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INTENDED MODEL THEORY

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The aim of the present paper is to develop a theory of intended models as an extension of standard model theory, using the pragmatics of standard languages [7] to represent the "intending." To this end, new foundations for standard pragmatic theory are formulated in section **1**, and pragmatic theory is further developed in section **2**. Section **3** contains semantic foundations for the theory of intended models of section **4**.

1 *New foundations for standard pragmatics* Standard pragmatics is developed in [7], where it is shown that an algebra of formulas which represents a standard first order predicate calculus may be induced over the expressions over an arbitrary finite alphabet, by appropriate verbal behavior of the users of the expressions. The snytax of the formulas of the algebra, as well as their logical (polyadic Boolean) structure, is induced pragmatically. In the present section it is shown that the part of standard pragmatics which determines logical as distinct from syntactic structure is derivable from two assumptions expressing the rationality of the degrees of belief of the users of the expressions of the syntax.

The primitive terms of standard pragmatics as developed in [7] are a set **L** (of expressions), a set $\mathbf{V} = \{0, 1, 2\}$ (whose elements represent the pragmatic values *reject*, *accept*, and *withhold*, respectively), a set **C** (of conditions under which expressions of **L** are valued in **V**), a set **U** (of users of **L**), and a set **W** (of times of valuation of expressions of **L**). An obvious generalization of the conditions of valuation **C**, although not required for the proof of proposition (1.2) below, is useful for the theory of intended models of section **4**. Intuitively, the generalization consists in accommodating possible as well as actual conditions of valuation of the expressions of **L**. This is accomplished by letting **C** be the domain of a Boolean algebra $\langle \mathbf{C}, \wedge, ' \rangle$, where **C** contains a distinguished element b, distinct from the unit element c_1 and from the zero element c_0 of **C**. Intuitively, a non-zero element of **C** may be regarded as a possible condition of pragmatic valuation, an element $c \geqslant b$ may be regarded as an actual condition of valuation, the unit element of **C** may be regarded as the necessary

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condition, and the zero element of **C** as the impossible condition. The distinguished condition b may be regarded as the total actual evidence for valuing expressions of **L** in **V**. Thus b serves the purpose of **C** itself in [7], and in the present paper we shall understand all definitions of [7] with b in place of **C**. All proofs of [7] go through with **C** replaced everywhere by b.

We shall now adapt the pragmatic theory of [7] to the above generalization of **C**, at the same time emending slightly the exposition of [7]. All terms and symbols introduced without definition are intended in the sense of [7], with conditions of pragmatic valuation understood in the above generalized sense.

Let $\mathbf{D} = \mathbf{V}^{\mathbf{U}\times\mathbf{W}\times\mathbf{C}}$, the set of all functions from the Cartesian product of **U**, **W**, and **C** to **V**. Intuitively, an element (u, w, c, v) of $\mathsf{d} \in \mathbf{D}$ is a disposition of user u, at time w, under condition c, to perform valuation v. It is useful to refer to non-zero elements of **C** as *proper conditions*. For $n = 0, 1, 2$, and $(u, w, c) \in \mathbf{U} \times \mathbf{W} \times \mathbf{C}$, we then define:

$$\mathsf{d}_n(u, w, c) = \begin{cases} n, \text{ if } c \text{ is proper,} \\ 2 \text{ otherwise.} \end{cases}$$

Thus d_2 is the constant function in **D** with value 2. Intuitively, d_2 is the uniform set of dispositions of users of **L** to perform the valuation *withhold* under any condition at any time.

Let Π be a function from **L** to **D**. Such a function assigns to each expression e of **L** a set of valuing dispositions $\Pi(e) \in \mathbf{D}$, and suggests the following definition.

D1. Π is a *pragmatic interpretation* (of **L** in **D**) iff:

I. $\Pi(e)(u, w, c_0) = 2$, for all $e \in \mathbf{L}$, $u \in \mathbf{U}$, $w \in \mathbf{W}$.
II. $\Pi(e)(u, w, b) \neq 2$, for some $e \in \mathbf{L}$, $u \in \mathbf{U}$, $w \in \mathbf{W}$.
III. If $\Pi(e)(u, w, c_1) \neq 2$, then $\Pi(e)(u, w, c) = \Pi(e)(u, w, c_1)$, for all $e \in \mathbf{L}$, $u \in \mathbf{U}$, $w \in \mathbf{W}$, proper $c \in \mathbf{C}$.

Clause I of D1 says that the impossible condition is not germane to any expression. Clause II says that the distinguished actual condition is germane to some expression. Finally, clause III says that if the necessary condition is germane to an expression, then this valuation is preserved under all proper conditions.

Further requirements on pragmatic interpretations are necessary for the pragmatic determination of standard syntax, but before considering this development it should be observed that pragmatic interpretations accommodate the concept of stimulus meaning, in a sense generalized from that of Quine [4]. We may define the *affirmative* (*negative*) *stimulus meaning* of an expression e of **L**, for user u at time w, to be the set $\{c \in \mathbf{C}: \Pi(e)(u, w, c) = 1(0)\}$. Then we may define the stimulus meaning of e, for u at w, to be the ordered pair of its affirmative and negative stimulus meanings for u at w. It is not assumed that the stimulus meaning of an expression is identified with its entire meaning; for we wish to investigate the relation between pragmatically and semantically characterized meaning. At the end of this section we shall consider in more detail the concept of intersubjective stimulus meaning, and its relation to pragmatic synonymy.

The first step in the pragmatic determination of standard syntax is as follows. Since the empty expression is a member of **L**, we shall refer to non-empty expressions of **L** as *proper* expressions, and we let $\mathbf{L}^*$ be the set of proper expressions of **L**.

D2. Π is a *proto-sentential interpretation* iff Π is a pragmatic interpretation and there are distinct and unique expressions & and $\sim$ of $\mathbf{L}^*$ such that for all $s, s' \in \mathbf{L}$; $u, u' \in \mathbf{U}$; $w, w' \in \mathbf{W}$:

I. If $\Pi(s) \neq \mathsf{d}_2$, then $\Pi(\& \; s \sim s) = \mathsf{d}_0$.
II. If $\Pi(s)(u, w, c) = 2 = \Pi(s')(u, w, c)$, then $\Pi(\& \; s \; s')(u, w, c) = 2$ or $\Pi(\& \; s \; s')(u, w, c_1) \neq 2$.
III. If $\Pi(s)(u, w, c) = 1 = \Pi(s')(u', w', c)$, then $\Pi(\& \; s \; s')(u'', w'', c) = 1$ for some $u'' \in \mathbf{U}$, $w'' \in \mathbf{W}$.
IV. If $\Pi(s)(u, w, c) \neq 2 \neq \Pi(s)(u', w', c)$, then $\Pi(s)(u, w, c) = \Pi(s)(u', w', c)$.
V. If $\Pi(\& \; s \sim s) = \mathsf{d}_0 = \Pi(\& \; s' \sim s')$, then $\Pi(\sim s)$ and $\Pi(\& \; s \; s')$ are (partially) fixed at all arguments according to the tables of D1 of [7].

Clauses II and III of the definition of a proto-sentential interpretation in [7] are omitted in D2, because the former follows from clause I of D3 below, and the latter follows from clauses II and III of D4 of [7]. Clauses III and IV of D2 above are conditions of intersubjectivity on the valuations of the users of **L**; if **U** and **W** are unit sets, these clauses become redundant. We shall return to the concept of intersubjectivity at the end of this section. By clause IV, the core of a proto-sentential interpretation exists.

Let π be the core of a proto-sentential interpretation Π. The non-uniqueness in the table for & of clause **V** is clarified if we define $\mathbf{N} = \{s \in \mathbf{L}: \pi(s, c_1) = 1\}$. Intuitively, **N** is the set of sentences accepted under the necessary condition. By clause I of D2, $\mathbf{N} \subseteq \mathbf{S}$, and by clause III of D1, $s \in \mathbf{N}$ iff $\pi(s, c) = 1$ for all proper c. As will be seen, **N** may be regarded as the set of analytic sentences of the language induced over **L**. (Analytic sentences are represented in a slightly different way in [7], but the present approach is preferable.) By clause II of D2, together with clause III of D1:

(1.1) *If* $\Pi(s)(u, w, c) = 2 = \Pi(s')(u, w, c)$, *then* $\Pi(\& \; s \; s')(u, w, c) = 2$ *or* $\sim \& \; s \; s' \in \mathbf{N}$.

