, tautologies, *modus ponens*, and the part of Theorem 25 corresponding to 2 of Corollary 5 that $k'' \to \sim \vee xF$ is **L**-provable and $f(n)$ is not consistent. Consequently, $i = 0$ via mathematical induction and K is not consistent. That is a contradiction and so

1. K' is consistent.

Hence,

2. K' implies F just when $F \in K'$, and $F \notin K'$ just when $\sim F \in K'$.

Let n be the positive integer such that F is the n^{th} formula. Clearly, K' implies F if $F \in K'$. Also, if K' implies F, then $F \in g(n) \subseteq K'$ since K' is otherwise not consistent. For the same reason, $F \notin K'$ if $\sim F \in K'$. Finally, if $F \notin K'$, then $g(n - 1)$ implies $\sim F$ and so $\sim F \in K'$.

Now let $\mathrm{ID}(t)$ be the set of all u such that $t\mathrm{I}u \in K'$. Since $t\mathrm{E} \leftrightarrow t\mathrm{I}t$, $t\mathrm{I}u \to t\mathrm{E}$, $t\mathrm{I}u \leftrightarrow u\mathrm{I}t$, and $t\mathrm{I}u \wedge u\mathrm{I}v \to t\mathrm{I}v$ are **L**-provable via Theorem 25, it follows from 2 that:

3. $\mathrm{ID}(t)$ is empty just when $\sim t\mathrm{E} \in K'$. Also, one of $\mathrm{ID}(t)$ and $\mathrm{ID}(u)$ is not empty just when $t\mathrm{I}u \in K'$ and $\mathrm{ID}(t) = \mathrm{ID}(u)$ are equivalent.

4. $\mathrm{ID}(t)$ is not empty just when there is an individual constant $k \in \mathrm{ID}(t)$ such that $\wedge x\langle x\mathrm{M} \to x \ulcorner k \mathrm{\ I\ } k\rangle \wedge k\mathrm{E} \in K'$.

For some individual constant k, $\vee yt\mathrm{I}y \to \wedge y\langle y\mathrm{M} \to y \ulcorner k \mathrm{\ I\ } k\rangle \wedge k\mathrm{E} \wedge t\mathrm{I}k$ is in K'. Also, $t\mathrm{I}u \to \vee yt\mathrm{I}y$ is $\mathbf{L}$-provable for y not occurring in t by Theorem 25. Since x is not free in k, y can then be replaced with x in the universal generalization by the axiom of $\mathbf{L}$ corresponding to Theorem 22.

Now let s be the set of all e such that, for some t, $\mathrm{ID}(t)$ is not empty and $e = \mathrm{ID}(t)$. If $e \in a$, let $\mathrm{N}(e)$ = the first individual constant k such that $k \in e$ and $\wedge x\langle x\mathrm{M} \to x \ulcorner k \mathrm{\ I\ } k\rangle \wedge k\mathrm{E} \in K'$ for some x. Similarly, if k is a natural number and $e \in s^k$, then $\mathrm{N}(e)$ = the k-term sequence d such that $d_j = \mathrm{N}(e_j)$ for positive $j \leqslant k$. It is significant that:

5. $\wedge xF \in K'$ just when ${}^{\;x}_{\mathrm{N}(e)}F \in K'$ for any $e \in s$.

