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AN EXTENDED JOINT CONSISTENCY THEOREM FOR FREE LOGIC WITH EQUALITY

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1 *Introduction* In this paper, proof is given of an extended joint consistency theorem for free logic with equality using techniques sketched by Hintikka in a proof of a similar theorem for standard first order logic.[1] The Robinson consistency theorem and the Craig and Lyndon interpolation theorems are immediate corollaries, and proof of the Beth definability theorem is easily had. Certain related theorems of standard first order logic, though, do not hold in free logic.[2] To obtain the extended joint consistency theorem, I modify the tree method as presented by Leblanc and Wisdom for standard first order logic,[3] adapting it to free logic with equality. The completeness and soundness of a free method for free logic with equality are then demonstrated informally by showing that it is equivalent to an axiomatization known to be sound and complete. Since an extension of this tree method can be applied to any infinite set of wffs, the extension is strongly complete and strongly sound. Free logics are of interest, first, because of the insights they provide concerning a number of philosophical issues. Axiomatic versions of certain modal and tense logics have free logics as their quantificational base, and the results presented here carry over into some of these logics. Second, since free logics provide one means of formalizing partial functions, free logics are of potential use as the underlying logics of mathematical theories, and they can be applied in the analysis of algorithms. To determine the suitability of free logics for such mathematical applications, it is desirable to establish results of the sort presented here.

2 *The Leblanc tree method* The grammar and semantics of **TQC***=, a tree method for free logic with equality, are to be those of Leblanc's **QC***=, a version of free logic with equality which employs in its deductive component axiom schemas and modus ponens as the only rule of inference.[4] **QC***= is known to be complete and sound. The primitive logical constants of **QC***= are $\sim$, $\supset$, $\forall$, and $=$. For notational convenience, I shall also use $\exists$, which is defined in the usual manner. Let T, T_1, and T_2 be individual

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parameters of **QC***=, let x and y be individual variables of **QC***=, let $\forall xA, B$, and C be wffs of **QC***=, and let $\mathscr{S}$ be a set of wffs of **QC***=. Following the notation of Leblanc and Wisdom,[5] *the rules of* **TQC***= can be schematized as:

$$\begin{array}{cccc}
(1) & (2) & (3) & (4) \\
\sim\sim B\,\checkmark & B \supset C\,\checkmark & \sim(B \supset C)\,\checkmark & \sim\forall xA\,\checkmark \\
\bullet & \bullet & \bullet & \bullet \\
B & \swarrow \quad \searrow & B & \exists x(x = T) \\
 & \sim B \qquad C & \sim C & \sim A(T/x)
\end{array}$$

$$\begin{array}{ccccccc}
 & (5) & & & & (6) & \\
\forall xA & & \exists y(y = T) & & T_1 = T_2 & & B \\
\vdots & \text{or} & \vdots & & \vdots & \text{or} & \vdots \\
\exists y(y = T) & & \forall xA & & B & & T_1 = T_2 \\
\bullet & & \bullet & & \bullet & & \bullet \\
A(T/x) & & A(T/x) & & B(T_2/\!\!/T_1) & & B(T_2/\!\!/T_1)
\end{array}$$

Rules (4)-(6) are subject to the following restrictions. In (4), $\sim\forall xA$ must not be of the form $\exists x(x = T_1)$, and T must be foreign to $\mathscr{S}$ and to every branch on which $\exists x(x = T)$ and $\sim A(T/x)$ are to be entered.[6] In (5), $A(T/x)$ is entered on a branch when $\forall xA$ and $\exists y(y = T)$, where x and y may be identical, already occur on that branch. In (6), $B(T_2/\!\!/T_1)$ is entered on a branch when $T_1 = T_2$ and B already occur on that branch, where B is an atomic wff (which may be an identity) or the negation of an atomic wff and $B(T_2/\!\!/T_1)$ is the wff obtained by substituting T_2 for zero or more occurrences of T_1 in B.