According to (1.1), the table for & determines the pragmatic value of a conjunction from the values of its conjuncts, unless the conjunction is analytically false.

The next step in the pragmatic determination of standard syntax is as follows.

D3. Π is a *lexical interpretation* iff Π is a proto-sentential interpretation and there is a unique expression $\exists$ of $\mathbf{L}^*$, distinct from & and $\sim$, such that the set

$$\mathbf{I} = \{i \in \mathbf{L}^*: \Pi(\exists ip) = \mathrm{d}_0 \text{ for some } p \in \mathbf{L}\}$$

is infinite, and for all e, e', F, $\mathbf{G}$ in $\mathbf{L}$ and i in $\mathbf{I}$:

I. Any expression of the form $\exists \& e$, $\exists \sim e$, $\exists\exists e$, $e \,\&\, ie'$, $e \sim ie'$, $e\&$, $e\sim$, $e\exists$, or which contains only &, $\sim$, $\exists$, and i, is valued d_2 by Π.
II. If $\Pi(\exists i \,\&\, ei \ldots i \sim ei \ldots i) = \mathrm{d}_0$, then any expression which differs from the displayed one by changing the number of iterated i's is valued d_2.
III. If $\Pi(\exists i \,\&\, Fi \ldots i \sim Fi \ldots i) = \mathrm{d}_0 = \Pi(\exists i \,\&\, \mathbf{G}i \ldots i \sim \mathbf{G}i \ldots i)$, then $\Pi(eF\mathbf{G}e') = \mathrm{d}_2$.
IV. If $\Pi(\&e \sim e) = \mathrm{d}_0$, then every occurrence of i in e is in a part of e of the form $\exists ie'$ for some e' in $\mathbf{L}^*$.

A lexical interpretation determines a standard lexicon of predicates and individual constants, as shown after D3 of [7], which corresponds to D3 above. By clause I of D3, the set $\mathbf{P}$ of predicates is disjoint from the set $\mathbf{I}$ of variables. By clause IV, $\mathbf{I}$ is disjoint from the set $\mathbf{K}$ of individual constants. By clause III, $\mathbf{P}$ and $\mathbf{K}$ are disjoint. These results supplement those of [7] concerning the lexicon determined by a lexical interpretation.

The final step in the pragmatic determination of standard syntax is as follows. Π is defined to be a *proto-polyadic interpretation* iff Π is a lexical interpretation which satisfies the definiens of D4 of [7], with the added requirement that in clause IV, q and r are proper expressions. It is shown in [7] that a proto-polyadic interpretation induces a standard syntax over $\mathbf{L}$, and that a polyadic extension of such an interpretation induces standard logic on the formulas of this syntax. We now consider a probabilistic foundation for polyadic interpretations.

Let Π be a proto-polyadic interpretation. For each sentence s of the syntax determined by Π, and for each user u, time w, and condition c, we assume that the strength of belief in s on the part of u, at w under c, admits of quantitative measure, as represented, for example by the betting dispositions of u. We understand the *coherence* of a belief function in the usual sense of subjective probability theory. From this viewpoint we define a proto-polyadic interpretation Π to be *coherent* iff, under any proper condition at any time:

I. The degrees of belief of any user in the sentences determined by Π are coherent.
II. Every user accepts those sentences in which his degree of belief is maximum.

Coherent interpretations determine standard logic provided that:

(1.2) *Every coherent interpretation is polyadic.*

Proof: Let Π be a coherent interpretation. Since Π is proto-polyadic, the sentences determined by Π have standard syntax. Then by clause I above and the well-known theorem of Ramsey [5] and de Finetti [2] relating coherence and probability, the degrees of belief in sentences of **S** are probabilities, for any user at any time under any proper condition. Let t be any logical truth of **S**. Then $\Pi(t) = \mathrm{d}_1$, by clause II of the definition of a coherent interpretation. Then by clause VI of D2, $\Pi(\sim t) = \mathrm{d}_0$. In particular, $\Pi(\exists(p \mathbin{\&} \sim\exists(J)p \mathbin{\&} q)) = \mathrm{d}_0$, so that clause I of the definition of a polyadic interpretation holds.

To verify clause II, suppose that for fixed u, w, and c, $\exists(p \mathbin{\&} \sim q)$ and $\exists(q \mathbin{\&} \sim r)$ are valued 0, but $\exists(p \mathbin{\&} \sim r)$ is not. Then by clause VI of D2, $\forall(p \rightarrow q) \mathbin{\&} \forall(q \rightarrow r) \mathbin{\&} \sim\forall(p \rightarrow q)$ is not valued 0, against the above result that contradictory sentences are always valued 0. Thus clause II holds, and in the same way clauses III, VII, and IX may also be shown to hold.

In order to prove clause V, we observe that, since the syntax determined by Π is standard, $e(p \mathbin{\&} q)$ is a sentence iff $e(q \mathbin{\&} p)$ is. If neither is a sentence, then both are valued d_2, by clause I of D2. If both are sentences, then $e(p \mathbin{\&} q) \leftrightarrow e(q \mathbin{\&} p)$ is logically true, by standard logic. Now suppose that $\Pi(e(p \mathbin{\&} q)) \neq \Pi(e(q \mathbin{\&} p))$. There are three possible cases, for fixed u, w, c.

Case 1. $e(p \mathbin{\&} q)$ is valued 1 and $e(q \mathbin{\&} p)$ is valued 0. Then $\sim e(q \mathbin{\&} p)$ and therefore also $e(p \mathbin{\&} q) \mathbin{\&} \sim e(q \mathbin{\&} p)$ is valued 1. It follows that $(e(p \mathbin{\&} q) \leftrightarrow e(q \mathbin{\&} p)) \mathbin{\&} e(p \mathbin{\&} q) \mathbin{\&} \sim e(q \mathbin{\&} p)$ is valued 1, against the coherence of Π.

Case 2. $e(p \mathbin{\&} q)$ is valued 1 and $e(q \mathbin{\&} p)$ is valued 2. Then $\sim e(p \mathbin{\&} q)$ and hence $(e(p \mathbin{\&} q) \leftrightarrow e(q \mathbin{\&} p)) \mathbin{\&} e(p \mathbin{\&} q) \mathbin{\&} \sim e(q \mathbin{\&} p)$ is valued 2, against the coherence of Π.

Case 3. $e(p \mathbin{\&} q)$ is valued 0 and $e(q \mathbin{\&} p)$ is valued 2. Then $\sim e(p \mathbin{\&} q)$ is valued 1, so that $(e(p \mathbin{\&} q) \leftrightarrow e(q \mathbin{\&} p)) \mathbin{\&} \sim e(p \mathbin{\&} q) \mathbin{\&} e(q \mathbin{\&} p)$ is valued 2, against the coherence of Π.

Thus clause V holds, and in the same way all the remaining clauses of the definition of a polyadic interpretation may be shown to hold. (1.2) is thereby established.

It should be noted that although logical (polyadic Boolean) properties of the formulas of the syntax determined by Π are essential to the above proof, knowledge of these properties is not attributed (nor denied) to the users of the formulas. (1.2) speaks only of verbal behavior, and in particular of dispositions to bet and to value expressions in the set **V**. An idealized linguist studying that behavior must first be able to recognize its coherence, before his hypotheses about the logical knowledge of the users, as an explanation of the coherence, may thereby have interest. From the viewpoint of (1.2), the idealized verbal behavior characterized by D1-D4 is a natural extension of the coherent behavior characterized by subjective provability theory. Indeed such idealization is a natural extension of that embodied in the concept of formalized languages themselves. It may be regarded as representing the long run of scientific investigation contemplated by Peirce. The possibility that such idealized behavior may distinguish the intended models of a theory as well as determine its logical structure thereby suggests itself. The realization of this possibility is the purpose of the remainder of the present paper.

We shall conclude this section by considering the concept of intersubjective stimulus meaning and its relation to pragmatic synonymy. Given a pragmatic interpretation Π, the *affirmative* (*negative*) *intersubjective stimulus meaning* of an expression e of $\mathbf{L}$ may be defined as the set of conditions c such that $\pi(e, c) = 1(0)$, where π is the core of Π. The *intersubjective stimulus meaning* of e relative to Π, $\Sigma_\Pi(e)$, may then be defined as the ordered pair of the affirmative and negative intersubjective stimulus meanings of e relative to Π. If π does not exist, there are no intersubjective stimulus meanings. If Π is a coherent interpretation, then π exists and:

(1.3) *If* $\mathbf{E}_\Pi(e, e')$, *then* $\Sigma_\Pi(e) = \Sigma_\Pi(e')$.

