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ON SUBSTITUTION FOR VARIABLE ONE-PLACE FUNCTORS

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In this paper* I develop rules of substitution for variable one-place functors (δ) as they appear in Łukasiewicz's Ł-modal system [2]. I also prove that the first of these rules preserves validity and that the second preserves invalidity. The need for the formulation of these rules and the proof of their validity- or invalidity-preserving characteristic became apparent to me upon questioning whether Ł was sound. Smiley, in his proof that this system has a characteristic matrix, slides over the problem of the soundness of Ł, noting that

> it is easy to show that every theorem is verified by (never takes an undesignated value in) **M** [5]

and instead concentrates on the difficult problem of proving Ł complete. It turns out, however, that, like so many things in logic which seem easy to prove, the soundness of Ł requires a bit of work to prove.

The difficulty may be briefly seen if we consider the inference from

(1) $\vdash CEpqC\delta p\delta q$

to

(2) $\vdash CEpqCCppCqp$.

On an intuitive level it is clear that if (1) is valid[1] in Ł, i.e., if it is verified by Łukasiewicz's $\mathfrak{M}9$ matrix for all assignments to δ of constant one-place functors definable in Ł, then (2) is also valid, as are

(3) $\vdash CEpqCAKprEpsAKprEqs$
(4) $\vdash CEpqCK\Delta pNpK\Delta pNq$,

and so on. Hence, it is intuitively clear that the rule, (which we have not

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yet formulated) which permits deducing (2), (3), and (4) from (1), does preserve validity.

Unfortunately, this intuitive clarity is not enough in a soundness proof of Ł, any more than our *intuitive belief* that the rule of detachment for assertions preserves validity would, by itself, suffice in such a proof. We need, quite simply, to *prove* that the rule of δ-substitution for assertions (the rule which would permit the move from (1) to (2)) preserves validity and that the corresponding rule of δ-substitution for rejections in invalidity preserving. I shall first attempt to formulate both of these rules and then shall prove their validity- or invalidity-preserving characteristic. After this, a proof that Ł is sound will indeed be easy.

1 Oddly, Łukasiewicz never formalizes his δ-substitution rule for assertions, even though he does make heavy use of this sort of substitution and does formalize several other transformation rules. The most he ever does is to give examples of such substitution [2]. Even more oddly, no one else, e.g., Prior [4] or Meredith [3], who makes use of variable one-place functors ever actually attempts to give such a formulation—at least not as far as I have been able to determine.[2] Furthermore, although several logicians use but do not formulate a rule of δ-substitution for *assertions*, the corresponding δ-substitution rule for rejections has not, until now, either been formulated or used. In this section I try to fill in these gaps.

The first thing which must be noted is that what are substituted for δ in Ł are wff *fragments*. One is provided, for example, with warrants such as δ/Cp' in justifying the inference from (2) to (1). But not all wff fragments may be substituted for δ: δ/Cp'' would make no sense as a warrant in a deduction in Ł, although δ/C'' or δ/CCp'' would do nicely. Such wff fragments as Cp', C'', and CCp'' I call *congenial wff fragments* (cwfffs, for short), and state that a wff fragment is a cwfff in Ł iff the result of replacing any placeholders (′) in it by a wff of Ł is itself a wff. The following recursion clauses are designed to provide a decision procedure for determining whether a wff fragment is congenial or not:

a. ′ is a cwfff.
b. If θ is a cwfff, then $N\theta$ is a cwfff.
c. If θ is a cwfff, then $\Delta\theta$ is a cwfff.
d. If θ is a cwfff, then $\delta\theta$ is a cwfff.
e. If θ and ϕ are cwfffs, then $C\theta\phi$ is a cwfff.
f. If θ is a cwfff and B is a wff, then $C\theta B$ is a cwfff.[3]
g. If θ is a cwfff and B is a wff, then $CB\theta$ is a cwfff.