Leblanc and Wisdom's ground rules[7] are modified to become *the ground rules of* **TQC***=:

A. If both an atomic wff and its negation occur on a branch or if a wff of the form $\sim(T = T)$ occurs on a branch, the branch is closed.
B. When applying any one of the rules of **TQC***= according to ground rules C through F, *omit any wff if an occurrence of that wff already appears on the branch.*
C. Rules (1)-(4) are applied only to unchecked wffs. If a wff is decomposed using any of the rules (1)-(4), the wff must be checked off and the wffs yielded by the decomposition entered on every open branch through the wff.
D. When applying rule (4) add $\exists x(x = T)$ and the instantiation $\sim A(T/x)$ of the negated universal quantification $\sim\forall xA$ on each open branch through $\sim\forall xA$, where T is an individual parameter foreign to S and to the branch. If a pair of wffs of the form $\exists y(y = T_1)$ and $\sim A(T_1/x)$ already occur on a branch, $\exists x(x = T)$ and $\sim A(T/x)$ are not added to that branch.
E. When applying rule (5), instantiate the universal quantification on each open branch through it by means of each term T which has occurred so far along the branch in a wff of the form $\exists y(y = T)$.
F. When applying rule (6), on each open branch through the identity $T_1 = T_2$, apply it n times to each atomic wff or negated atomic wff B on the branch, where n is the number of occurrences of T_1 in B, adding each of the n wffs of the form $B(T_2/\!\!/T_1)$ to the branch.[8]

Leblanc and Wisdom's routine[9] requires more extensive modification so that infinite sets of wffs can be tested for consistency. It is more convenient to present the routine for **TQC***= in the form of a sequence of instructions (steps), as is frequently done in the specification of an algorithm, rather than in the form of a flowchart.[10]

Call a set of wffs $\mathscr{J}$ of **QC***= *infinitely extendible* if there are $\aleph_0$ individual parameters foreign to $\mathscr{J}$, and call a wff B belonging to $\mathscr{J}$ *activated* if B has been activated in step 2 of the routine for **TQC***=, presented below. Let $\mathscr{J}$ be infinitely extendible, and let $\mathrm{e}(\mathscr{J})$ be an enumeration B_0, B_{-1}, B_{-2}, . . . of the wffs in $\mathscr{J}$. Understand that, if $\mathscr{J}$ is empty, $\mathrm{e}(\mathscr{J})$ is the null sequence; if $\mathscr{J}$ contains just one wff, B_0, $\mathrm{e}(\mathscr{J}) = B_0$; if just two wffs, B_0 and B_{-1}, are in $\mathscr{J}$, $\mathrm{e}(\mathscr{J}) = B_0, B_{-1}$; and so on. The enumeration $\mathrm{e}(\mathscr{J})$ determines the order in which the routine for **TQC***= activates the wffs in $\mathscr{J}$. One wff in $\mathscr{J}$ is activated in step 2 during each *iteration* through the loop beginning at step 2 and ending at step 7. In particular, if $\mathscr{J}$ contains one or more wffs, B_0 is activated during the first iteration in step 2; if $\mathscr{J}$ contains two or more wffs and the routine does not declare $\mathscr{J}$ to be inconsistent and halt in step 6 during the first ($i = 1$) iteration, B_{-1} is activated during the second ($i = 2$) iteration in step 2; etc.

The Routine for **TQC***=

1. Set i to 1. If $\mathscr{J}$ is empty, then (a) declare $\mathscr{J}$ to be *consistent* and (b) halt.
2. If there are one or more wffs in $\mathscr{J}$ which have not been activated (if there are i or more wffs in $\mathscr{J}$), then activate $B_{-(i-1)}$ and list $B_{-(i-1)}$ at the *top* of the tree.
3. If there is on an open branch (a) an unchecked wff to which one of the rules (1)-(4) of **TQC***= is applicable or (b) a pair of wffs to which rule (6) is applicable, then apply the relevant rule. Continue applying rules (1)-(4) and (6) in this manner until none of these rules can be applied.
4. If there is on an open branch a pair of wffs to which rule (5) is applicable, then apply rule (5). Continue applying rule (5) in this manner until it can no longer be applied.
5. If every wff in $\mathscr{J}$ has been activated and there is an open branch to which no line was added in steps 3 and 4, then (a) declare $\mathscr{J}$ to be *consistent* and (b) halt.
6. If every branch is closed, then (a) declare $\mathscr{J}$ to be *inconsistent* and (b) halt.
7. (a) Set i to $i + 1$, and (b) go to step 2.