That is, pragmatically synonymous expressions have the same stimulus meaning. (1.3) is a purely pragmatic analogue of propositions (5.1) and (7.9) of [7], which assert a relation between pragmatic and semantic synonymy. In section **4** we shall consider further analogues of these propositions, and their relation to Peirce's semiotic theory.

2 *Pragmatic foundations of intended model theory* In this section the pragmatic theory of coherent interpretations is further developed, to provide a foundation for the theory of intended models of section **4**. Throughout this section, Π with or without subscripts or primes is understood to be a coherent interpretation, unless otherwise indicated, and π is its core. We shall refer to the algebra of formulas $\mathbf{L}_\Pi$ (determined by Π in the sense of Theorem 1 of [7]) together with its semantic interpretations, as a *language*. So conceived, a language $\mathbf{L}_\Pi$ has a pragmatic, as well as a syntactic and a semantic, dimension, the coherent interpretation Π.

We first define a condition c to be *positively* (*negatively*) *relevant* to a sentence s (relative to Π) iff $\pi(s, c) = 1(0)$ and $\pi(s, c_1) = 2$. Then we define c to be *relevant* to s iff c is either positively or negatively relevant to s. It follows that:

(2.1) *c is relevant to s iff c is relevant to $\sim s$.*

(2.2) *If c is not relevant to s nor to s', then c is not relevant to s & s'.*

Proposition (2.2) is a corollary of (1.1). If c is relevant to s and to s', it does not follow that c is relevant to s & s', for s' may be $\sim s$. If $s \epsilon \mathbf{N}$, then no condition is relevant to s.

Let $\mathbf{T}_\mathbf{G}$ be the set of closed logical truths of $\mathbf{L}_\Pi$. Since $\mathbf{L}_\Pi$ is a polyadic algebra with respect to $\mathbf{E}_\Pi$, logically equivalent formulas of $\mathbf{L}_\Pi$ are $\mathbf{E}_\Pi$-equivalent. It follows, by definition of $\mathbf{E}_\Pi$, that $\pi(t, b) = 1$ for all $t \epsilon \mathbf{T}_\mathbf{G}$. For any proper condition c, the proof that $\mathbf{L}_\Pi$ is polyadic goes through if b is replaced everywhere by c. Thus $\pi(t, c) = 1$ for all $t \epsilon \mathbf{T}_\mathbf{G}$ and proper conditions c, so that $\Pi(t) = \mathrm{d}_1$. It also follows that $\mathbf{N}$ is closed under logical consequence. By the above and clause III of D1:

(2.3) $\mathbf{T}_G \subseteq \mathbf{N} \subseteq \mathbf{T}_S$.

That is, logical truths are analytic, and analytic sentences are contained in the theory of the language $\mathbf{L}_\Pi$.

An analysis of the conditions $\mathbf{C}$ of pragmatic valuation is required for distinguishing the intended models of $\mathbf{L}_\Pi$ by means of its pragmatics. Let $\mathbf{Z}$ by a countable set, disjoint from the primitive terms of standard pragmatics. If the conditions of valuation in $\mathbf{C}$ are regarded as propositions, then functions from $\mathbf{Z}^\mathbf{L}$ to $\mathbf{C}$ may be regarded as propositional functions. (*Cf.* [3], p. 102.) The set of all such functions is a Boolean algebra, defined pointwise with respect to $\mathbf{C}$. From this viewpoint, each condition in $\mathbf{C}$ has the form $\mathrm{c}(z)$ for $z \in \mathbf{Z}^\mathbf{L}$ and propositional function c on $\mathbf{Z}^\mathbf{L}$. Intuitively, $\mathrm{c}(z)$ may be regarded as the outcome of an experiment c on the sample z, indexed for expression in a language over $\mathbf{L}$. In this generalized sense of "experiment," a condition in $\mathbf{C}$ may be the outcome of distinct experiments on distinct samples, but no confusion will result from using the same symbols for conditions in $\mathbf{C}$ and for propositional functions from $\mathbf{Z}^\mathbf{L}$ to $\mathbf{C}$.

The intended models of a language, in the sense of section **4**, will be relative to the choice of $\mathbf{Z}$. From the viewpoint of an idealized linguist studying the language $\mathbf{L}_\Pi$, $\mathbf{Z}$ is the set of things the observation of which is associated with dispositions, expressed by Π, to value expressions of $\mathbf{L}$ in $\mathbf{V}$. For example, $\mathbf{Z}$ may contain rabbits, rabbit-parts, or timelike rabbit-intervals, according to the hypotheses of the linguist. (*Cf.* [4], p. 465.) We shall also consider an example in which $\mathbf{Z}$ contains representations of numbers by tallies for computations whose outcomes are conditions of valuation for expressions of $\mathbf{L}$. Thus an element $z \in \mathbf{Z}^\mathbf{L}$ may contain the representations on a computing machine tape of the arguments and value (indexed by expressions of $\mathbf{L}$) of a functional relation c, so that $c(z)$ is a condition of pragmatic valuation. It is not necessary for a linguist studying a language $\mathbf{L}_\Pi$ to understand the language in order for him to analyze the conditions of valuation in the above way. Rather such analysis is required to understand $\mathbf{L}_\Pi$, in the sense of distinguishing its intended models.

Each element $z \in \mathbf{Z}^\mathbf{L}$ has the form (e, z_e), for $e \in \mathbf{L}$, $z_e \in \mathbf{Z}$. We define an individual constant $\mathrm{a} \in \mathbf{K}$ to be *basic* iff (1) some $c \in \mathbf{C}$ is relevant to an atomic sentence containing a, and (2) if $\mathrm{c}(z)$ and $\mathrm{c}(z')$ are relevant to an atomic sentence containing a, then $z_\mathrm{a} = z'_\mathrm{a}$. We define $\mathbf{K}_0$ to be the set of basic constants of $\mathbf{L}_\Pi$. Let a be a basic constant and $\mathrm{c}(z)$ be relevant to some atomic sentence containing a. We define $\mathrm{a}^* = z_\mathrm{a}$. a^* exists by the first clause in the definition of a basic constant and the fact that $\mathbf{L}$ is the domain of z; a^* is unique by the second clause in the definition of a basic constant.

If s is a sentence of $\mathbf{L}_\Pi$, we define s to be *basic* iff (1) s is an atomic sentence over basic constants, and (2) there is a condition c which is positively relevant to s, such that for any $c' \geqslant c$, c' is positively relevant to s' only if $s \rightarrow s'$ is analytic. The condition c in the definiens is *specific* for s in the sense that c is positively relevant only to analytic consequences of s; c is *minimal* with respect to s in the sense that any weaker conditions c' are positively relevant only to analytic consequences of s. Minimality of c in this sense accommodates the possibility that $c' = c \vee c''$, where c'' as well as c is positively relevant to s; in this case we should not wish to exclude the possibility that c' is also positively relevant to s. By clause (2), basic sentences are not analytic.

The fundamental intuitive idea of intended model theory is that the degree of monomorphism of the set of intended models of a theory is preserved under certain extensions of the theory. In order to characterize such extensions the following definitions are useful.

If F is a predicate of $\mathbf{L}_\Pi$, we define F to be *basic* iff F occurs in some basic sentence of $\mathbf{L}_\Pi$ or is definable in the theory of $\mathbf{L}_\Pi$ by predicates which occur in basic sentences. A basic predicate of $\mathbf{L}_\Pi$ may be, for example, a qualitative predicate underlying a quantitative predicate of $\mathbf{L}_\Pi$ in accordance with the theory of measurement.

The identity predicate is accommodated in the following way. We define Π to be *cylindric* iff there is a diadic predicate E of $\mathbf{L}_\Pi$ such that the closures of Eii and $p(i)$ & $Eij \rightarrow p(j)$ are analytic, for all variables i and j and formulas $p(i)$ in which j is free for i and $p(j)$ is the result of this substitution. By (2.3), the mapping $(i,j) \rightarrow Eij$ from $\mathbf{I}^2$ to $\mathbf{Q}$ is an equality on $\mathbf{L}_\Pi$, so that $\mathbf{L}_\Pi$ together with this equality is a cylindric algebra. (*Cf.* [3], p. 216-217.)