Assume first $\wedge xF \in K'$ and $e \in s$. By 4, $\wedge x\langle x\mathrm{M} \to x \ulcorner \mathrm{N}(e) \mathrm{\ I\ } \mathrm{N}(e)\rangle \wedge \mathrm{N}(e)\mathrm{E} \in K'$ and so ${}^{\;x}_{\mathrm{N}(e)}F \in K'$ by the weak schema of universal instantiation of $\mathbf{L}$. On the other hand, if $\wedge xF \notin K'$, then $\vee x \sim F \in K'$ by 2 and quantifier negation through Theorem 25. But then there is an individual constant k not occurring in F such that $\wedge x\langle x\mathrm{M} \to x \ulcorner k \mathrm{\ I\ } k\rangle \wedge k\mathrm{E} \wedge \sim {}^{x}_{k}F \in K'$ by the definition of K'. By 2 and 3, $e = \mathrm{ID}(k)$ is not empty and $k \mathrm{\ I\ } \mathrm{N}(e) \in K'$. But $\wedge x\langle x\mathrm{M} \to x \ulcorner \mathrm{N}(e) \mathrm{\ I\ } \mathrm{N}(e)\rangle \wedge \mathrm{N}(e)\mathrm{E} \in K'$ as well by 4 and so $\sim {}^{\mathrm{N}(e)}_{k}\ {}^{\;x}_{\mathrm{N}(e)}F \leftrightarrow \sim {}^{\;x}_{\mathrm{N}(e)}F \in K'$ by the weak schema of indiscernibility of $\mathbf{L}$ and the part of Theorem 25 corresponding to 1 of Corollary 14. Since ${}^{x}_{k}F = {}^{\mathrm{N}(e)}_{k}\ {}^{\;x}_{\mathrm{N}(e)}F$, it follows that $\sim {}^{\;x}_{\mathrm{N}(e)}F \in K'$ and ${}^{\;x}_{\mathrm{N}(e)}F \notin K'$ by 2.

Let $\circ$ = the least ordinal $\notin s$, let m = the set of all $e \in s$ such that, for some $t \in e$, $t\mathrm{M} \in k'$, and let n = the union of m and $\{\circ\}$. Also, let $\mathrm{V}_\circ(t) = \{\mathrm{ID}(t)\}$ if $t\mathrm{E} \in K'$ and $\{\ \}$ if $\sim t\mathrm{E} \in K'$. Also, let $\mathrm{V}_\circ(F) = 1$ if $F \in K'$ and 0 if $\sim F \in K'$. On the other hand, if $e \in s$, let $\mathrm{V}_e(t) = \{\mathrm{ID}(\mathrm{N}(e) \ulcorner t)\}$ if $\mathrm{N}(e) \ulcorner t\mathrm{E} \in K'$, and $\{\ \}$ if $\sim \mathrm{N}(e) \ulcorner t\mathrm{E} \in K'$. Finally, $\mathrm{V}_e(F) = 1$ if $\mathrm{N}(e) \vdash F \in K'$ and 0 if $\sim \mathrm{N}(e) \vdash F \in K'$. These clauses are proper by 2 through 4. If there is a c such that $\mathrm{CN}klmcxtF$, then $\mathrm{SP}(xtF)$ = the n-spread f in s of type klm such that, for any $p \in n$, positive $j \leqslant l + m$, and $e \in s^k$, $(f(p)(j))(e) = \mathrm{V}_p({}^{\;x}_{\mathrm{N}(e)}(t^\frown F)_j)$.

6. If $\mathrm{CN}klmcxtF$, $\mathrm{CN}klmcyuG$, $\mathrm{SP}(xtF) = \mathrm{SP}(yuG)$, and $p \in n$, then $\mathrm{V}_p(c(xtF)) = \mathrm{V}_p(c(yuG))$.

Assume the antecedent and let $z^\frown\langle z'\rangle$ be the sequence of the first $k + 1$ variables not occurring in x, y, t, u, F, or G. If $e \in s^k$ and T is a value of one of t through G, then ${}^{\;x}_{\mathrm{N}(e)}T = {}^{\;z}_{\mathrm{N}(e)}\ {}^{x}_{z}T$. Also, if $1 \leqslant j \leqslant l + m$, then $\mathrm{V}_q({}^{\;x}_{\mathrm{N}(e)}(t^\frown F)_j) = \mathrm{V}_q({}^{\;y}_{\mathrm{N}(e)}(u^\frown G)_j)$ for any $q \in n$ since $\mathrm{SP}(xtF) = \mathrm{SP}(yuG)$. Hence, if $1 \leqslant j \leqslant l$, $H(e) = {}^{\;x}_{\mathrm{N}(e)}t_j\mathrm{E} \vee {}^{\;y}_{\mathrm{N}(e)}u_j\mathrm{E} \to {}^{\;x}_{\mathrm{N}(e)}t_j \mathrm{\ I\ } {}^{\;y}_{\mathrm{N}(e)}u_j$, and $H'(z) = {}^{x}_{z}t_j\mathrm{E} \vee {}^{y}_{z}u_j\mathrm{E} \to {}^{x}_{z}t_j \mathrm{\ I\ } {}^{y}_{z}u_j$, then $\mathrm{N}(q)\mathrm{M} \to \mathrm{N}(q) \vdash H(e) \in K'$ for any $q \in s$ and $H(e) \in K'$. Via 5, it follows that $\wedge z'\langle z'\mathrm{M} \to z' \vdash H(e)\rangle \wedge H(e) \in K'$ for any $e \in s^k$. Similarly, by iterated application of 5, it follows that $\mathrm{C}(\wedge z\ \wedge z'\langle z'\mathrm{M} \to$