Once we have defined what a cwfff is and given procedures for determining congeniality of a wff fragment, it is an easy matter to formulate the rule of δ-substitution for assertion, the rule of which Łukasiewicz makes heavy use: (In what follows $A_{\delta|B_1,\ldots,\delta|B_n}$ is used to represent any wff which contains δ's. $B_1, \ldots, B_n$ are the (not necessarily distinct) arguments of δ in A.)

Rule of δ-substitution for assertions: from $\vdash A_{\delta B_1,\ldots,\delta B_n}$ one may infer (as an assertion thesis) the result of universally substituting any cwfff for δ.

It is only after one has provided such a formulation of this rule that the sort of discussion Łukasiewicz provides in the appendix to [2], concerning the actual method of δ-substitution, is in order.

Since there are in the Ł-modal system transformation rules of sentential variable substitution for both assertions and rejections and rules of detachment for both assertions and rejections, one would expect there to be a rule of δ-substitution for rejections in addition to the rule given above. Although, as mentioned above, Łukasiewicz does not make use of such a rule, I do think that he could have used the following rule if he had so desired: (In what follows θ_{B_i} represents the result of substituting some cwfff θ for δ in δB_i.)

Rule of δ-substitution for rejections: from $\dashv A_{\theta_{B_1},\ldots,\theta_{B_n}}$ where θ is any cwfff, one may infer $\dashv A_{\delta B_1,\ldots,\delta B_n}$.

Using this rule one could, for example, infer

(5) $\dashv KC\delta pqC\delta qp$

from

(6) $\dashv KCKrpqCKrqp$

and, once I have proved that this rule preserves invalidity, one will be able to justly conclude that if (6) is invalid in Ł, then so is (5).

2 Before beginning my proof that the first of the above rules preserves validity while the second preserves invalidity, I shall make a few remarks about the notions of validity and invalidity in Ł of wffs containing δ's. It is clear that a wff such as

(7) $C\delta pC\delta Np\delta q$

is not determined to be either valid or invalid in Ł *merely* by seeing whether, for all assignments of truth values to p and q (for all interpretations), it takes the designated value 1. For example, the assignment of 1 to p and 3 to q tells us nothing about the validity of (7), because

(8) $C\delta 1C\delta N1\delta 3$

does not have a truth value. It still contains a (nonsentential) variable, viz., δ. Something needs to be assigned to δ, but certainly not a truth value. Rather, since δ is a variable one-place functor, we assign to it constant one-place functors, definable in Ł, e.g., N, V (verum), T (constant 3 functor), etc.

This is not a startling development, and Łukasiewicz himself says much the same thing ([2], pp. 126-127). What is important, though, especially for understanding the proofs which I shall give, is that we think of an interpretation of a wff in Ł as not only assigning truth values to sentential variables contained in the wff, but also as assigning constant one-place functors definable in Ł to any δ's in the wff. I propose, if only to keep this second sort of assignment in clear focus, to call this new type of interpretation a δ-interpretation.[4] A wff will thus be valid in Ł iff it takes the value 1 under every δ-interpretation, i.e., for every possible assignment of truth values to its sentential variables and constant one-place functors definable in Ł to the δ's it contains (if any). Likewise, a wff will be invalid in Ł iff it takes a non-designated value under some δ-interpretation. It is in these terms that my claims that the above rules preserve validity or invalidity are to be understood.

Finally, since I shall be speaking of the δ-interpretations of Ł, I must list the constant one-place functors definable in Ł which shall be assigned by these interpretations:

f_1 - 1111	f_5 - 2121	f_9 - 3311	f_{13} - 4321
f_2 - 1133	f_6 - 2143	f_{10} - 3333	f_{14} - 4343
f_3 - 1212	f_7 - 2222	f_{11} - 3412	f_{15} - 4422
f_4 - 1234	f_8 - 2244	f_{12} - 3434	f_{16} - 4444.