Let Π be cylindric. We define $\mathbf{P}_0$ to be the set of basic predicates of $\mathbf{L}_\Pi$ together with E. We also define $\mathbf{B}$ to be the set of singular (variable-free) sentences over $\mathbf{P}_0 \cup \mathbf{K}_0$ which are in the theory of $\mathbf{L}_\Pi$. We next define a cylindric interpretation to be *basic* iff $\mathbf{P} = \mathbf{P}_0$ is finite, $\mathbf{K} = \mathbf{K}_0$ is finite, and $\mathbf{K} \cup \mathbf{P}$ is included in the lexicon of $\mathbf{B} - \mathbf{N}$. If Π is basic, then $\mathbf{K} \neq \Lambda \neq \mathbf{B} \not\subseteq \mathbf{N}$, and $\mathbf{B}$ has a finite model. If Π is basic we shall say that $\mathbf{L}_\Pi$ is a basic language; it will also be convenient to apply similar properties of Π to the corresponding language $\mathbf{L}_\Pi$.

Let $\Pi_j < \Pi_i$. Every set $\mathbf{A}$ of sentences of $\mathbf{L}_{\Pi_i}$ determines an equivalence relation $\mathbf{E}_\mathbf{A}$ on the sentences of $\mathbf{L}_{\Pi_j}$, as follows: $\mathbf{E}_\mathbf{A}(s, s')$ iff $\mathbf{A} \Vdash s \leftrightarrow s'$, for all sentences s and s' of $\mathbf{L}_{\Pi_j}$. Let $(\mathbf{A})^F$ be the restriction of $\mathbf{A}$ to sentences not containing the predicate F of $\mathbf{A}$. Let $\mathbf{T}_i^j$ be the restriction of the theory $\mathbf{T}_i$ of $\mathbf{L}_{\Pi_i}$ to sentences which are not equivalent to any sentences over the predicates of $\mathbf{L}_{\Pi_j}$.

Let F be a predicate of $\mathbf{P}_i - \mathbf{P}_j$, i.e., a predicate of $\mathbf{L}_{\Pi_i}$ which is not a predicate of $\mathbf{L}_{\Pi_j}$. We define F to be *creative relative to* $\mathbf{L}_{\Pi_j}$ iff either $\mathbf{E}_{\mathbf{T}_i^j} \neq \mathbf{E}_{(\mathbf{T}_i^j)^F}$ or $\mathbf{E}_{\mathbf{T}_i^j} \neq \mathbf{E}_{(\mathbf{T}_i^j)^G}$ for some predicate G of $\mathbf{P}_i - \mathbf{P}_j$ which is definable in $\mathbf{T}_i$ by (a formula over) F. Let $\mathbf{P}_i^j$ be the set of predicates of $\mathbf{P}_i$ which are creative relative to $\mathbf{L}_{\Pi_j}$. We then define Π_i to be *minimal* *relative to* Π_j iff any predicate of $\mathbf{P}_i - \mathbf{P}_i^j$ is definable in $\mathbf{T}_i$ by predicates of $\mathbf{P}_i^j \cup \mathbf{P}_j$. We may now define Π to be *minimal* iff there exists a sequence of cylindric interpretations $\Pi_0, \Pi_1, \ldots, \Pi_n = \Pi$ such that (1) Π_0 is basic, (2) all predicates of Π_0 are basic predicates of Π, (3) all individual constants of Π are individual constants of Π_0 or definable in the theory of $\mathbf{L}_\Pi$, and (4) Π_i is minimal relative to Π_{i-1} for all i such that $0 < i \leqslant n$.

It is with respect to extension to minimal languages that, in the theory of section **4**, degree of monomorphism of intended models is preserved. It is an immediate consequence of the definition of minimality that every basic language is minimal.

We shall close this section by applying the relation of synonymy among theories, in the sense of K. de Bouvère [1], to the corresponding pragmatics. Let $\mathbf{T}$ be a standard theory, and let d be an explicit definition of a new predicate by a formula over the lexicon of $\mathbf{T}$. $\mathbf{T}'$ is defined to be a *definitional extension* of $\mathbf{T}$ iff $\mathbf{T}' = \mathbf{T} + \{\mathrm{d}\}$, the least deductive system over the lexicon of $\mathbf{T}$ and d which contains $\mathbf{T}$ and d. $\mathbf{T}'$ is defined to be an *extension* by *definitions* of $\mathbf{T}$ iff $\mathbf{T}'$ is obtained from $\mathbf{T}$ by a finite number of definitional extensions (*Cf.* [6], p. 60.) We may adapt these ideas to coherent interpretations by defining Π' to be an *extension by definitions* of Π iff $\Pi < \Pi'$, both interpretations are minimal, and the theory of Π' is an extension by definitions of the theory of Π. Finally we define Π and Π' to be *synonymous* iff some coherent interpretation is an extension by definitions of both Π and Π'. This terminology is motivated by the fact that, if Π and Π' are synonymous, then so are their theories, in the sense of [1]. (*Cf.* [6], p. 67.) In section **4** we shall consider the relation between the sets of intended models of $\mathbf{L}_{\Pi_1}$ and $\mathbf{L}_{\Pi_2}$ when Π_1 and Π_2 are synonymous.

3 *Model-theoretic foundations* Let μ be an interpretation of the theory of $\mathbf{L}_\Pi$ in a relational structure $(\mathbf{X}, \mathbf{R})$, in the sense of [7], where μ is onto $\mathbf{R}$. Then μ is a structure for the language $\mathbf{L}_\Pi$, in the usual model-theoretic sense. (*Cf.* [6], p. 18.) If $\mathbf{P}$ is finite and $\mathbf{K}$ is empty, the structure μ may be represented in the form $(\mathbf{X}, \mu F_1, \ldots, \mu F_n)$, for $F_i \in \mathbf{P}$. In the terminology of Tarski [8], μ in this case is of finite order. We shall extend this terminology to include the case in which $\mathbf{K}$ is finite. Then by Theorem (1.3) of Tarski [8], structures of finite order have the following property. We understand normal structures to be ones in which the equality predicate is interpreted as identity.

(3.1) *Let μ_1 and μ_2 be universally equivalent normal structures for a cylindric language $\mathbf{L}_\Pi$ with finite lexicon. Then any finite structure is embedded in μ_1 iff it is embedded in μ_2.*

For let μ_1 and μ_2 satisfy the hypotheses of (3.1). Replace each individual constant a of $\mathbf{L}_\Pi$ with the monadic predicate F^{a}, calling the set of formulas over the lexicon so obtained $\mathbf{Q}^*$. Replace each constant element $\mu_1\mathrm{a}$ of μ_1 with the property $\mu_1 F^{\mathrm{a}}(y) = 1$ iff $\mu_1\mathrm{a} = y$, and similarly for μ_2. We thereby obtain structures μ_1^* and μ_2^* to which Tarski's theorem directly applies.

Let p^* be any formula of $\mathbf{Q}^*$, and let p be p^* with $F^{\alpha}i$ replaced by $E\alpha i$. Then p is a formula of $\mathbf{L}_{\Pi}$. For any system of values x of μ_1, and hence of μ_1^*, x satisfies p^* under μ_1^* iff x satisfies p under μ_1, since this holds for $F^{\alpha}i$ and $E\alpha i$. μ_2 and μ_2^* are related in the same way. Thus by hypothesis, μ_1^* is universally equivalent to μ_2^*. Then by Tarski's theorem cited above, any finite structure is embedded in μ_1^* iff it is embedded in μ_2^*. Thus by the definitions of μ_1^* and μ_2^*, this holds also for μ_1 and μ_2, so that (3.1) is true.

The following proposition is a generalization, in one direction, of Theorem 2 of de Bouvère [1]. We shall refer to the set of models of a standard theory as its *variety*.

(3.2) *If* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are synonymous, then their varieties are coalescent.*

Proof: By hypothesis, there is an extension by definitions $\mathbf{T}$ of both $\mathbf{T}_1$ and $\mathbf{T}_2$. Then the variety of $\mathbf{T}$ is the required definitional enrichment of the varieties of $\mathbf{T}_1$ and $\mathbf{T}_2$. For any model of $\mathbf{T}_1$ is the reduct of a unique model of $\mathbf{T}$, and similarly for $\mathbf{T}_2$. (*Cf.* [6], p. 61.) Any reduct of a model of $\mathbf{T}$ which is in the similarity class of the variety of $\mathbf{T}_1$, is also a model of $\mathbf{T}_1$, and similarly for $\mathbf{T}_2$. (*Cf.* [6], p. 44, Lemma 1.) Thus the varieties of $\mathbf{T}_1$ and $\mathbf{T}_2$ are coalescent.