$z' \vdash H'(z)\rangle \wedge H'(z)) \in K'$. By analogous reasoning, the same holds when $1 \leqslant j \leqslant m$, $H(e) = {}^{x}_{\mathsf{N}(e)} F_j \leftrightarrow {}^{y}_{\mathsf{N}(e)} G_j$, and $H'(z) = {}^{x}_{z} F_j \leftrightarrow {}^{y}_{z} G_j$. From the absoluteness of the connectives, E and I, together with the axioms of **L** corresponding to the formulas of Theorems 16, 22, and 23, it follows that $\mathsf{V}_p(c(xtF)) = \mathsf{V}_p(c(z\ {}^{x}_{z}t\ {}^{x}_{z}F)) = \mathsf{V}_p(c(z\ {}^{y}_{z}u\ {}^{y}_{z}G)) = \mathsf{V}_p(c(yuG))$ for any $p \in n$.

If $p \in n$ and c is a constant of type Cklm, then $\mathsf{IN}_p(c)$ is a specific n-intension g in s of type Cklm. If CN$klmcxtF$ and $f = \mathsf{SP}(xtF)$, then $g(f) = \mathsf{V}_p(c(xtF))$. For other n-spreads f in s of type klm, if c is a basic logical constant other than A, $g(f)$ is determined by clauses 6, 7, 8, 10, and 12 through 16 of the definition of interpreter with $i(c,p) = g$. If $c =$ A and $f(p) = \mathsf{SP}(xtF)(p)$, then $g(f) = g(\mathsf{SP}(xtF))$. Otherwise, $g(f) = 1$ just when $p \neq \mathrm{o}$ and $f(p) = \langle$the $\{p\}$-function$\rangle$. For the remaining constants c, $g(f) = \{\ \}$ if c is term-making and $g(f) = 0$ if c is formula-making. This specification of $\mathsf{IN}_p(c)$ is proper by 6 and the **L**-provability of $F \wedge \langle v\mathrm{M} \rightarrow v \vdash F\rangle$ where $F = \langle t\mathrm{E} \vee u\mathrm{E} \rightarrow t\mathrm{I}u\rangle \rightarrow \langle t\mathrm{A} \leftrightarrow u\mathrm{A}\rangle$.

Now let i be the function defined on the set of all pairs c,p with c a constant and $p \in n$ and such that $i(c,p) = \mathsf{IN}_p(c)$ for any given c,p. It must next be shown that

7. i is an interpreter.