3 *Completeness and soundness* Regarding the routine for **TQC***=, note the following facts:

(1) If $\mathscr{J}$ is empty, the routine declares $\mathscr{J}$ to be consistent and terminates in step 1.

During each iteration $i \geqslant 1$ through the loop beginning at step 2 of the routine:

(2) if there are i or more wffs in $\mathscr{J}$, $B_{-(i-1)}$ will be activated in step 2;

(3) each wff or pair of wffs to which one of the rules (1)-(4) or (6) is applicable will have the rule(s) applied to it in step 3 a finite number of times, which is the maximum number of times permissible according to ground rules A-D and F in step 3 during iteration i;
(4) each pair of wffs on an open branch to which rule (5) is applicable will have rule (5) applied to it a finite number of times, which is the maximum number of times permissible according to ground rule E in step 4 during iteration i;
(5) if every wff in $\mathscr{J}$ has been activated and no line was added to some open branch in steps 3 and 4, $\mathscr{J}$ will be declared to be consistent and the routine will terminate in step 5 (since nothing more can be done to close that open branch);
(6) if every branch is closed, $\mathscr{J}$ will be declared to be inconsistent and the routine will terminate in step 6;
(7) if the routine has not terminated in step 5 or step 6, (since steps 2-6 involve a finite sequence of finite operations) the routine will reach step 7, where it will return to step 2 for the i + 1-th iteration.

A *descendant of A* is recursively defined as follows:

(1) if B is a wff obtained by applying one of the rules (1)-(4) of **TQC***= to A or rules (5) or (6) of **TQC***= to a pair of wffs, one of which is A, then B is a descendant of A;
(2) if C is a descendant of B and B is a descendant of A, then C is a descendant of A.

An inductive argument on the iteration i employing facts (1)-(7), above, establishes the additional facts that, for any wff B_{-i} in the enumeration $e(\mathscr{J})$, if $\mathscr{J}$ is not declared to be inconsistent in some iteration h of the routine for **TQC***= such that $1 \leqslant h \leqslant i$, (a) the i + 1-th iteration occurs and B_{-i} is activated in that iteration and (b) for each iteration j such that $j \geqslant i + 1$, the rules of **TQC***= are applied to B_{-i} and each of its descendants on an open branch the maximum number of times permissible according to ground rules A-D and F in step 3 and the maximum number of times permissible according to ground rule E in step 4.

Let $\mathscr{J}$ be a set of wffs, and let $\mathscr{C}$ be a set consisting of zero or more sets of wffs. Call a logic *strongly complete with respect to* $\mathscr{C}$ if in that logic, for every $\mathscr{J}$ belonging to $\mathscr{C}$, if $\mathscr{J}$ is syntactically consistent, then $\mathscr{J}$ is semantically consistent. Call a logic *strongly sound with respect to* $\mathscr{C}$ if in that logic, for every $\mathscr{J}$ in $\mathscr{C}$, if $\mathscr{J}$ is semantically consistent, then it is syntactically consistent. Let $\mathscr{E}$ be the set consisting of all and only those sets of wffs of **QC***= which are infinitely extendible. Since **QC***= is strongly complete and strongly sound, facts (a) and (b) about the routine for **TQC***= insure that, if **TQC***= is weakly complete and weakly sound, then it is also strongly complete and strongly sound with respect to $\mathscr{E}$. By demonstrating that to each of the axioms and the rule of inference of **QC***= there corresponds a closed tree generated by the routine for **TQC***=, we prove the weak completeness of **TQC***=; hence the strong completeness of **TQC***= with respect to $\mathscr{E}$ follows. The weak soundness of **TQC***=—hence the strong soundness of **TQC***= with respect to $\mathscr{E}$—is had by demonstrating that if the set of wffs on an open branch before applying one of the rules of **TQC***= is syntactically consistent in **QC***=, then the set of wffs on the branch after applying one of the rules of **TQC***= is syntactically consistent in **QC***=.

For the weak completeness of **TQC***=, I shall only consider the quantificational axioms of **QC***=, since the results for the other axioms and the rule of inference can be retrieved from the literature on tree methods for standard first order logic with equality. The following proofs employ the derived ground rule that permits the closing of a branch as soon as *any* wff and its negation appear on a branch.[11] For each set consisting of the negation of a quantificational axiom of **QC***=, the routine for **TQC***= generates a closed tree.