K. de Bouvère's definition of coalescence of varieties may be generalized to subvarieties (i.e., subclasses of varieties). Let $\mathbf{M}$ and $\mathbf{M}'$ be subvarieties of standard theories. We define $\mathbf{M}'$ to be a *definitional enrichment* of $\mathbf{M}$ iff (1) every model of $\mathbf{M}$ is a reduct of a unique model of $\mathbf{M}'$, and (2) every reduct of a model of $\mathbf{M}'$ which is in the similarity class of the models of $\mathbf{M}$ is also in $\mathbf{M}$. Then we define $\mathbf{M}$ and $\mathbf{M}'$ to be *coalescent* iff there is a definitional enrichment of both $\mathbf{M}$ and $\mathbf{M}'$. In section **4** we shall consider the conditions under which synonymous theories have coalescent sets of intended models.

It will be useful to define μ_1 to be a *subreduct* of μ_2 iff some reduct of μ_2 is an extension of μ_1. Thus if μ_1 is a subreduct of μ_2, there is a unique reduct of μ_2 which is an extension of μ_1. We shall also say that μ_1 is a *maximal subreduct* in $\mathbf{M}$ of μ_2 iff $\mu_1 \epsilon \mathbf{M}$ and for any subreduct μ in $\mathbf{M}$ of μ_2, μ is a substructure of μ_1.

We next define $\mathbf{M}$ to be a set of *compatible structures* iff there exists a structure μ such that every structure in $\mathbf{M}$ is a substructure of μ. Then we define $\mathbf{M}$ to be *bounded* iff every set of compatible structures which is included in $\mathbf{M}$ contains an upper bound with respect to the substructure relation.

Finally we define μ and μ' to be *projectively equivalent models* of sentences $\mathbf{A}$ iff μ and μ' are countable models of $\mathbf{A}$ of finite order, any finite model of the singular (variable-free) sentences of $\mathbf{A}$ is embedded in μ iff it is embedded in μ', and $\mathbf{A}$ contains some singular sentence. If $\mathbf{A}$ has projectively equivalent models, its lexicon is finite, so that there exists a finite model of the singular sentences of $\mathbf{A}$.

4 *Intended model theory* In the present section we shall investigate the manner in which the intended models of a theory may be distinguished by its pragmatics. By proposition (8.1) of [7], it is indifferent whether we speak of a model of a language $\mathbf{L}_\Pi$, or of the theory of $\mathbf{L}_\Pi$. As indicated in section **2**, the pragmatic determination of intended models should be such that pragmatic interpretations which are related in certain ways (e.g. by the subinterpretation relation) determined languages whose intended models are related in corresponding ways. Considerations of this kind suggest that intended model theory should be developed relative to the family of all coherent interpretations (of $\mathbf{L}$ in $\mathbf{D}$). Further reflection suggests that the set of intended models of the theory $\mathbf{T}$ a language $\mathbf{L}_\Pi$ might be regarded as the significance of $\mathbf{T}$, for Π as interpretant in the sense of Peirce's semiotic theory. We are thereby led to formulate intended model theory as a semiotic theory, in the spirit of Peirce.

Let $\boldsymbol{\mathcal{P}}$ be the set of all coherent interpretations (of $\mathbf{L}$ in $\mathbf{D}$). Let $\boldsymbol{\mathcal{L}}$ be the set of all polyadic algebras $\mathbf{L}_\Pi$ for $\Pi \in \boldsymbol{\mathcal{P}}$. Let $\mathbf{M}_\Pi$ be the variety of the theory of $\mathbf{L}_\Pi \in \boldsymbol{\mathcal{L}}$, and let $\mathfrak{M}$ be the set of such sets $\mathbf{M}_\Pi$. We then define:

(4.1) $\langle \boldsymbol{\mathcal{L}}, \mathfrak{M}, \boldsymbol{\mathcal{P}}, \sigma\rangle$ is a *system of standard semiotics* iff σ is a function from $\boldsymbol{\mathcal{P}}$ to $\mathfrak{M}$ such that for all $\Pi, \Pi' \in \boldsymbol{\mathcal{P}}$ and $\mu, \mu' \in \mathfrak{M}$:

I. $\Lambda \neq \sigma(\Pi) \subseteq \mathbf{M}_\Pi$.
II. If Π is cylindric, then each model in $\sigma(\Pi)$ is normal.
III. If $\Pi < \Pi'$ and Π is minimal, then each model in $\sigma(\Pi')$ has a subreduct in $\sigma(\Pi)$.
IV. If $\Pi < \Pi'$ and both Π and Π' are minimal, then for any models μ and μ' which are maximal subreducts in $\sigma(\Pi)$ of elementarily equivalent models in $\sigma(\Pi')$, μ is elementarily equivalent to μ'.
V. If $\Pi < \Pi'$ and both Π and Π' are minimal, then for any models μ and μ' in $\sigma(\Pi')$ which have isomorphic maximal subreducts in $\sigma(\Pi)$, μ is isomorphic to μ'.
VI. If Π' is an extension by definitions of Π, then $\mu \in \sigma(\Pi)$ iff both $\mu \in \mathbf{M}_\Pi$ and μ is a reduct of some model of $\sigma(\Pi')$.

We shall refer to the elements of $\sigma(\Pi)$ as *intended models* of the theory of $\mathbf{L}_\Pi$, and to the corresponding Boolean models (by (8.1) of [7]) as intended models of $\mathbf{L}_\Pi$. In the terminology of Peirce's semiotic theory, we shall also say that $\sigma(\Pi)$ is the *significance* of $\mathbf{L}_\Pi$, for the *interpretant* Π. Thus clause I requires that the objects actually signified by $\mathbf{L}_\Pi$ are included among its possible objects. In the limiting case in which $\sigma(\Pi) = \mathbf{M}_\Pi$, the requirement that $\sigma(\Pi)$ be non-empty follows from the consistency of the theory of $\mathbf{L}_\Pi$. (*Cf*. [7], Theorem 4.) In this sense, clause I is a generalization of standard model theory. If $\sigma(\Pi) = \mathbf{M}_\Pi$, clause II expresses a standard requirement for theories with identity, and is in this sense a generalization of standard model theory.

The size and structure of $\sigma(\Pi)$ represent the precision, or degree of monomorphism, of the significance of the language $\mathbf{L}_\Pi$. Without defining such precision, clauses III-V may be understood to express the ways in which precision in this intuitive sense is preserved by extension. Thus clause III restricts the ways in which $\sigma(\Pi)$ may be "enlarged" by minimal extension. In the limiting case in which $\sigma(\Pi) = \mathbf{M}_\Pi$, clause III is a consequence of the fact that the reduct of any model of $\sigma(\Pi')$ to the similarity class of $\mathbf{M}_\Pi$ is a member of $\mathbf{M}_\Pi$. In this sense, clause III is a generalization of standard model theory. Distinguishability of models of $\sigma(\Pi)$ by sentences of $\mathbf{L}_\Pi$ represents "potential smallness" of $\sigma(\Pi)$. Clause IV describes a condition in which such distinguishability is preserved by extension. Clause IV is no requirement on $\sigma(\Pi)$ if there are no maximal reducts of the kind described. Clause V expresses an obvious sense in which the degree of monomorphism of $\sigma(\Pi)$ is preserved by extension. Clause V is no requirement on $\sigma(\Pi)$ if there are no maximal subreducts of the kind described. In the limiting case in which $\sigma(\Pi) = \mathbf{M}_\Pi$ and $\sigma(\Pi') = \mathbf{M}_{\Pi'}$, clause VI expresses no requirement on the intended models described. For in this case $\sigma(\Pi')$ is a definitional enrichment of $\sigma(\Pi)$, by the reasoning in the proof of (3.2) above. In this sense, clause VI is a generalization of standard model theory.