It is clearly sufficient to show that, if CN$klmcxtF$, c is a basic logical constant, $p \in n$, and $f = \mathsf{SP}(xtF)$, then $i(c,p)(f)$ satisfies the relevant clauses of 6 through 16 of the definition of interpreter. Let $r = i(c,p)(f) = \mathsf{V}_p(c(xtF))$ under these assumptions. Assume first that $p = \mathrm{o}$. If $c =$ E, $r = 1$ just when $t_1\mathrm{E} \in K'$ and so just when there is an $e \in s$ such that $\mathsf{V}_p(t_1) = \{e\}$ by 2 and 3. If $c =$ I, $r = 1$ just when $t_1\mathrm{I}\ t_2 \in K'$ and so just when there is an $e \in s$ such that $\mathsf{V}_p(t_1) = \mathsf{V}_p(t_2) = \{e\}$ through the **L**-provability of $t\mathrm{I}u \rightarrow t\mathrm{E} \wedge u\mathrm{E}$ and 2 through 4. If $c =$ M, $r = 1$ just when $t_1\mathrm{M} \in K'$ and so just when there is an $e \in m$ such that $\mathsf{V}_p(t_1) = \{e\}$ through the **L**-provability of $u\mathrm{M} \rightarrow u\mathrm{E}$ and $u\mathrm{I}v \rightarrow \langle u\mathrm{M} \leftrightarrow v\mathrm{M}\rangle$ and both 2 and 3. If $c =$ A, $r = 1$ just when $t_1\mathrm{A} \in K'$ and so only if there is an $e \in s$ such that $\mathsf{V}_p(t_1) = \{e\}$ through the **L**-provability of $u\mathrm{A} \rightarrow u\mathrm{E}$, 2, and 3. Also, if f' is an n-spread in s of type 010 and $f'(p) = \langle$the $\mathsf{V}_p(t_1)$-function$\rangle$, then $i(c,p)(f') = r$ by the definition of i. If $c =$ B, $r = 1$ just when $t_1\mathrm{B} \in K'$ and so just when $\mathsf{V}_p(t_1\mathrm{E}) = 1$ and $\mathsf{V}_p(t_1\mathrm{A}) = 0$ by the **L**-provability of $u\mathrm{B} \leftrightarrow u\mathrm{E} \wedge \sim u\mathrm{A}$. If $c =$ A and $t_1\mathrm{M} \in K'$, $\sim t_1\mathrm{A} \in K'$ by 2 since $u\mathrm{M} \rightarrow \sim u\mathrm{A}$ is **L**-provable and so $r \neq 1$. If $c = \sim$, $r = 1$ just when $\sim F_1 \in K'$ and so just when $\mathsf{V}_p(F_1) = 0$. If $c = \rightarrow$, $r = 0$ just when $\sim\langle F_1 \rightarrow F_2\rangle \in K'$ and so just when $\mathsf{V}_p(F_1) = 1$ and $\mathsf{V}_p(F_2) = 0$ by the **L**-provability of tautologies. The cases for $c \in \{\wedge \vee \leftrightarrow\}$ are dealt with in the same sort of way. If $c = \bigwedge$, $r = 1$ just when $\bigwedge x_1F_1 \in K'$ and so just when $\mathsf{V}_p({}^{x_1}_{\mathsf{N}(e)}\ F_1) = 1$ for any $e \in s$ by 5. Consequently, if $c = \bigvee$, $r = 0$ just when $\sim\bigvee x_1F_1 \in K'$ and so just when $\mathsf{V}_p({}^{x_1}_{\mathsf{N}(e)}\ F_1) = 0$ for any $e \in s$ by the **L**-provability of $\sim\bigvee yG \leftrightarrow \bigwedge y\sim G$. If $c = \mathrm{ɿ}$, $r = \{e\}$ just when $e = \mathsf{ID}(\mathrm{ɿ}x_1F_1) \neq \{\ \}$ and so just when $\bigvee y \bigwedge x_1\langle F_1 \leftrightarrow x_1\ \mathrm{I}\ y\rangle \in K'$ for some y not occurring in $\mathrm{ɿ}x_1F_1$ by 2, 3, and the **L**-provability of $\mathrm{ɿ}yG\mathrm{E} \leftrightarrow \bigvee z \bigwedge y\langle G \leftrightarrow y\mathrm{I}z\rangle$ for z not occurring in $\rceil yG$. Consequently, if $r = \{e\}$, there is an $e' \in s$ such that, for any $e'' \in s$, $\frac{x_1}{\mathsf{N}(e'')} F_1 \in K'$ just when $e'' = e'$. Hence, $\mathsf{V}_p(\frac{x_1}{\mathsf{N}(e'')} F_1) = 0$ if $e'' \neq e'$. Also, since $\mathsf{N}(e')$ I $\mathsf{N}(e') \in K'$ by 2, 3, and the **L**-provability of $u\mathrm{E} \leftrightarrow u\mathrm{I}u$, it follows from 2, 3, and the **L**-provability of weak existential generalization that $\bigvee y\langle \bigwedge x_1 \langle F_1 \leftrightarrow x_1 \mathrm{~I~} y\rangle \wedge \mathsf{N}(e') \mathrm{~I~} y\rangle \in K'$. Hence, by the description schema of **L** and 3, $e' = e$ and so $\mathsf{V}_p(\frac{x_1}{\mathsf{N}(e)} F_1) = 1$ if $r = \{e\}$. On the other hand, if $e \in s$ and $\mathsf{V}_p(\frac{x_1}{\mathsf{N}(e'')} F_1) = 1$ just when $e'' = e$ for $e'' \in s$, $\bigvee y\langle \bigwedge x_1\langle F_1 \leftrightarrow x_1 \mathrm{~I~} y\rangle \wedge$ $\mathsf{N}(e) \mathrm{~I~} y\rangle \in K'$ and $r = \{e\}$ by similar reasoning. If $c = \ulcorner$ and there is no $q \in m$ such that $\mathsf{V}_p(t_1) = \{q\}$, $\sim t_1\mathrm{M} \in K'$ by the **L**-provability of $u\mathrm{I}v \rightarrow \langle u\mathrm{M} \leftrightarrow v\mathrm{M}\rangle$ and so $r = \{\ \}$ by 2 and the **L**-provability of $\sim u\mathrm{M} \rightarrow \sim u \ulcorner v$ E. On the other hand, if there is a $q \in m$ such that $\mathsf{V}_p(t_1) = \{q\}$, $r = \mathsf{V}_q(t_2)$ by 2 through 4 since $\mathsf{N}(q) \mathrm{~I~} t_1 \rightarrow \mathsf{N}(q) \ulcorner t_2 \mathrm{~E} \vee t_1 \ulcorner t_2 \mathrm{~E} \rightarrow \mathsf{N}(q) \ulcorner t_2 \mathrm{~I~} t_1 \ulcorner t_2$ is **L**-provable. The case in which $c = \vdash$ is dealt with analogously.