Axiom 1 $\forall x(A \supset B) \supset (\forall xA \supset \forall xB)$

Proof: The first lines generated by the routine for **TQC***= are:

0.	$\sim(\forall x(A \supset B) \supset (\forall xA \supset \forall xB))$	✓
1.	$\forall x(A \supset B)$	from 0
2.	$\sim(\forall xA \supset \forall xB)$	from 0 ✓
3.	$\forall xA$	from 2
4.	$\sim\forall xB$	from 2 ✓

which are added in step 3 during the first iteration.

If $A = B$, the tree closes. If $A \neq B$, consider the case when $\sim\forall xB$ is not of the form $\exists x(x = T_1)$. (The other case is similar.) Lines 5 and 6 are also added in step 3 during the first iteration:

5.	$\exists x(x = T)$	from 4
6.	$\sim B(T/x)$	from 4

If $B(T/x)$ is an identity of the form $(T_1 = T_1)$, the tree closes. Suppose $B(T/x) \neq (T_1 = T_1)$. If $m \geqslant 0$ is the number of times rule (2) is applied to the descendants of $\sim B(T/x)$ in step 3 during the first iteration, there are at most 2^m open branches on entering step 4. For any branch b such that $1 \leqslant b \leqslant 2^m$, b either closes in step 3 or is still open when step 4 is entered in the first iteration. Suppose it is still open. Let $k \geqslant m \geqslant 0$ be twice the number of times rules (3) and (4) are applied plus the number of times rules (1), (2), and (6) are applied on branch b to $\sim B(T/x)$ and its descendants in step 3. In step 4 of the first iteration, line $7 + k$ is added to branch b:

$7 + k$.	$A(T/x)$	from 3, 5

and, if branch b does not close at this point, line $8 + k$ is also added:

$8 + k$.	$A(T/x) \supset B(T/x)$	from 1, 5 ✓

Branch b does not close after line $8 + k$ since $\sim(A(T/x) \supset B(T/x))$ cannot occur on branch b. Let $l \geqslant 0$ be the number of times rule (5) is applied to the descendants of $\forall xA$ and $\sim B(T/x)$ on branch b in step 4 of the first iteration. After line $8 + k + l$, branch b may close. Suppose that it does not. Then, step 5 is entered, and the routine does not terminate by fact (5) since $8 + k + l > 0$ lines were added to branch b in steps 3 and 4. Step 6 is entered, and the routine does not terminate by fact (6) since, by assumption, branch b is not closed. So, by fact (7), the routine executes step 7 and returns to step 2 for the second iteration. In step 3 of the second iteration, the following line is added:

	/	\	
$9 + k + l$.	$\sim A(T/x)$	$B(T/x)$	from $8 + k$
	X	X	
	$7 + k$	6	

and branch b is closed. So, if a branch does not close before step 3 of the second iteration, it will close then. Since there are no open branches, the routine procedes through step 4 without adding any lines and through step 5 without halting. In step 6, the routine declares the set consisting of the negation of axiom 1 to be inconsistent and halts by fact 6, since every branch is closed. So, if the routine does not declare the set consisting of the negation of axiom 1 to be inconsistent and halt in step 6 during the first iteration, it will do so in step 6 during the second iteration.

Axiom 2 $A \supset \forall xA$

Proof:

0.	$\sim(A \supset \forall xA)$	✓
1.	A	from 0
2.	$\sim \forall xA$	from 0 ✓
3.	$\exists x(x = T)$	from 2
4.	$\sim A$	from 2
	X	
	1	

Note that since A is a wff, the quantification $\forall xA$ is vacuous and $\sim A(T/x) = \sim A$.