Several of these functors, of course, appear in Ł under a different symbol: e.g., f_2 is the Δ functor, f_8 is the Γ functor, etc. In any case, Łukasiewicz proves that f_1-f_{16} are the only constant one-place functors definable in Ł ([2], pp. 126-127).

I now begin my proof that the rule of δ-substitution for assertions preserves validity.

Lemma 1: *For any wff B of Ł,* $f_j f_k \equiv f_1 B$, *where* f_j, f_k *and* f_1 *are all one-place functors definable in Ł.*[5]

Proof: This can be proved by checking each of the 256 possibilities. In each case $f_j f_k B$ will be equivalent to some $f_1 B$, no matter which functors f_j and f_k are. For example, $f_8 f_{11} B \equiv f_{15} B$, $f_{13} f_3 \equiv f_{14} B$, and so on.

Lemma 2: *For any wff B of Ł,* $Nf_j B \equiv f_1 B$, *where* f_j *and* f_1 *are both one-place functors definable in Ł.*

Proof: This follows directly from Lemma 1, since N is f_{13}.

Lemma 3: *For any wff B of Ł,* $\Delta f_j B \equiv f_1 B$, *where* f_j *and* f_1 *are both one-place functors definable in Ł.*

Proof: This follows directly from Lemma 1, since Δ is f_2.

Lemma 4: *For any wff B of Ł,* $Cf_j B f_k B \equiv f_1 B$, *where* f_j, f_k *and* f_1 *are all one-place functors definable in Ł.*

Proof: This may be proved by checking each of the 256 possibilities. In each case $Cf_j B f_k B$ will be equivalent to some $f_1 B$, no matter which functors f_j and f_k are. For example, $Cf_6 B f_{12} B \equiv f_{11} B$, $Cf_2 B F_{14} B \equiv f_{13} B$, and so on.

In the following, θ_D and θ_E are the results of substituting a cwfff θ of Ł for δ in δD and δE.

Lemma 5: *For any δ-interpretation* Σ_i *of Ł, for any cwfff* θ *of Ł and for* *any two wffs D and E of Ł, there is some one-place functor f_j definable in Ł such that f_jD has the same value under Σ_i as θ_D, and f_jE has the same value under Σ_i as θ_E.*

Proof: We use induction on the number of connectives in θ_D or in θ_E occurring outside of D or E.

Case α: θ_D and θ_E have no connectives occurring outside of D or E and so are D and E. It is clear that, whatever value D has under Σ_i, f_4D will have that same value under Σ_i, since f_4 does not change the value of its argument under any δ-interpretation. Likewise, no matter what value E has under Σ_i, f_4E will have the same value under Σ_i. Hence the lemma holds for this case.

Case β: Assume that θ_D and θ_E have k connectives occurring outside of D or E. We must now consider the following subcases:

Subcase 1: θ_D is of the form $N\phi_D$ and θ_E is of the form $N\phi_E$. By the assumption of induction the lemma holds for both ϕ_D and ϕ_E since both contain fewer than k connectives occurring outside of D or E. Thus, there is some f_j definable in Ł such that f_jD has the same value under some Σ_i as ϕ_D and f_jE has the same value as ϕ_E under this Σ_i. If this is the case, then clearly Nf_jD and $N\phi_D$ will have the same value under Σ_i and Nf_jE and $N\phi_E$ will also have the same value under Σ_i. But, by Lemma 2 Nf_jD is equivalent to f_1D for some f_1 definable in Ł, and Nf_jE is equivalent to f_1E. Thus, f_1D, Nf_jD and $N\phi_D$ (i.e., θ_D) will all take the same value under Σ_i, and f_1E, Nf_jE and $N\phi_E$ (i.e., θ_E) will all take the same value under Σ_i. Hence the lemma holds for this subcase.