It follows that 6 through 16 of the definition of interpreter hold for i when $p = \mathrm{o}$. When $p \neq \mathrm{o}$, the cases for when c is one of the absolute constants E, I, M, $\sim$, $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$, $\bigwedge$, $\bigvee$, and $\rceil$ follow from the corresponding cases with $p = \mathrm{o}$ by the various absoluteness axioms of **L**. Similarly, the cases with $p \neq \mathrm{o}$, $k = \mathsf{N}(p)$, and $c \in \{\ulcorner \vdash\}$ follow from the cases with $p = \mathrm{o}$ via the **L**-provability of $k \ulcorner \langle t_1 \ulcorner t_2\rangle\mathrm{E} \vee \langle k \ulcorner t_1\rangle \ulcorner t_2 \mathrm{~E} \rightarrow k \ulcorner \langle t_1 \ulcorner t_2\rangle \mathrm{~I~} \langle k \ulcorner t_1\rangle \ulcorner t_2$ and $k \vdash t_1 \vdash F_1 \leftrightarrow k \ulcorner t_1 \vdash F_1$. Also, if $c = \mathrm{A}$, $r = 1$ only if $k \vdash t_1 \mathrm{A} \in K'$ and so only if there is an $e \in s$ such that $\mathsf{V}_p(t_1) = \{e\}$ via the absoluteness of the sentential connectives, the **L**-provability of $k\mathrm{M} \rightarrow k \vdash \langle t_1\mathrm{A} \rightarrow t_1\mathrm{E}\rangle$, 2, and 3. Similarly, if $c = \mathrm{B}$, $r = 1$ just when $\mathsf{V}_p(t_1\mathrm{E}) = 1$ and $\mathsf{V}_p(t_1\mathrm{A}) = 0$ by the **L**-provability of $k\mathrm{M} \rightarrow k \vdash \langle t_1\mathrm{B} \leftrightarrow t_1\mathrm{E} \wedge \sim t_1\mathrm{A}\rangle$ and the absoluteness of the sentential connectives. Also, if $c = \mathrm{A}$, f' is an n-spread in s of type 010, and $f'(p) = \langle$the $\mathsf{V}_p(t_1)$-function$\rangle$, then $i(c,p)(f') = r$ by the definition of i. Finally, if $k \vdash t_1\mathrm{M} \in K'$, $r = 1$ just when $\mathsf{V}_p(t_1) = \{p\}$ via the **L**-provability of $k \vdash \langle t_1\mathrm{M} \wedge t_1\mathrm{A}\rangle \leftrightarrow k \ulcorner t_1 \mathrm{~I~} k$ and 2 through 4. Thus, 6 through 16 of the definition of interpreter hold for i and 7 is established.

