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A SIMPLIFICATION PROCEDURE FOR ALTERNATIONAL NORMAL SCHEMATA

JOHN RISER

In his *Methods of Logic*, Quine remarks, with regard to alternational normal schemata: "A simplification technique that did not depend on any exhaustive survey of the prime implicants would be a boon." [1], p. 67. In what follows, such a simplification procedure is proposed.

Consider an alternational normal schema in n distinct letters. There are 2^n different combinations of truth values assignable to the letters, and the schema comes out true (otherwise false) in $k\,(0 \leqslant k \leqslant 2^n)$ of those combinations. Each clause of the schema contains $i\ (0 \neq i \leqslant n)$ literals. Let an unnegated letter represent (or be associated with) the value of truth and a negated letter represent that of falsity. Then if $i = n$, the assignment of truth values represented by the clause occurs only once within the 2^n possible combinations; if $i = n - 1$, the assignment occurs twice; if $i = n - 2$, then four times; and, in general, where $i = n - m$, the assignment occurs 2^m times within all possible combinations.

Suppose the 2^n combinations were organized in such a way that a hypothetical ordering of them, in which the letters were listed alphabetically and in which, say, T and $\bot$ were used to represent truth values, would have listed under the first letter $2^n/2$ Ts followed by $2^n/2$ $\bot$s, under the second letter $2^n/4$ Ts alternated with $2^n/4$ $\bot$s until 2^n values were listed, and so on.

Now take some alternational normal schema; for example [1], p. 65:

$$p\bar{q} \vee \bar{p}q \vee q\bar{r} \vee \bar{q}r$$

Each clause contains 2, that is, $3 - m$, literals; and thus each clause represents a particular combination of truth values that would occur twice within all possible combinations. For example, $p\bar{q}$ represents a combination of truth values that would occur in rows 3 and 4 of the supposed truth table for the n letters of the schema. $\bar{p}q$ would be associated with rows 5 and 6, $q\bar{r}$ with rows 2 and 6, and $\bar{q}r$ with rows 3 and 7. (The systematic mode of organization of the truth-value combinations permits routine calculation of the relevant rows, apart from any actually constructed truth table.) This information can be exhibited as follows:

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$$\begin{array}{ccccccc} 3,4 & & 5,6 & & 2,6 & & 3,7 \\ p\bar{q} & \vee & \bar{p}q & \vee & q\bar{r} & \vee & \bar{q}r \end{array}$$

Each pair of numerals represents rows in a truth table for the schema in which the clause would come out true, and all the numerals represent all (and only) those rows in which the entire schema would come out true. Thus, ignoring redundant numerals, the schema would come out true in rows 2-7 (and false in rows 1 and 8).

To simplify the given schema so as to produce a shortest equivalent schema, without having to survey exhaustively prime implicants, it is only necessary that the clauses of the latter schema be associated with exactly the same numerals (and, hence, with the same k rows of a truth table) as the clauses of the former. For the given schema, simplification can be effected in the following manner.

Select a redundant numeral, for example: 3. Cancel it either in the pair 3,4 or in the pair 3,7. Suppose it is cancelled in 3,7. Then this clause is now associated only with row 7 and should, accordingly, be rewritten as $\bar{p}\bar{q}r$ (since 3 literals are required for representing a combination of truth values that occurs only once within 2^3 possible combinations). Now cancel an occurrence of the redundant numeral 6—say, in the second clause. Rewriting the second clause, so that it is associated solely with row 5, yields $\bar{p}qr$. The two rewritten clauses taken together—that is, $\bar{p}\bar{q}r \vee \bar{p}qr$—simplify obviously to $\bar{p}r$. The resulting, and finally simplified, schema is therefore $p\bar{q} \vee q\bar{r} \vee \bar{p}r$. The simplification procedure may be shown in this way:

$$\begin{array}{ccccccc} 3,4 & & 5,\cancel{6} & & 2,6 & & \cancel{3},7 \\ p\bar{q} & \vee & \bar{p}q & \vee & q\bar{r} & \vee & \bar{q}r \\ & & | & & & & | \\ & & \bar{p}qr & & & & \bar{p}\bar{q}r \\ & & & \searrow & & \swarrow & \\ p\bar{q} & \vee & q\bar{r} & \vee & \bar{p}r & & \end{array}$$

If, in cancelling the redundant numeral 6, its occurrence in the third, rather than the second, clause had been selected, no simplification could have been carried out. For $q\bar{r}$ would have had to be rewritten as $pq\bar{r}$; and this and the previously rewritten clause $\bar{p}\bar{q}r$ together admit of no further simplification. That this is so is signalized by the fact that it is not possible for a clause to be associated with the pair 2,7, since any iterated combination of truth values in the assumed table never occurs in an even/odd-numbered sequence of rows.

Another equally short simplification of the schema considered can be found by varying the (effective) cancellations:

$$\begin{array}{ccccccc} \cancel{3},4 & & 5,6 & & 2,\cancel{6} & & 3,7 \\ p\bar{q} & \vee & \bar{p}q & \vee & q\bar{r} & \vee & \bar{q}r \\ | & & & & | & & \\ p\bar{q}\bar{r} & & & & pq\bar{r} & & \\ & \searrow & & \swarrow & & & \\ & & p\bar{r} & \vee & \bar{p}q & \vee & \bar{q}r \end{array}$$

The procedure described herein can be employed in other phases of simplification:

1. Eliminating a clause that is redundant, as shown by the fact that each of its associated numerals is also associated with (an)other clause(s). Thus, using an example from [1], p. 64:

$$\begin{array}{ccccccc} 10,12, & & 13,14, & & 2,4, & & \\ 14,16 & & 15,16 & & 10,12 & & 3,7 \\ \overline{ps} & \vee & \overline{pq} & \vee & \bar{s}q & \vee & \bar{r}ps \end{array}$$

The first clause is clearly redundant.

2. Eliminating a literal that is redundant, as shown by the fact that its cancellation from a clause adds no associated numerals to that revised clause that are not already associated with other clauses. Thus, again using an example from [1], p. 64:

$$\begin{array}{ccccc} 1,2 & & 3 & & 8 \\ pq & \vee & p\bar{q}r & \vee & \overline{pqr} \end{array}$$

simplifies to:

$$\begin{array}{ccccc} 1,2 & & 1,3 & & 8 \\ pq & \vee & pr & \vee & \overline{pqr} \end{array}$$

The elimination of any other literal from the example would have added numerals not already associated with its clauses.

3. Selecting an adequate minimum of prime implicants. Using an example from [1], p. 66:

$$\begin{array}{ccccccccccc} 3,4 & & 5,6 & & 2,6 & & 3,7 & & 2,4 & & 5,7 \\ p\bar{q} & \vee & \bar{p}q & \vee & q\bar{r} & \vee & \bar{q}r & \vee & p\bar{r} & \vee & \bar{p}r \end{array}$$

The fifth and sixth clauses can immediately be seen to be redundant—as can, in turn, seven other possible pairs of clauses. The four remaining non-redundant clauses can be further simplified as in preceding examples.

It may be noted in passing that an alternational normal schema in n letters is valid if and only if its clauses can be associated (in the sense described above) with the numerals 1, 2, . . . n, in which case the shortest normal equivalent is, of course, $p \vee \bar{p}$ [2], p. 174.

REFERENCES

[1] Quine, W. V., *Methods of Logic*, Third Edition, Holt, Rinehart and Winston, New York, 1972.

[2] Quine, W. V., *Selected Logic Papers*, Random House, New York, 1966.

Florida Technological University
Orlando, Florida