Subcase 2: θ_D is of the form $\Delta\phi_D$ and θ_E is of the form $\Delta\phi_E$. The proof here is the same as in Subcase 1, except Lemma 3 is used instead of Lemma 2.

Subcase 3: θ_D is of the form $\delta\phi_D$ and θ_E is of the form $\delta\phi_E$. Σ_i is a δ-interpretation which assigns some one-place functor f_k to δ. Thus, we are in this subcase really considering θ_D under the form $f_k\phi_D$ and θ_E under the form $f_k\phi_E$. The proof here is the same as in Subcase 1, except Lemma 1 is used instead of Lemma 3.

Subcase 4: θ_D is of the form $C\phi_DB$ and θ_E is of the form $C\phi_EB$. B is the same wff in both $C\phi_DB$ and $C\phi_EB$, and so will take the same value under Σ_i in either wff. Whatever value B takes under Σ_i, call it v_m, there is an f_n (n = 1, 7, 10, 16) which gives the value v_m under every δ-interpretation, no matter what its argument is. Thus, for some constant functor f_n, f_nD and f_nE will take the same value under Σ_i as B. For example, if B takes the value 3 under Σ_i, then $f_{10}D$ and $f_{10}E$ also take this value under Σ_i. This being so, it is clear that $C\phi_Df_nD$ takes the same value under Σ_i as $C\phi_DB$, for some constant functor f_n definable in Ł. Likewise, it is clear that $C\phi_Ef_nE$ takes the same value under Σ_i as $C\phi_EB$. Both ϕ_D and ϕ_E fall under the assumption of induction, and so there is an f_j such that f_jD has the same value under Σ_i as ϕ_D and f_jE has the same value under Σ_i as ϕ_E. If this is so, then obviously Cf_jDf_nD takes the same value under Σ_i as $C\phi_Df_nD$ (and the same value as $C\phi_DB$), and Cf_jEf_nE takes the same value under Σ_i as $C\phi_Ef_nE$ (and the same value as $C\phi_EB$). But by Lemma 4, Cf_jDf_nD is equivalent to some f_1D and Cf_jEf_nE is equivalent to f_1E. Thus, f_1D, Cf_jDf_nD, $C\phi_Df_nD$ and $C\phi_DB$ (i.e., θ_D) will all take the same value under Σ_i. Likewise, f_1E, Cf_jEf_nE, $C\phi_Ef_nE$ and $C\phi_EB$ (i.e., θ_E) will all take the same value under Σ_i. Hence, the lemma is proved for this subcase.

Subcase 5: θ_D is of the form $CB\phi_D$ and θ_E is of the form $CB\phi_E$. The proof of this subcase is analogous to that given in Subcase 4.

Subcase 6: θ_D is of the form $C\psi_D\phi_D$ and ϕ_E is of the form $C\psi_E\phi_E$. The proof of this subcase is analogous to that given in Subcase 4, except there is no need to introduce the constant functor f_n.

In the following B_{A_D} represents any wff B of $Ł$ which contains an embedded wff A (which itself contains an embedded wff D).

Lemma 6: *For any wff B_{A_D} of $Ł$ containing an embedded wff A_D, for any δ-interpretation Σ_i of $Ł$ and for any one-place functor f_j definable in $Ł$, if A_D is replaced in B_{A_D} by f_jD, then, if A_D and f_jD have the same value under Σ_i, B_{A_D} and B_{f_jD} will also take the same value under Σ_i.*

Proof: We use induction on the number of connectives in B_{A_D} or B_{f_jD} not occurring in A_D or f_jD.

Case α: B_{A_D} and B_{f_jD} have no connectives occurring outside of A_D of f_jD. In this case, B_{A_D} must be A_D and B_{f_jD} must be f_jD. It is obvious that the lemma holds in this case.

