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TWO AXIOM SYSTEMS FOR RELATION ALGEBRAS

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It has long been known that, in principle, conversion can be eliminated from the primitive operations of a relation algebra. In [2], and more recently in [1], it is shown that $r^{\smile}$ is the largest element x such that $x;r' \leqslant \mathrm{e}'$ ($r^{\smile}$ is the converse of r, e is the identity element, the accent denotes complementation, and the semicolon denotes relative product). This characterization of $r^{\smile}$ is in fact not a necessary ingredient in the elimination of conversion as a primitive operation. In Definition 2 below conversion is eliminated by axiomatizing a relation algebra in terms of the operation $r^{\smile};s$ (just as inverses may be eliminated from the definition of a group by using the operation $a^{-1}\cdot b$). Definition 3 goes one better: it eliminates not only conversion but also complementation, by using the operation $r^{\smile};s'$. Definition 1, taken from [2], is used as standard; it is shown that Definitions 2 and 3 are each equivalent to Definition 1. Also the independence of the axioms in Definitions 2 and 3 is established.

Definition 1 A relation algebra is an algebra $\langle R, +, ', ;, {}^{\smile}, \mathrm{e}\rangle$ satisfying the following axioms:

A1 $\langle R, +, '\rangle$ is a Boolean algebra
A2 $(r;s);t = r;(s;t)$
A3 $(r + s);t = r;t + s;t$
A4 $r;\mathrm{e} = r$
A5 $r^{\smile\smile} = r$
A6 $(r + s)^{\smile} = r^{\smile} + s^{\smile}$
A7 $(r;s)^{\smile} = s^{\smile};r^{\smile}$
A8 $r^{\smile};(r;s)' + s' = s'$.

Note that

$$\mathrm{e}^{\smile} = \mathrm{e}^{\smile};\mathrm{e} = \mathrm{e}^{\smile};\mathrm{e}^{\smile\smile} = (\mathrm{e}^{\smile};\mathrm{e})^{\smile} = \mathrm{e}^{\smile\smile} = \mathrm{e}$$

by A4, A5 and A7. Also, using the same axioms, we get

$$\mathrm{e};r = (r^{\smile};\mathrm{e}^{\smile})^{\smile} = (r^{\smile};\mathrm{e})^{\smile} = r^{\smile\smile} = r.$$

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Definition 2 A relation algebra is an algebra $\langle R, +, ', \times, \mathrm{e}\rangle$ satisfying the following axioms:

B1 $\langle R, +, '\rangle$ is a Boolean algebra
B2 $(r \times s) \times t = s \times ((r \times \mathrm{e}) \times t)$
B3 $(r + s) \times t = r \times t + s \times t$
B4 $(r \times \mathrm{e}) \times \mathrm{e} = r$
B5 $(r \times \mathrm{e}) \times (r \times s)' + s' = s'$.

Let $\langle R, +, ', ;, ^\smile, \mathrm{e}\rangle$ be a relation algebra as defined in Definition 1, and write

$$r \times s \text{ for } r^\smile;s.$$

B1 follows immediately from A1. For B2, note that $r \times \mathrm{e} = r^\smile;\mathrm{e} = r^\smile$, by A4. Then

$$\begin{aligned}(r \times s) \times t = (r^\smile;s)^\smile;t &= (s^\smile;r^{\smile\smile});t && \text{by A7}\\ &= s^\smile;(r^{\smile\smile};t) && \text{by A2}\\ &= s \times (r^\smile \times t) = s \times ((r \times \mathrm{e}) \times t).\end{aligned}$$

B3 follows from A6 and A3, and B4 is just a translation of A5. B5 is a translation of

$$(r^\smile;\mathrm{e})^\smile;(r^\smile;s)' + s' = s'$$

and this follows from A4, A5 and A8. Hence $\langle R, +, ', \times, \mathrm{e}\rangle$, the translation of the original algebra, is a relation algebra as defined by Definition 2. Now let $\langle R, +, ', \times, \mathrm{e}\rangle$ be a relation algebra as defined by Definition 2, and write

$$\begin{aligned}&r^\smile \text{ for } r \times \mathrm{e}\\ &r;s \text{ for } (r \times \mathrm{e}) \times s.\end{aligned}$$

A1 follows immediately from B1. A2 is a translation of

$$(((r \times \mathrm{e}) \times s) \times \mathrm{e}) \times t = (r \times \mathrm{e}) \times ((s \times \mathrm{e}) \times t).$$