Axiom 3 $\forall y \exists x(x = y)$

Proof:

0.	$\sim \forall y \exists x(x = y)$	✓
1.	$\exists y(y = T)$	from 0
2.	$\sim \exists x(x = T)$	from 0
3.	$\forall x \sim (x = T)$	from 2
4.	$\sim (T = T)$	from 1, 3
	X	

Axiom A4 *Any wff $\forall xA$ such that $A(T/x)$ is an axiom and T is foreign to $\forall xA$.*

Proof: Suppose that $A(T/x)$ is an axiom, that T is foreign to $\forall xA$, and that the routine for **TQC*=** generates a closed tree for $\sim A(T/x)$. That is:

0. $\sim A(T/x)$
$\vdots \quad \vdots$
$\quad$ **X**

Then, since T is foreign to $\forall xA$:

0.	$\sim \forall xA$	✓
1.	$\exists x(x = T)$	from 0
2.	$\sim A(T/x)$	from 0
$\vdots$	$\vdots$	
	X	

by fact (b). So, if the routine for **TQC*=** generates a closed tree for $\sim A(T/x)$ and T is foreign to $\sim \forall xA$, it also generates a closed tree for $\sim \ xA$.

To establish the weak soundness of **TQC*=**, it is sufficient to consider only the quantificational rules of **TQC*=**, as the results for the other rules are in the literature. Fof each of the rules (4) and (5) of **TQC*=**, if the set of wffs on an open branch before applying the rule is syntactically consistent in **QC*=**, then the set of wffs on the branch after applying the rule is syntactically consistent in **QC*=**:

Rule 4 If $\mathscr{J}$ is syntactically consistent, then, if $\sim \forall xA$ belongs to $\mathscr{J}$, $\mathscr{J} \cup \{\exists x(x = T), \sim A(T/x)\}$ is syntactically consistent for any T foreign to $\mathscr{J}$.

Proof: Suppose that $\mathscr{J}$ is syntactically consistent and that $\sim \forall xA$ belongs to $\mathscr{J}$. Now suppose that $\mathscr{J} \cup \{\exists x(x = T), \sim A(T/x)\}$ is syntactically inconsistent, where T is foreign to $\mathscr{J}$. Then, substituting y for x in $\exists x(x = T)$, $\mathscr{J} \vdash \exists y(y = T) \supset A(T/x)$. Since T is foreign to $\mathscr{J}$, $\mathscr{J} \vdash \forall x(\exists y(y = x) \supset A)$; Hence, by Axiom 1 and modus ponens, $\mathscr{J} \vdash \forall x \exists y(y = x) \supset \forall xA$. Since $\forall x \exists y(y = x)$ is an axiom, $\mathscr{J} \vdash \forall xA$. But, since $\sim \forall xA$ belongs to $\mathscr{J}$, $\mathscr{J} \vdash \sim \forall xA$, hence $\mathscr{J}$ is syntactically inconsistent, contradicting the hypothesis. So $\mathscr{J} \cup \{\exists x(x = T), \sim A(T/x)\}$ is syntactically consistent.

Rule 5 If $\mathscr{J}$ is syntactically consistent, then, if $\{\forall xA, \exists y(y = T)\} \subseteq \mathscr{J}$, $\mathscr{J} \cup \{A(T/x)\}$ is syntactically consistent.

Proof: Suppose that $\mathscr{J}$ is consistent and that $\{\forall xA, \exists y(y = T)\} \subseteq \mathscr{J}$. Since $\exists y(y = T) \supset (\forall xA \supset A(T/x))$ is a theorem of **QC*=**,[12] $\mathscr{J} \vdash A(T/x)$. Now suppose $\mathscr{J} \cup \{A(T/x)\}$ is syntactically inconsistent. Then $\mathscr{J} \vdash \sim A(T/x)$. Hence $\mathscr{J}$ is syntactically inconsistent, contradicting the hypothesis. So $\mathscr{J} \cup \{A(T/x)\}$ is syntactically consistent.