Case β: Assume that B_{A_D} and B_{f_jD} have k connectives occurring outside of A_D or f_jD. We must consider the following subcases:

Subcase 1: B_{A_D} is of the form NE_{A_D} and B_{f_jD} is of the form NE_{f_jD}. E_{A_D} and E_{f_jD} both contain less than k connectives occurring outside of A_D or f_jD, and, thus, both fall under the assumption of induction. Therefore, if A_D and f_jD both take the same value under Σ_i, then E_{A_D} and E_{f_jD} will also take the same value under Σ_i. But if E_{A_D} and E_{f_jD} take the same value under Σ_i, then clearly NE_{A_D} (i.e., B_{A_D}) and NE_{f_jD} (i.e., B_{f_jD}) will also take the same value under Σ_i. Hence, the lemma holds for this subcase.

Subcase 2: B_{A_D} is of the form ΔE_{A_D} and B_{f_jD} is of the form ΔE_{f_jD}. The proof here is analogous to that given in Subcase 1.

Subcase 3: B_{A_D} is of the form δE_{A_D} and B_{f_jD} is of the form δE_{f_jD}. Since Σ_i is a δ-interpretation which assigns some one place functor to δ, we are in this subcase really considering B_{A_D} under the form $f_kE_{A_D}$ and B_{f_jD} under the form $f_kE_{f_jD}$. The proof of this subcase is analogous to that given in Subcase 1.

Subcase 4: B_{A_D} is of the form $CE_{A_D}G$ and B_{f_jD} is of the form $CE_{f_jD}G$. E_{A_D}

and E_{f_jD} both fall under the assumption of induction, and so if A_D and f_jD take the same value under Σ_i, then E_{A_D} and E_{f_jD} will also take the same value under Σ_i. G will take the same value under Σ_i in both $CE_{A_D}G$ and $CE_{f_jD}G$. Thus, if A_D and f_jD take the same value under Σ_i, then the antecedents of $CE_{A_D}G$ and $CE_{f_jD}G$ will take the same value under Σ_i, as will their consequents. If this is so, then clearly $CE_{A_D}G$ (i.e., B_{A_D}) and $CE_{f_jD}G$ (i.e., B_{f_jD}) will themselves take the same value under Σ_i, and, hence, the lemma is proved for this subcase.

Subcase 5: B_{A_D} is of the form CGE_{A_D} and B_{f_jD} is of the form CGE_{f_jD}. The proof of this subcase is analogous to that given in Subcase 4.

Subcase 6: B_{A_D} is of the form $CE_{A_D}G_{A_D}$ and B_{f_jD} is of the form $CE_{f_jD}G_{f_jD}$. The proof of this subcase is analogous to that given in Subcase 4, except that E_{A_D}, E_{f_jD}, G_{A_D} and G_{f_jD} all fall under the assumption of induction.

In the following $A_{\theta_{B_1},\dots,\theta_{B_n}}$ represents the result of substituting some cwfff θ of $Ł$ for δ in $A_{\delta B_1,\dots,\delta B_n}$.

Lemma 7: *For any wff of* $Ł$ *containing* δ*'s* $A_{\delta B_1,\dots,\delta B_n}$ *and for any cwfff* θ *of* $Ł$, *if* $A_{\theta_{B_1},\dots,\theta_{B_n}}$ *takes an undesignated value under a given* δ*-interpretation* Σ_i, *then, for some one-place functor* f_j *definable in* $Ł$, $A_{f_jB_1,\dots,f_jB_n}$ *also takes that value under* Σ_i.

Proof: Assume the opposite, i.e., that for some Σ_i $A_{\theta_{B_1},\dots,\theta_{B_n}}$ takes an undesignated value, but there is no f_j such that $A_{f_jB_1,\dots,f_jB_n}$ takes that value under Σ_i. By Lemma 5 we know that there is an f_1 such that f_1B_1 takes the same value under Σ_i as θ_{B_1}, f_1B_2 takes the same value under Σ_i as θ_{B_2}, and so on. If this is so, then, since each θ_{B_k} is an embedded wff in $A_{\theta_{B_1},\dots,\theta_{B_n}}$, by Lemma 6 we know that the *result* of replacing each θ_{B_k} in $A_{\theta_{B_1},\dots,\theta_{B_n}}$ by f_1B_k, i.e., $A_{f_1B_1,\dots,f_1B_n}$, must take the same (undesignated) value under Σ_i as $A_{\theta_{B_1},\dots,\theta_{B_n}}$. But this contradicts our original assumption, hence the lemma is proved.