Proof: $((r \times \mathrm{e}) \times s) \times \mathrm{e} = s \times (((r \times \mathrm{e}) \times \mathrm{e}) \times \mathrm{e}) = s \times (r \times \mathrm{e})$, by B2 and B4. Hence,

$$(((r \times \mathrm{e}) \times s) \times \mathrm{e}) \times t = (s \times (r \times \mathrm{e})) \times t = (r \times \mathrm{e}) \times ((s \times \mathrm{e}) \times t), \text{ by B2.}$$

A3 is a translation of

$$((r + s) \times \mathrm{e}) \times t = (r \times \mathrm{e}) \times t + (s \times \mathrm{e}) \times t$$

and this follows by successive applications of B3. A4 and A5 are both translations of B4, and A6 follows from B3. A7 is a translation of

$$((r \times \mathrm{e}) \times s) \times \mathrm{e} = ((s \times \mathrm{e}) \times \mathrm{e}) \times (r \times \mathrm{e}).$$

Proof: $((r \times \mathrm{e}) \times s) \times \mathrm{e} = s \times (((r \times \mathrm{e}) \times \mathrm{e}) \times \mathrm{e} = s \times (r \times \mathrm{e})$, by B2 and B4
$= ((s \times \mathrm{e}) \times \mathrm{e}) \times (r \times \mathrm{e})$, by B4.

A8 is a translation of

$$((r \times \mathrm{e}) \times \mathrm{e}) \times ((r \times \mathrm{e}) \times s)' + s' = s'$$

which is a direct consequence of B5. Hence $\langle R, +, ', ;, ^{\smile}, \mathrm{e}\rangle$, the translation of $\langle R, +, ', \times, \mathrm{e}\rangle$, is a relation algebra as defined by Definition 1. It remains to show that a relation algebra, as defined by either definition, can be recaptured from its translation. Let $\langle R, +, ', ;, ^{\smile}, \mathrm{e}\rangle$ be a relation algebra as defined by Definition 1, and let $\langle R, +, ', \times, \mathrm{e}\rangle$ be its translation. Now let $\langle R, +, ', \otimes, \hat{}, \mathrm{e}\rangle$ be the translation of this translation. Then

$$r \otimes s = (r \times \mathrm{e}) \times s = (r^{\smile};\mathrm{e})^{\smile};s = r^{\smile\smile};s = r;s \text{ by A4 and A5.}$$
$$r^{\wedge} = r \times \mathrm{e} = r^{\smile};\mathrm{e} = r^{\smile}.$$

Hence $\langle R, +, ', \otimes, \hat{}, \mathrm{e}\rangle$ is just the original algebra $\langle R, +, ', ;, ^{\smile}, \mathrm{e}\rangle$. In the same way, by using B4, it can be shown that a relation algebra as defined by Definition 2 can be recaptured from its translation. This completes the proof of the equivalence of Definitions 1 and 2.

Definition 3 A relation algebra is an algebra $\langle R, +, :, \mathrm{e}\rangle$ satisfying the following axioms, in which r' abbreviates $\mathrm{e}:r$ and $r^{\smile}$ abbreviates $r:\mathrm{e}'$:

C1 $\langle R, +, '\rangle$ is a Boolean algebra
C2 $((r:s):t)^{\smile} = (r^{\smile}:t):s$
C3 $(r + s):t = r:t + s:t$
C4 $r^{\smile}:(r:s) + s = s$.

From Definition 3 we derive the following consequences:

T1 $\mathrm{e}:(\mathrm{e}:r) = r$.

Proof: $\mathrm{e}:(\mathrm{e}:r) = \mathrm{e}:r' = r'' = r$, by C1.

T2 $\mathrm{e}^{\smile} = \mathrm{e}$.

Proof: $\mathrm{e}^{\smile} = \mathrm{e}:\mathrm{e}' = \mathrm{e}:(\mathrm{e}:\mathrm{e}) = \mathrm{e}$, by T1.

T3 $(r:s)^{\smile} = s':r'$.

Proof: $(r:s)^{\smile} = ((\mathrm{e}:(\mathrm{e}:r)):s)^{\smile}$, by T1
$= (\mathrm{e}^{\smile}:s):(\mathrm{e}:r)$, by C2
$= (\mathrm{e}:s):(\mathrm{e}:r)$, by T2
$= s':r'$.

T4 $r:(s:t)' = (s^{\smile}:r'):t$.