So, since **TQC*=** is weakly complete and weakly sound, it is strongly complete and strongly sound with respect to $\mathscr{E}$. We now extend **TQC*=** to a system **TQC*=+** and show that **TQC*=+** is strongly complete and strongly sound. Let $\mathscr{P}$ be the set of all individual parameters of **QC*=**, let g be a one-one function $g:\mathcal{P} \rightarrow \mathcal{P}$ such that $\mathcal{P} - \{g(T):T \in \mathcal{P}\}$ has cardinality $\aleph_0$,[13] let $g(A) = A(g(T_1), g(T_2), \ldots, g(T_n)/T_1, T_2, \ldots, T_n)$, where $T_1, T_2, \ldots, T_n$ are all of the n distinct individual parameters occurring in A, let $\mathcal{J}$ be a set of wffs which need not be infinitely extendible, and let $g(\mathcal{J}) = \{g(A):A \in \mathcal{J}\}$. Clearly, $g(\mathcal{J})$ is infinitely extendible. Then, as Leblanc has shown,[14] $\mathcal{J}$ is syntactically inconsistent in **QC***= if and only if $g(\mathcal{J})$ is. By the strong completeness and soundness of **TQC***= with respect to $\mathcal{C}$, $\mathcal{J}$ is syntactically inconsistent in **QC***= if and only if the routine for **TQC***= declares $g(\mathcal{J})$ to be inconsistent.

Modify **TQC***= to obtain **TQC***=+ as follows. Take the individual parameter T of rule (4) and ground rule D to be any individual parameter belonging to $\mathcal{P} - \{g(T):T \in \mathcal{P}\}$ which is foreign to the branch. In step 2 of the routine, instead of listing $B_{-(i-1)}$ at the top of the tree, list $g(B_{-(i-1)})$ there. **TQC***=+ can be applied to any set of wffs of **QC***=, whether infinitely extendible or not. By the result of Leblanc's mentioned above, $\mathcal{J}$ is syntactically inconsistent in **QC***= if and only if the routine for **TQC***=+ declares $\mathcal{J}$ to be inconsistent. Since the computation of $g(B_{-(i-1)})$ is effective, the finite character of **TQC***= in the generation of closed trees is retained in **TQC***=+. So, **TQC***=+ is strongly complete and strongly sound with respect to the power set of the set of wffs of **QC***=, i.e., **TQC***=+ is strongly complete and strongly sound.

4 *The extended joint consistency theorem* We shall use **TQC***= (and not **TQC***=+) in proving the extended joint consistency theorem. Let $\mathcal{J}$ be an infinitely extendible set of wffs of **QC***=, and let $\mathrm{e}(\mathcal{J})$ be an enumeration of $\mathcal{J}$. If the routine for **TQC***= declares $\mathcal{J}$ to be inconsistent, understand *the length of a closed tree for $\mathcal{J}$ under the enumeration* $\mathrm{e}(\mathcal{J})$, $l(\mathrm{e}(\mathcal{J}))$, to be $n - (j + 1)$, where n is the largest m such that m is the number of nodes (wffs) on a branch and B_{-j} in the enumeration $\mathrm{e}(\mathcal{J})$ is the root node (wff at the top) of the tree.[15]

Let $\mathrm{C}(\mathcal{J})$ consist of all and only those individual, sentence, and predicate parameters that belong to $\mathcal{J}$. In case $\mathcal{J}$ is the singleton $\{A\}$, we write $\mathrm{C}(A)$ as short for $\mathrm{C}(\{A\})$. Call A a *quasi-wff* if $A = B(x_1, x_2, \ldots, x_n/T_1, T_2, \ldots, T_n)$, B is a wff, $n \geq 1$, and $\{T_1, T_2, \ldots, T_n\} \subseteq \mathrm{C}(B)$. Let f be a predicate or sentence parameter, and let A, B, C be wffs or quasi-wffs of **QC***=. *f occurs positively in A* and *f occurs negatively in A* are defined recursively as follows:

(1) $A = f(T_1, T_2, \ldots, T_j, x_1, x_2, \ldots, x_k)$, where $j \geq 0$ and $k \geq 0$. f occurs positively in A.
(2) $A = \sim B$ or $A = B \supset C$.
(a) If f occurs positively in B, then f occurs negatively in A.
(b) If f occurs negatively in B, then f occurs positively in A.
(3) $A = B \supset C$ or $A = \forall xC$.
(a) If f occurs positively in C, then f occurs positively in A.
(b) If f occurs negatively in C, then f occurs negatively in A.