Lemma 8: *For any wff of* $Ł$ *containing* δ*'s* $A_{\delta B_1,\dots,\delta B_n}$ *and for any cwfff* θ *of* $Ł$, *if, for some* Σ_i, $A_{\theta_{B_1},\dots,\theta_{B_n}}$ *is invalid, then, for some* f_j *definable in* $Ł$, $A_{f_jB_1,\dots,f_jB_n}$ *is also invalid.*

Proof: This follows directly from Lemma 7 and the definition of invalidity.

MT 1: *The rule of* δ*-substitution for assertions preserves validity.*

Proof: Assume the contrary, i.e., that $A_{\delta B_1,\dots,\delta B_n}$ is valid in $Ł$, i.e., takes the value 1 under all δ-interpretations, but that $A_{\theta_{B_1},\dots,\theta_{B_n}}$ is invalid for some cwfff θ of $Ł$. By Lemma 8 we then know that for some f_j definable in $Ł$, $A_{f_jB_1,\dots,f_jB_b}$ is also invalid. But if this is the case, then there is a δ-interpretation under which $A_{\delta B_1,\dots,\delta B_n}$ is invalid, viz., the δ-interpretation which assigns f_j to δ. But this contradicts our original assumption, hence the rule of δ-substitution for assertions preserves validity.

MT 2: *The rule of* δ*-substitution for rejections preserves invalidity.*

Proof: If $A_{\theta B_1,\ldots,\theta B_n}$ takes an undesignated value under some δ-interpretation Σ_i, then we know by Lemma 7 that there is an f_j definable in Ł such that $A_{f_j B_1,\ldots,f_j B_n}$ takes the same (undesignated) value under Σ_i. If this is so, then $A_{\delta B_1,\ldots,\delta B_n}$ is invalid since it takes an undesignated value under at least one δ-interpretation, viz., the one which assigns f_j to δ. Hence, the rule of δ-substitution for rejections preserves invalidity.

Now that we have proved MT 1 and MT 2, it is a simple matter to prove that Ł is sound, i.e., to prove that every assertion thesis of Ł is valid and that every rejection thesis of Ł is invalid. Once that has been done, one can then use Smiley's proof that Ł is complete and further prove that the 𝔐9 matrix is characteristic of Ł.

NOTES

1. I think a case could be made for the position that one does not show that an *assertion thesis* is valid or invalid, but rather only what follows the '⊢' sign—the "component wff" of the assertion thesis. (The same point could, of course, be made with regard to rejection theses.) I am not prepared, however, to press the point here and shall continue to speak of an assertion (or rejection) thesis as itself being valid or invalid.

2. Leśniewski [1], I am told, formulated a rule of substitution for variables of all logical types, and thus, indirectly, the rule of δ-substitution. The need for the explicit formulation of the rule of substitution in this simpler case remains.

3. Contrary to standard practice, I have consistently used capital *Roman* letters (rather than Greek) as meta-variables for wffs. I have done this since Greek letters, both capital and small, are used for several other purposes throughout this paper: i.e., as modal functors, meta-variables for cwfffs, and so on.

4. A δ-interpretation will, of course, only assign truth values to a wff which does not contain δ's.

5. It is assumed here and throughout the rest of the paper that, with regard to f_j, f_k and f_1, $1 \leqslant j \leqslant 16$, $1 \leqslant k \leqslant 16$, and $1 \leqslant 1 \leqslant 16$.

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