Proof: $r:(s:t)' = r'':(s:t)' = ((s:t):r')^{\smile}$, by T3
$= (s^{\smile}:r'):t$, by C2.

T5 $r^{\smile\smile} = r$.

Proof: $r^{\smile\smile} = (r:\mathrm{e}')^{\smile} = \mathrm{e}'':r' = \mathrm{e}:r' = r'' = r$, by T3 and C1.

T6 $r^{\smile}:\mathrm{e}' = r$.

Proof: $r^{\smile}:\mathrm{e}' = r^{\smile\smile} = r$, by T5.

T7 $(r + s)^{\smile} = r^{\smile} + s^{\smile}$.

Proof: $(r + s)^{\smile} = (r + s){:}\mathrm{e}' = r{:}\mathrm{e}' + s{:}\mathrm{e}' = r^{\smile} + s^{\smile}$, by C3.

T8 $(r^{\smile}{:}s')^{\smile}{:}t' = r^{\smile}{:}(s^{\smile}{:}t')'$.

Proof: $(r^{\smile}{:}s')^{\smile}{:}t' = (s''{:}r^{\smile\prime}){:}t'$, by T3
$= (s{:}r^{\smile\prime}){:}t'$, by C1
$= (s^{\smile\smile}{:}r^{\smile\prime}){:}t'$, by T5
$= r^{\smile}{:}(s^{\smile}{:}t')'$, by T4.

T9 $(r + s)^{\smile}{:}t' = r^{\smile}{:}t' + s^{\smile}{:}t'$.

Proof: By T7 and C3.

T10 $(r^{\smile}{:}s')^{\smile} = s^{\smile\smile}{:}r^{\smile\prime}$.

Proof: $(r^{\smile}{:}s')^{\smile} = s''{:}r^{\smile\prime}$, by T3
$= s{:}r^{\smile\prime}$, by C1
$= s^{\smile\smile}{:}r^{\smile\prime}$, by T5.

T11 $r^{\smile\smile}{:}(r^{\smile}{:}s')'' + s' = s'$.

Proof: By C1 and C4.

Let $\langle R, +, ', ;, {}^{\smile}, \mathrm{e}\rangle$ be a relation algebra as defined in Definition 1 and write

$$r{:}s \text{ for } r^{\smile};s'.$$

Then $\mathrm{e}{:}r = \mathrm{e}^{\smile};r' = \mathrm{e};r' = r'$ and $r{:}\mathrm{e}' = r^{\smile};\mathrm{e}'' = r^{\smile};\mathrm{e} = r^{\smile}$.
This justifies the abbreviations used in Definition 3. C1 now follows immediately from A1. C2 is a translation of

$$((r^{\smile};s')^{\smile};t')^{\smile} = (r^{\smile\smile};t')^{\smile};s'.$$

Proof: $((r^{\smile};s')^{\smile};t')^{\smile} = t'^{\smile};(r^{\smile};s')^{\smile\smile}$, by A7
$= (t'^{\smile};r^{\smile});s'$, by A5 and A2
$= (r^{\smile\smile};t')^{\smile};s'$, by A7 and A5.

C3 is a translation of

$$(r + s)^{\smile};t' = r^{\smile};t' + s^{\smile};t'$$

which follows from A6 and A3. And C4 is a translation of

$$r^{\smile\smile};(r^{\smile};s')' + s = s$$

which is a consequence of A8. Hence $\langle R, +, :, \mathrm{e}\rangle$, the translation of $\langle R, +, ', ;, {}^{\smile}, \mathrm{e}\rangle$, is a relation algebra as defined by Definition 3. Now let $\langle R, +, :, \mathrm{e}\rangle$ be a relation algebra as defined by Definition 3, and write

$$\begin{aligned} r' &\text{ for } \mathrm{e}{:}r \\ r^{\smile} &\text{ for } r{:}\mathrm{e}' \\ r;s &\text{ for } r'{:}s'. \end{aligned}$$

Then C1, T8, T9, T6, T5, T7, T10 and T11, when rewritten in this way, are precisely the axioms A1-A8. Hence $\langle R, +, ', ;, {}^{\smile}, \mathrm{e}\rangle$, the translation of $\langle R, +, :, \mathrm{e}\rangle$, is a relation algebra as defined by Definition 1. From A1 and

A5 it follows that the translation of the translation of a relation algebra as defined by Definition 1 is again that same algebra. By C1 and T5 the same result holds for a relation algebra as defined by Definition 3. This concludes the proof that Definition 1 is equivalent to Definition 3.