Understand that if a wff A belongs to a set of wffs $\mathcal{J}$, then (a) if f occurs positively in A, *f occurs positively in $\mathcal{J}$*, and (b) if f occurs negatively in A, *f occurs negatively in* $\mathscr{J}$. Also understand that (a) if f occurs positively (negatively, positively and negatively) in $\mathscr{J}$ when f occurs positively (negatively, positively and negatively) in A, *f occurs the same way in* $\mathscr{J}$ *as in* A, and (b) if f occurs negatively (positively, positively and negatively) in $\mathscr{J}$ when f occurs positively (negatively, positively and negatively) in A, *f occurs the opposite way in* $\mathscr{J}$ *as in* A.

Extended Joint Consistency Theorem (Craig-Lyndon-Robinson) *The set of wffs* $\mathscr{J}_1 \cup \mathscr{J}_2$ *is inconsistent if and only if there is a wff F such that:*

(1) $\mathscr{J}_1 \cup \{F\}$ *and* $\mathscr{J}_2 \cup \{\sim F\}$ *are inconsistent,*
(2) $\mathrm{C}(F) \subseteq \mathrm{C}(\mathscr{J}_1) \cap \mathrm{C}(\mathscr{J}_2)$,

and

(3) *for every predicate and sentence parameter* $f \in \mathrm{C}(F)$, *f occurs in* $\mathscr{J}_1$ *the opposite way as in F, and f occurs in* $\mathscr{J}_2$ *the same way as in F.*

Proof[16]*:* The "if" part of the proof is immediate and we turn to the "only if" part.

Case 1: $\mathscr{J}_1 \cup \mathscr{J}_2$ is infinitely extendible. The proof proceeds by induction on $l(\mathrm{e}(\mathscr{J}_1 \cup \mathscr{J}_2))$. When it is said regarding an application of rule (1), rule (5), or rule (6) that the induction hypothesis is applied to enumerations of the form $\mathrm{e}(\mathscr{J}_1 \cup \{G\} \cup \mathscr{J}_2)$, where G is on the first line under B_0 (the node labeled B_0 is the father of the node labeled G) in the closed tree for $\mathscr{J}$ under the enumeration $\mathrm{e}(\mathscr{J}_1 \cup \mathscr{J}_2) = B_0, \ldots$, understand $\mathrm{e}(\mathscr{J}_1 \cup \{G\} \cup \mathscr{J}_2)$ to be B_0', $B_{-1}', \ldots = G, B_0, \ldots$. Hence, $l(\mathrm{e}(\mathscr{J}_1 \cup \mathscr{J}_2)) = l(\mathrm{e}(\mathscr{J}_1 \cup \{G\} \cup \mathscr{J}_2)) + 1$. When rule (2) is applied, the tree branches, and G is one of two wffs on the first line under B_0. In this case, $l(\mathrm{e}(\mathscr{J}_1 \cup \mathscr{J}_2)) \geqslant l(\mathrm{e}(\mathscr{J}_1 \cup \{G\} \cup \mathscr{J}_2) + 1$.[17] Similar remarks hold when it is said regarding an application of rule (3) or rule (4) that the hypothesis of induction is applied to enumerations of the form $\mathrm{e}(\mathscr{J}_1 \cup \{G,H\} \cup \mathscr{J}_2)$, where G is the first line and H is the second line under B_0 in the enumeration $\mathrm{e}(\mathscr{J}_1 \cup \mathscr{J}_2) = B_0, \ldots$; understand $\mathrm{e}(\mathscr{J}_1 \cup \{G,H\} \cup \mathscr{J}_2)$ to be $G, H, B_0, \ldots$, and $l(\mathrm{e}(\mathscr{J}_1 \cup \mathscr{J}_2)) = l(\mathrm{e}(\mathscr{J}_1 \cup \{G,H\} \cup \mathscr{J}_2)) + 2$.