If $d = \{\ \}$, let $\mathsf{N}'(d) = \rceil y \sim y\mathrm{E}$ where y is the first variable. On the other hand, if there is an $e \in s$ such that $d = \{e\}$, let $\mathsf{N}'(d) = \mathsf{N}(e)$. For any assigner a in $\mathrm{U}i = s$ and any T, if x is the sequence in standard order of the variables free in T and k is the sequence with the same domain as x such that $k_j = \mathsf{N}'(a(x_j))$ for j in the domain of k, let $T^a = {}^x_k T$. Clearly,

8. If a is an assigner in $\mathrm{U}i$ and $p \in \mathrm{N}i$, then $\mathrm{Int}_{iap}(x) = \mathsf{V}_p(x^a) = \mathrm{Int}_{iap}(x^a)$.

For $x^a = \mathsf{N}'(a(x))$, $\mathrm{Int}_{iap}(x) = a(x) = \mathsf{V}_p(\mathsf{N}'(a(x)))$, and $\mathsf{V}_p(x^a) = \mathrm{Int}_{iap}(x^a)$ under the assumption of 8 by the definitions of i and N'. Also,

9. If $\mathrm{CN}klmcxtF$ and $\mathrm{Int}_{iap}((t^\frown F)_j) = \mathsf{V}_p({(t^\frown F)_j}^a) = \mathrm{Int}_{iap}({(t^\frown F)_j}^a)$ for any assigner a in $\mathrm{U}i$, $p \in \mathrm{N}i$, and positive $j \leqslant l + m$, then $\mathrm{Int}_{iap}(c(xtF)) = \mathsf{V}_p(c(xtF)^a) = \mathrm{Int}_{iap}(c(xtF)^a)$ for any assigner a in $\mathrm{U}i$ and $p \in \mathrm{N}i$.

Assume the antecedent. If a is an assigner in $\mathrm{U}i$, let t'^a and F'^a be l- and m-term sequences such that $(t'^{a\frown} F'^a)_j = {}^y_{k'}(t^\frown F)_j$ for positive $j \leqslant l + m$

where y is the sequence in standard order of variables free in $(t^\frown F)_j$ which are not values of x and k' is the sequence with the same domain as y such that $k'_h = \mathsf{N}'(a(y_h))$ for h in the domain of k'. Also, if $e \in \mathrm{U}i^k$, let $a^e = a(^{x}_{e'})$ where e' is the k-term sequence such that $e'(h) = \{e(h)\}$ for positive $h \leqslant k$. By the assumption of 9, $\mathsf{Int}_{ia^e p}((t^\frown F)_j) = \mathsf{V}_p((t^\frown F)_j{}^{a^e}) = \mathsf{Int}_{ia^e p}((t^\frown F)_j{}^{a^e})$ for any positive $j \leqslant l + m$, $e \in \mathrm{U}i^k$, assigner a in $\mathrm{U}i$, and $p \in \mathrm{N}i$. But, if a is an assigner in $\mathrm{U}i$ and $e \in \mathrm{U}i^k$, then $\mathsf{Int}_{ia^e p}(\mathsf{N}'(a^e(x_h))) = \mathsf{Int}_{ia^e p}(x_h)$ for any positive $h \leqslant k$ and any $p \in \mathrm{N}i$. Hence, by iterated applications of Lemma 5, $\mathsf{Int}_{ia^e p}((t^\frown F)_j{}^{a^e}) = \mathsf{Int}_{ia^e p}((t'^{a\frown} F'^{a})_j)$ for positive $j \leqslant l + m$ and such a and p when $e \in \mathrm{U}i^k$. Consequently, $\mathsf{SP}(iaxtF) = \mathsf{SP}(iaxt'^a F'^a)$ if a is an assigner in $\mathrm{U}i$. Since $c(xtF)^a = c(xt'^a F'^a)$, it follows that $\mathsf{Int}_{iap}(c(xtF)) = \mathsf{Int}_{iap}(c(xtF)^a) = \mathsf{V}_p(c(xtF)^a)$ when $p \in \mathrm{N}i$ and 9 holds. With the induction principle for terms and formulas, 8 and 9 imply