To establish the independence of each axiom in Definition 2 or 3 an example is given of an algebra in which that axiom fails while the other axioms hold. In each of the examples given only the failure of the relevant axiom will be pointed out, verification of the other axioms always being a routine computation.

For Definition 2 we have the following examples. Firstly, let $R = \{\mathrm{e}, b, a\}$ where $\mathrm{e} \leqslant b \leqslant a$, $r + s = \max\{r, s\} = r \times s$ and $r' = a$. Then B1 fails since R has three elements. Secondly, let $\langle R, +, {}'\rangle$ be any Boolean algebra, let $\mathrm{e} = 0$ and let $r \times s = r \cdot s'$. Then B2 fails since $(1 \times 0) \times 0 = 1 \cdot 1 \cdot 1 = 1$, whereas $0 \times ((1 \times 0) \times 0) = 0 \cdot (1 \cdot 1 \cdot 1)' = 0$. Thirdly, let R be the two-element Boolean algebra $\{0, 1\}$, let $\mathrm{e} = 1$ and let $0 \times 0 = 1 = 1 \times 1$ and $1 \times 0 = 0 = 0 \times 1$. Then B3 fails since $(0 + 1) \times 0 = 1 \times 0 = 0$, whereas $0 \times 0 + 1 \times 0 = 1 + 0 = 1$. Fourthly, let $\langle R, +, {}'\rangle$ be any Boolean algebra, let $\mathrm{e} = 1$ and let $r \times s = s$. Then B4 fails, since $(0 \times \mathrm{e}) \times \mathrm{e} = (0 \times 1) \times 1 = 1 \neq 0$. Fifthly, let $\langle R, +, {}'\rangle$ be any Boolean algebra, let $\mathrm{e} = 0$ and let $r \times s = r + s$. Then B5 fails, since $(1 \times \mathrm{e}) \times (1 \times 1)' + 1' = (1 + 0) + (1 + 1) + 0 = 1 + 0 = 1 \neq 0$.

For Definition 3 we have the following examples. Firstly, let $R = \{\mathrm{e}, b, a\}$, where $\mathrm{e} \leqslant b \leqslant a$. Let $r{:}s = \mathrm{e}$ and let $r + s = \max\{r, s\}$, then $r' = \mathrm{e}{:}r = \mathrm{e}$ and $r^{\smile} = r{:}\mathrm{e}' = r{:}\mathrm{e} = \mathrm{e}$. C1 fails since R has three elements. Secondly, let R be the four-element Boolean algebra $\{0, \mathrm{e}, n, 1\}$, where $n = \mathrm{e}'$ and : is defined by

:	0	e	n	1
0	0	0	0	0
e	1	n	e	0
n	0	0	0	0
1	1	n	e	0

Note that $\mathrm{e}{:}r = r'$. This justifies the abbreviation used in Definition 3. C2 fails since

$$((1{:}0){:}n)^{\smile} = (1{:}n)^{\smile} = \mathrm{e}^{\smile} = \mathrm{e}{:}n = \mathrm{e}$$

whereas

$$(1^{\smile}{:}n){:}0 = ((1{:}n){:}n){:}0 = (\mathrm{e}{:}n){:}0 = \mathrm{e}{:}0 = 1.$$

Thirdly, let R be the two-element Boolean algebra $\{0, 1\}$, let $\mathrm{e} = 1$ and let $0{:}0 = 0 = \mathrm{e}{:}\mathrm{e}$ and $\mathrm{e}{:}0 = \mathrm{e} = 0{:}\mathrm{e}$. Note that $\mathrm{e}{:}r = r'$, this justifies the abbreviation used in Definition 3. C3 fails since $(0 + \mathrm{e}){:}\mathrm{e} = \mathrm{e}{:}\mathrm{e} = 0$, whereas $0{:}\mathrm{e} + \mathrm{e}{:}\mathrm{e} = \mathrm{e} + 0 = \mathrm{e}$. Fourthly, let $\langle R, +, {}'\rangle$ be any Boolean algebra, let $\mathrm{e} = 0$ and let $r{:}s = r + s'$. Then $\mathrm{e}{:}r = 0 + r' = r'$ and this justifies the abbreviation used in Definition 3. C4 fails since

$$1^{\smile}{:}(1{:}0) + 0 = (1{:}\mathrm{e}'){:}(1{:}0) = (1 + 0) + (1 + 1)' = 1 + 0 \neq 0.$$

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