Before investigating the structure of the set of intended models $\sigma(\Pi)$ of the language $\mathbf{L}_\Pi$, we shall consider more deeply the idea that $\sigma(\Pi)$ represents the significance of $\mathbf{L}_\Pi$ as fixed by Π. Sufficient conditions for $\sigma(\Pi)$ to be a unit set are considered later in this section. If $\sigma(\Pi)$ is a unit set, then all descriptive signs of $\mathbf{L}_\Pi$ in the customary intuitive sense may be understood as signs in the sense of definition (4.1). For let $\sigma(\Pi) = \{\mu\}$, and let μ^* correspond to μ by proposition (8.1) of [7]. We then define the function σ_Π on formulas p, predicates F, and individual constants $\mathfrak{a}$ of $\mathbf{L}_\Pi$ as follows:

(4.2) $\sigma_\Pi(p) = \mu^*(p)$.
$\sigma_\Pi(F) = \mu(F)$.
$\sigma_\Pi(\mathfrak{a}) = \mu(\mathfrak{a})$.

We may understand the definienda of (4.2) to be the objects signified by p, F, and $\mathfrak{a}$, for Π as interpretant. To apply Peirce's terminology more accurately to this situation, we should say that Π is the "entire general intended interpretant" of $\mathbf{L}_\Pi$ regarded as a sign, and of each formula, predicate, and individual constant of $\mathbf{L}_\Pi$ when $\mathbf{L}_\Pi$ has a unique intended model. The significance of the descriptive signs of $\mathbf{L}_\Pi$ depends on the Π-valuations of all the expressions of $\mathbf{L}$, in the sense that these valuations determine the structure of $\mathbf{L}_\Pi$ and thus of $\mathbf{M}_\Pi$.

Let $\sigma(\Pi) = \{\mu\}$. Then μ^* as above is a homomorphism of $\mathbf{L}_\Pi$. Let 1 and 0 be the unit and zero elements of the range of μ^*. Then by Theorem 4 of [7]:

(4.3) *If* $p \in \mathbf{T}_S$, *then* $\sigma_\Pi(p) = 1$ *and* $\sigma_\Pi(\sim p) = 0$.
If $\sim p \in \mathbf{T}_S$, *then* $\sigma_\Pi(p) = 0$ *and* $\sigma_\Pi(\sim p) = 1$.

The constant functions 1 and 0 of μ^* may be regarded as the True and the

False, in Frege's sense, if μ^* is the distinguished model of $\mathbf{L}_\Pi$ in which sentences valued 1 are true.

It is shown in [7] (propositions (5.1) and (7.9)) that pragmatically synonymous predicates and formulas of $\mathbf{L}_\Pi$ are semantically synonymous. These propositions hold *a fortiori* for intended models of $\mathbf{L}_\Pi$, and so restricted admit of the following analogue for individual constants. Let a and b be individual constants of a cylindric language $\mathbf{L}_\Pi$. For all intended models μ of $\mathbf{L}_\Pi$:

(4.4) *If* $\mathbf{E}_\Pi(\mathsf{a}, \mathsf{b})$, *then* $\mu\mathsf{a} = \mu\mathsf{b}$.

For by hypothesis the sentence $E\mathsf{ab}$ is in the theory of $\mathbf{L}_\Pi$, and (4.4) follows by clause II of (4.1).

We now consider an analogue of the above propositions for synonymous theories.

(4.5) *If* $\mathbf{L}_{\Pi_1}$ *and* $\mathbf{L}_{\Pi_2}$ *are synonymous, and both languages are minimal, then* $\sigma(\Pi_1)$ *and* $\sigma(\Pi_2)$ *are coalescent.*

Proof: By hypothesis, there is a definitional extension Π of Π_1 and Π_2, such that the theories $\mathbf{T}_1$ and $\mathbf{T}_2$ of $\mathbf{L}_{\Pi_1}$ and $\mathbf{L}_{\Pi_2}$ are synonymous. Then by the proof of (3.2), the variety $\mathbf{M}_\Pi$ of the theory of $\mathbf{L}_\Pi$ is a definitional enrichment of the varieties $\mathbf{M}_{\Pi_1}$ and $\mathbf{M}_{\Pi_2}$ of $\mathbf{T}_1$ and $\mathbf{T}_2$. Now let $\mu \in \sigma(\Pi_1)$. Then μ is a reduct of some model of $\sigma(\Pi)$, by clause VI. Since $\mathbf{M}_\Pi$ is a definitional enrichment of $\mathbf{M}_{\Pi_1}$, μ must be the reduct of a unique model of $\sigma(\Pi)$. Now let $\mu \in \mathbf{M}_{\Pi_1}$ be a reduct of some model of $\sigma(\Pi)$. Then $\mu \in \sigma(\Pi_1)$, by clause VI. Thus $\sigma(\Pi)$ is a definitional enrichment of $\sigma(\Pi_1)$, and by the same reasoning, of $\sigma(\Pi_2)$ also. Thus $\sigma(\Pi_1)$ and $\sigma(\Pi_2)$ are coalescent.

If Π_1 is not compatible with Π_2, the theories of $\mathbf{L}_{\Pi_1}$ and $\mathbf{L}_{\Pi_2}$ need not also be incompatible, since Π_1 and Π_2 may disagree germanely under a condition $c \neq b$, although they may agree germanely under b. But if the theories of $\mathbf{L}_{\Pi_1}$ and $\mathbf{L}_{\Pi_2}$ are not compatible, i.e., if their union is not consistent, then they are not synonymous. Thus the following proposition is a partial converse of (4.5).

(4.6) *If* $\sigma(\Pi_1)$ *and* $\sigma(\Pi_2)$ *are coalescent, then the theories of* $\mathbf{L}_{\Pi_1}$ *and* $\mathbf{L}_{\Pi_2}$ *are compatible.*

Proof: Let the theories $\mathbf{T}_1$ and $\mathbf{T}_2$ of $\mathbf{L}_{\Pi_1}$ and $\mathbf{L}_{\Pi_2}$ be incompatible, so that their union has no model. Then there is no definitional enrichment of $\sigma(\Pi_1)$ and $\sigma(\Pi_2)$. For any such enrichment $\mathbf{M}$ would contain a model with a reduct in $\sigma(\Pi_1)$ and a model with a reduct in $\sigma(\Pi_2)$. But all models of $\mathbf{M}$ are similar, since $\mathbf{M}$ is a subvariety by definition. Then any model μ of $\mathbf{M}$ has a reduct in the similarity class of $\sigma(\Pi_1)$, which is therefore a model of $\mathbf{T}_1$; and μ also has a reduct in the similarity class of $\sigma(\Pi_2)$, which is therefore a model of $\mathbf{T}_2$. Thus μ must be a model of $\mathbf{T}_1 \cup \mathbf{T}_2$, against the hypothesis of their incompatibility.

(4.7) *If* Π_1 *and* Π_2 *have the same core and both are minimal, then* $\sigma(\Pi_1) = \sigma(\Pi_2)$.

Proof: By first hypothesis we may define

$$\Pi(e)(u, w, c) = \begin{cases} \Pi_1(e)(u, w, c), \text{ if } \Pi_1(e)(u, w, c) \neq 2, \\ \Pi_2(e)(u, w, c), \text{ if } \Pi_2(e)(u, w, c) \neq 2, \\ 2 \text{ otherwise.} \end{cases}$$

Π satisfies the intersubjectivity requirements of a cylindric interpretation by the first hypothesis, and the remaining ones by the second hypothesis. Now let F be a proper expression, distinct from all logical and non-logical symbols of $\mathbf{L}_\Pi$, and let $p(i)$ be a fixed formula of $\mathbf{L}_\Pi$ in the one free variable i. We then define:

$$\Pi'(e)(u, w, c) = \begin{cases} \Pi(e)(u, w, c), \text{ if } e \text{ does not contain } F, \\ 0, \text{ if } e = \exists iFi \ \& \sim Fi \text{ and } c \text{ is proper}, \\ 1, \text{ if } e = \forall(Fi \leftrightarrow p(i)) \text{ and } c \text{ is proper}, \end{cases}$$

and otherwise $\Pi'(e)(u, w, c) \neq 2$ only as required by the condition that Π' be coherent. Since Π is coherent, Π' may be so defined; for Π' is obtained from Π by changing valuations of the form $\Pi(e(F))(u, w, c) = 2$ to accommodate the new predicate F in the indicated manner.