10. If a is an assigner in $\mathrm{U}i$ and $p \in \mathrm{N}i$, then $\mathsf{Int}_{iap}(T) = \mathsf{V}_p(T^a)$.

Now let a' be the function defined on the variables such that $a'(x) = \mathsf{V}_{0i}(x)$ for any variable x. Via the axioms of $\mathbf{L}$ corresponding to the formulas of Theorem 17, a' is an assigner in $\mathrm{U}i$. Also, by the induction principle for terms and formulas,

11. $\mathsf{V}_{0i}(T) = \mathsf{V}_{0i}(T^{a'})$.

Hence, if $F \in K \subseteq K'$, it follows from 10 and 11 that

12. $\mathsf{Int}_{ia'}(F) = \mathsf{V}_{0i}(F^{a'}) = \mathsf{V}_{0i}(F) = 1$.

But then K is satisfiable and the theorem is proved.

NOTES

1. A k-term sequence can be understood to be a function defined on either the natural numbers $<k$ or the positive integers $\leqslant k$. We here employ the second alternative. An index of a sequence is an object in the domain of the sequence.

2. The idea of defining intensions as functions which assign extensional objects is from Schock [10]. However, both the arguments and values of the functions were there of a different sort. The arguments of the present study are essentially the objects used as arguments of the interpretations of variable binders in Schock [11]. Contrary to what is sometimes claimed in the literature, this device was not formulated earlier in Section 40 of Carnap [1] since Carnap there explicated propositions as sets of state descriptions, properties of individuals as functions from individual constants to sets of state descriptions, and individual concepts as functions from state descriptions to individual constants. The closest that Carnap came to the idea there was an unused intuition that an individual concept might be an assignment of individuals to states. In connection with matters of precedence, it is perhaps also worth mentioning that some central semantic and logical devices from Schock [11] have reappeared later in Scott [13], Corcoran and Herring [2], and Corcoran, Herring, and Hatcher [3].

3. As the reader has perhaps already observed, t is reminiscent of terms such as "here" or "now".

REFERENCES

[1] Carnap, R., *Meaning and Necessity*, The University of Chicago Press, Chicago, 1947.

[2] Corcoran, J. and J. Herring, "Notes on a semantic analysis of variable binding term operators," *Logique et Analyse*, vol. 55 (1971), pp. 644-657.

[3] Corcoran, J., J. Herring and W. Hatcher, "Variable binding term operators," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 18 (1972), pp. 177-182.

[4] Cresswell, M., "Second-order intensional logic," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 18 (1972), pp. 297-320.

[5] Gallin, D., "Systems of intensional logic" (abstract), *The Journal of Symbolic Logic*, vol. 36 (1971), p. 585.

[6] Garson, J., "Indefinite topological logic," *Journal of Philosophical Logic*, vol. 2 (1973), pp. 102-118.

[7] Montague, R., "Pragmatics," in *Contemporary Philosophy I*, R. Klibansky, ed., La Nuova Italia Editrice, Firenze, (1968), pp. 102-122.

[8] Montague, R., "Pragmatics and intensional logic," *Synthese*, vol. 22 (1970), pp. 68-94.

[9] Rescher, N., and A. Urquhart, *Temporal Logic*, Springer, Wien and New York, 1971.

[10] Schock, R., "A definition of event and some of its applications," *Theoria,* vol. 28 (1962), pp. 250-268.

[11] Schock, R., "On the logic of variable binders," *Archiv für mathematische Logik*, vol. 6 (1964), pp. 71-90.

[12] Schock, R., *Logics without Existence Assumptions*, Almqvist and Wiksell, Stockholm, 1968.

[13] Scott, D., "Existence and description in formal logic," in *Bertrand Russell: Philosopher of the Century* R. Schoenman, ed., London, (1967), pp. 181-200.

[14] Scott, D., "Advice on modal logic," in *Philosophical Problems in Logic*, K. Lambert, Ed., D. Reidel, Dordrecht, (1970), pp. 143-173.

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