Since Π is cylindric, so is Π'. There is a sequence $\Pi_0, \ldots, \Pi_1$, which establishes the minimality of Π_1. Consequently $\Pi_0, \ldots, \Pi_1, \Pi'$ establishes the minimality of Π'. For the positive relevancies of Π, and thus of Π_1, are preserved by Π', since $\exists iFi \ \& \sim Fi$ and $\forall(Fi \leftrightarrow p(i))$ are analytic in $\mathbf{L}_{\Pi'}$. Thus all predicates of Π_0 are basic predicates of Π' as well as Π_1. And since Π' and Π_1 have the same individual constants, all individual constants of Π' are individual constants of Π_0. Finally, since F is definable in the theory $\mathbf{T}'$ of $\mathbf{L}_{\Pi'}$, Π' is minimal relative to Π.

$\mathbf{T}'$ is a definitional extension of the theory of $\mathbf{L}_{\Pi_1}$, since $\mathbf{T}' = \mathbf{T}_1 + \{\exists(Fi \leftrightarrow p(i))\}$. Thus Π' is an extension by definitions of Π_1, since both are minimal. Similarly, Π' is an extension by definitions of Π_2. Thus μ_1 and Π_2 are synonymous, so that by (4.5), $\sigma(\Pi_1)$ and $\sigma(\Pi_2)$ are coalescent. Suppose $\sigma(\Pi_1) \neq \sigma(\Pi_2)$. We consider the case in which there is a model Π_1 of $\sigma(\Pi_1)$ which is not in $\sigma(\Pi_2)$. By the coalescence of $\sigma(\Pi_1)$ and $\sigma(\Pi_2)$ there is a definitional enrichment of $\sigma(\Pi_1)$ and $\sigma(\Pi_2)$, which contains an expansion μ of μ_1. By clause VI, every reduct of μ in the similarity class of $\sigma(\Pi_2)$ is in $\sigma(\Pi_2)$. μ_1 itself is such a reduct, so that $\mu_1 \in \sigma(\Pi_2)$, against the above supposition. (4.7) is thereby proved.

The following proposition is a consequence of clause III.

(4.8) *Let* $\Pi < \Pi'$ *and* $|\Pi| = |\Pi'|$. *If* $\sigma(\Pi) = \mathbf{M}_\Pi$, *then* $\sigma(\Pi') \subseteq \sigma(\Pi)$; *if* Π' *is minimal, then each model in* $\sigma(\Pi')$ *has a substructure in* $\sigma(\Pi)$.

Let $\Pi_1 < \Pi_2$ and both interpretations be minimal. Then clauses III-V say intuitively that $\mathbf{L}_{\Pi'}$ preserves the precision of significance of $\mathbf{L}_\Pi$, and otherwise any model of $\mathbf{L}_{\Pi'}$ may be an intended model. It thus appears reasonable to pursue the idea that the significance of $\mathbf{L}_{\Pi'}$ is "abstract" relative to that of $\mathbf{L}_\Pi$. $\mathbf{L}_{\Pi'}$ would then have special interest if the structures comprising the significance of $\mathbf{L}_\Pi$ were "concrete" in an appropriate sense.

We shall now suggest the following explication of the above ideas. Let **M** be a subvariety of **T**. We define **M** to be *concrete* iff the domains of all models in **M** are included in **Z**, projectively equivalent models in **M** are isomorphic, and **M** is bounded. The adequacy of this definition is suggested by the following proposition.

(4.9) *Let* $\Pi < \Pi'$, Π *be basic, and* Π' *be minimal. If* $\sigma(\Pi)$ *is concrete, then any two elementarily equivalent models of* $\sigma(\Pi')$ *are isomorphic.*

Proof: Let μ_1' and μ_2' be elementarily equivalent models of $\sigma(\Pi)$. Then by clause III, and the boundedness of $\sigma(\Pi)$, there are maximal subreducts μ_1 and μ_2 in $\sigma(\Pi)$ of μ_1' and μ_2'. Then by clause IV, μ_1 is elementarily equivalent to μ_2, so that μ_1 and μ_2 are universally equivalent. Since Π is basic, $\mathbf{L}_\Pi$ is a cylindric language with finite lexicon. By clause II, μ_1 and μ_2 are normal models of the theory of $\mathbf{L}_\Pi$. Then by (3.1), any finite structure is embedded in μ_1 iff it is embedded in μ_2.

It follows that μ_1 and μ_2 are projectively equivalent models of the theory **T** of $\mathbf{L}_\Pi$. For they are countable, by the concreteness of $\sigma(\Pi)$. Since Π is basic, **T** contains some singular sentence, and μ_1 and μ_2 are of finite order. By the above, any finite model of the singular sentences of **T** is embedded in μ_1 iff it is embedded in μ_2. Thus μ_1 and μ_2 are projectively equivalent models of **T**, so that by the concreteness of $\sigma(\Pi)$, μ_1 is isomorphic to μ_2. Then by clause V, μ_1' and μ_2' are isomorphic. (4.9) is thereby established.

If the theory of $\mathbf{L}_{\Pi'}$ in (4.9) is complete, then $\mathbf{L}_{\Pi'}$ has a unique intended model up to isomorphism. The consequent of (4.9) may thus be understood to say that the significance of $\mathbf{L}_{\Pi'}$ is *potentially monomorphic.*

(4.9.1) Corollary. *If* Π *is basic and* $\sigma(\Pi)$ *is concrete, then* $\mathbf{L}_\Pi$ *is potentially monomorphic.*

Proof: By hypothesis Π is minimal. Thus since $\Pi < \Pi$, the conclusion follows by (4.9).

We now consider sufficient conditions for a language to have concrete significance. We first define $\sigma(\Pi)$ to be *quasi-ostensive* iff, for each model μ of $\sigma(\Pi)$, μ maps **K** onto the universe of μ, and $\mu a = a^*$ for each $a \in \mathbf{K}$. If $\sigma(\Pi)$ is quasi-ostensive and the universe of every model in $\sigma(\Pi)$ is included in **Z**, we shall say that $\sigma(\Pi)$ is *ostensive.* Finally, we define a theory **T** to be *data-complete* iff, for each singular sentence s of the language of **T**, either s or $\sim s$ is in **T**.

(4.10) *Let* Π *be basic and* $\sigma(\Pi)$ *be quasi-ostensive. If the theory of* $\mathbf{L}_\Pi$ *is data-complete, then it has a unique intended model.*

Proof: Let $F \in \mathbf{P}^n$ and $a_1, \ldots, a_n \in \mathbf{K}$, for some n. Since F is basic, by hypothesis, such constants exist. By hypothesis, either $Fa_1 \ldots a_n$ or $\sim Fa_1 \ldots a_n$ is in the theory **T** of $\mathbf{L}_\Pi$. Let μ, $\mu' \in \sigma(\Pi)$. Since **T** holds in these models, $\mu F(\mu a_1, \ldots, \mu a_n) = \mu' F(\mu' a_1, \ldots, \mu' a_n)$ is 1 or 0 according as $Fa_1 \ldots a_n$ or $\sim Fa_1 \ldots a_n$ is in $\mathbf{T}$. Also by hypothesis, $\mu a = a^* = \mu' a$, for each $a \in \mathbf{K}$, and each individual in the universe of μ has the form μa for $a \in \mathbf{K}$, and similarly for μ'. Thus $\mu F = \mu' F$, for each predicate F of $\mathbf{L}_\Pi$, so that by clause I $\sigma(\Pi)$ contains only one model $\mu = \mu'$.

If the hypotheses of (4.10) are strengthened so that $\sigma(\Pi)$ is ostensive, it follows that $\sigma(\Pi)$ is concrete.

We now consider an application of proposition (4.9) to formal arithmetic. Let Π be cylindric and let $\mathbf{L}_\Pi$ contain, in addition to the identity predicate, a single diadic predicate G and a constant 0. Let $\mathbf{T}$ contain the Peano postulates for arithmetic (with the induction postulate represented by an infinite number of sentences, one for each open formula in a single variable, and where the intuitive meaning of Gij is that j is the successor of i). What conditions on Π will restrict the intended models of $\mathbf{T}$ to standard models, i.e., to models isomorphic to the natural numbers under the successor relation?

Let $\mathbf{Z} = \{|, ||, |||, \ldots\}$, where $|$ is a tally like that of a computing machine. Every function $z \in \mathbf{Z}^{\mathbf{L}}$ whose range has two elements may be represented in the form $\{z_1, z_2\} \subseteq \mathbf{Z}$, where the subscripts indicate ordering by length. It will not matter that distinct functions may have the same representation of this kind. Let concatenation of elements of $\mathbf{Z}$ be expressed by $^\frown$. If $\mathbf{Z}$ and Π are as above, and if there is a propositional function c such that whenever $z = \{z_1, z_2\}$ and $z_2 = z_1{}^\frown|$, $c(z)$ is positively relevant to $\exists(Gij)$ (relative to Π), we shall say that Π is a *Peano interpretation*. It should be recalled that Π-valuations represent dispositions of the users of $\mathbf{L}$ to value expressions in $\mathbf{L}$, and not necessarily actual valuations.

(4.11) *Let* Π *be a Peano interpretation. Let* μ *be a model of the theory of* $\mathbf{L}_\Pi$ *such that* (1) $\mu 0 = |$, (2) *the domain of* μ *is included in* $\mathbf{Z}$, *and* (3) $\mu G(z_1, z_2) = 1$ *whenever* $z = \{z_1, z_2\}$, $c(z)$ *is positively relevant to* $\exists(Gij)$, *and* $z_2 = z_1{}^\frown|$. *Then* μ *is a standard model of the theory of* $\mathbf{L}_\Pi$.

Proof: Since Π is a Peano interpretation, the theory of $\mathbf{L}_\Pi$ requires that μG is a function, so that we may express $\mu G(z_1, z_2) = 1$ as $\mu G(z_1) = z_2$. It is sufficient to show that each element of the universe of μ is related to $\mu 0$ by $(\mu G)^n$, the nth iterate of μG, for some natural number n. For then the mapping $\phi n = (\mu G)\,(\mu 0)$ is one-one from the natural numbers onto the universe of μ. Moreover, for natural numbers n and m:

$$n = m + 1 \text{ iff } (\mu G)^n(\mu 0) = (\mu G((\mu G)^m(\mu 0)) \text{ iff } \phi n, = G(\phi m).$$

Thus μ is isomorphic to the system of natural numbers under the successor relation.

Let x be in the universe of μ; by hypothesis (2), $x \in \mathbf{Z}$. We may show by induction on the length of x that $x = (\mu G)^n(\mu 0)$. If $x = \mu 0$, then $x = (\mu G)^0(\mu 0)$. Let $x \neq \mu 0$. Then $x \neq |$, by hypothesis, so that $x = y^\frown|$ for $y \in \mathbf{Z}$. Since $\mathbf{Z}^{\mathbf{L}}$ contains all functions from $\mathbf{L}$ into $\mathbf{Z}$, it contains a function represented as $z = \{y, y^\frown|\}$. Since Π is a Peano interpretation, some condition $c(z)$ is positively relevant to $\exists(Gij)$, then by hypothesis of (4.11),

$\mu G(y) = y^\frown|$. Thus by the induction hypothesis, $x = \mu G(\mu G^m(\mu 0))$. (4.11) is thereby proved.

(4.11.1) Corollary. *Let the hypotheses of* (4.11) *hold. If* Π *is basic,* $\sigma(\Pi)$ *is concrete,* $\mu \in \sigma(\Pi)$, *and the theory of* $\mathbf{L}_\Pi$ *is complete, then all intended models of* $\mathbf{L}_\Pi$ *are standard.*

Proof: By hypothesis and the corollary of (4.9), $\sigma(\Pi)$ is monomorphic. Then by (4.11), all models of $\sigma(\Pi)$ are standard.

We now consider an application of (4.9) to quantitative languages. We define Π to be a *proto-quantitative extension* of Π_0 iff $\Pi_0 < \Pi$, Π_0 is basic, and the theory of $\mathbf{L}_\Pi$ consists of the consequences of (1) the theory of $\mathbf{L}_{\Pi_0}$, (2) a set of axioms for set theory together with definitions adequate to accommodate real number theory, none of whose predicates are in $\mathbf{P}_0$, and (3) axioms of the form:

$$(\exists i)(\exists j)(Fai \;\&\; Fbj \;\&\; {<}ij) \leftrightarrow Gab$$

for all individual constants a and b of $\mathbf{K}_0$, and for each predicate F of $\mathbf{P} - \mathbf{P}_0$ which is not number-theoretic and is distinct from the membership predicate ε and the identity predicate E, where G is some predicate of $\mathbf{P}_0$ corresponding to F and $<$ is the predicate for the ordering relation on the real numbers.

Let Π be an extension of a quantitative extension of Π_0, let R be the predicate of $\mathbf{L}_\Pi$ 'is a real number', and let μ be a model of the theory $\mathbf{T}$ of $\mathbf{L}_\Pi$. If the cardinality of $\{x\colon \mu R(x) = 1\}$ is that of the continuum, then we shall say that μ is a *proto-standard* model of $\mathbf{T}$.

(4.12) *Let* Π *be minimal relative to a proto-quantitative extension of* Π_0, *where clauses* (2) *and* (3) *of the definition of a minimal interpretation are satisfied. Let* $\sigma(\Pi)$ *contain a proto-standard model of the theory* $\mathbf{T}$ *of* $\mathbf{L}_\Pi$, *where* $\mathbf{T}$ *is complete. If* $\sigma(\Pi_0)$ *is concrete, then all models of* $\sigma(\Pi)$ *are proto-standard.*

Proof: By the first hypothesis, Π is minimal relative to some quantitative extension Π_1 of Π_0. Then Π_0 is basic, so that Π is minimal provided Π_1 is minimal relative to Π_0. Let F be a predicate of $\mathbf{P}_1 - \mathbf{P}_0$ which is not a number-theoretic predicate and is distinct from ϵ and E. Then for some predicate G of $\mathbf{P}_0$, the theory $\mathbf{T}_1$ of $\mathbf{L}_{\Pi_1}$ contains all sentences of the form:

$$(\exists i)(\exists j)(Fai \;\&\; Fbj \;\&\; {<}ij) \leftrightarrow Gab$$

where a and b and individual constants of $\mathbf{L}_{\Pi_0}$. Such sentences are not equivalent to any sentences of $\mathbf{L}_{\Pi_0}$, and are thus contained in $\mathbf{T}_1^0$. Since Π_0 is basic, $\mathbf{T}_0$ contains some sentence Gab, so that $\mathbf{T}_1$ and hence $\mathbf{T}_1^0$ contains $(\exists i)(\exists j)(Fai \;\&\; Fbj \;\&\; {<}ij)$. It follows that Gab is a consequence of $\mathbf{T}_1^0$ but not $(\mathbf{T}_1^0)^F$, so that the predicates F are creative relative to $\mathbf{L}_{\Pi_0}$. Similarly, the predicate $<$ is creative relative to $\mathbf{L}_{\Pi_0}$. Since $<$ is definable in $\mathbf{T}_1$ by ε, ε is creative relative to $\mathbf{L}_{\Pi_0}$. Thus the number-theoretic predicates of $\mathbf{L}_{\Pi_1}$ are definable by the creative predicate ε, so that Π_1 is minimal relative to Π_0. By the second hypothesis and (4.9), if $\sigma(\Pi_0)$ is concrete then all models of

$\sigma(\Pi)$ are isomorphic to some proto-standard model of $\mathbf{T}$, so that all models of $\sigma(\Pi)$ are proto-standard.

The theory of intended models has been developed in the spirit of Peirce's semiotic theory. The naturalness of this approach from the pragmatic point of view is suggested by proposition (1.2) and the ensuing discussion. Its naturalness from the model-theoretic point of view is suggested by the following observations, with which we conclude the present paper.

Standard model theory suggests a substantive theory of significance or referential meaning: we might identify the significance of a theory (in standard formalization) with its variety. Familiar results about non-standard models suggest a refinement of this approach: we may regard the significance of a theory as a distinguished subclass of its variety, which in the light of intended model theory we may take to be the class of intended models of the theory.

The Gödel-Henkin completeness theorem suggests a beautifully simple criterion of significance for theories with standard formalization: consistency. In the light of intended model theory, traditional scruples about significance may then be reconstructed as requirements for precision of significance, as represented by the size and structure of the set of intended models. For example, proposition (4.9) provides sufficient conditions for a theory to have potentially monomorphic significance. As illustrated in (4.11.1) Corollary, the problem of showing that the intended models of a theory are standard requires considerations specific to that theory. Intended model theory provides the context in which such considerations may be precisely formulated.

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