Basis: $l(\mathrm{e}(\mathscr{J}_1 \cup \mathscr{J}_2)) = 0$ There is an atomic wff $G = h(T_1, T_2, \ldots, T_m)$ such that $m \geqslant 0$ and (1) $\sim G = \sim h(T_1) = \sim(T_1 = T_1) \in \mathscr{J}_1 \cup \mathscr{J}_2$ or (2) $\{G, \sim G\} = \{h(T_1, T_2, \ldots, T_m), \sim h(T_1, T_2, \ldots, T_m)\} \subseteq \mathscr{J}_1 \cup \mathscr{J}_2$. Suppose $\sim(T_1 = T_1) \in \mathscr{J}_1$ or $\{G, \sim G\} \subseteq \mathscr{J}_1$. Since $\mathscr{J}_1$ is inconsistent, $\mathscr{J}_1 \cup \{\forall x(x = x)\}$ and $\mathscr{J}_2 \cup \{\sim \forall x(x = x)\}$ are inconsistent. Since $\mathrm{C}(\forall x(x = x))$ is null, $\mathrm{C}(\forall x(x = x)) \subseteq \mathrm{C}(\mathscr{J}_1) \cap \mathrm{C}(\mathscr{J}_2)$, and every predicate and sentence parameter $f \in \mathrm{C}(\forall x(x = x))$ occurs in $\mathscr{J}_1$ the opposite way as in $\forall x(x = x)$ and the same way in $\mathscr{J}_2$ as in $\forall x(x = x)$. Take F to be $\forall x(x = x)$. The case when $\sim(T_1 = T_1) \in \mathscr{J}_2$ or $\{G, \sim G\} \subseteq \mathscr{J}_2$ is similar.

Suppose $G \in \mathscr{J}_1$ and $\sim G \in \mathscr{J}_2$. Since $\mathrm{C}(G) = \mathrm{C}(\sim G) = \{h, T_1, T_2, \ldots, T_m\} - \{=\}$, $\mathrm{C}(\sim G) \subseteq \mathrm{C}(\mathscr{J}_1) \cap \mathrm{C}(\mathscr{J}_2)$, and every $f \in \mathrm{C}(\sim G)$ occurs the opposite way in $\mathscr{J}_1$ as in $\sim G$ and the same way in $\mathscr{J}_2$ as in $\sim G$. Take F to be $\sim G$. The case when $\sim G \in \mathscr{J}_1$ and $G \in \mathscr{J}_2$ is similar.

Induction Step: $l(\mathrm{e}(\mathscr{J}_1 \cup \mathscr{J}_2)) \geqslant 1$.

The proof is by cases, depending upon what rule of **TQC*=** was applied in obtaining the first line under B_0.

Case 1.1: Rule (1) applied to $\sim\sim G$. The first line under B_0 is a wff of the form G which was obtained by applying rule (1) of **TQC*=** to a wff of the form $\sim\sim G \in \mathscr{J}_1 \cup \mathscr{J}_2$. Suppose $\sim\sim G \in \mathscr{J}_1$. Then, applying the hypothesis of induction to $\mathrm{e}(\mathscr{J}_1 \cup \{G\} \cup \mathscr{J}_2)$, there is a wff H such that $\mathscr{J}_1 \cup \{G\} \cup \{H\}$ and $\mathscr{J}_2 \cup \{\sim H\}$ are inconsistent, $\mathrm{C}(H) \subseteq \mathrm{C}(\mathscr{J}_1 \cup \{G\}) \cap \mathrm{C}(\mathscr{J}_2)$, and each predicate or sentence parameter $f \in \mathrm{C}(H)$ occurs the opposite way in $\mathscr{J}_1 \cup \{G\}$ as in H and occurs the same way in $\mathscr{J}_2$ as in H. Since $\sim\sim G \in \mathscr{J}_1$ and $\mathscr{J}_1 \cup \{G\} \cup \{H\}$ is inconsistent, $\mathscr{J}_1 \cup \{H\}$ is inconsistent. Since $\sim\sim G \in \mathscr{J}_1$, $\mathrm{C}(\mathscr{J}_1) = \mathrm{C}(\mathscr{J}_1 \cup \{G\})$ and $\mathrm{C}(H) \subseteq \mathrm{C}(\mathscr{J}_1) \cap \mathrm{C}(\mathscr{J}_2)$. Since $\sim\sim G \in \mathscr{J}_1$ and every $f \in \mathrm{C}(H)$ occurs the opposite way in $\mathscr{J}_1 \cup \{G\}$ as in H, every $f \in \mathrm{C}(H)$ occurs the opposite way in $\mathscr{J}_1$ as in H. Take F to be H. The case when $\sim\sim G \in \mathscr{J}_2$ is similar.

Case 1.2: Rule (2) applied to $G \supset H$. The first line under B_0 on the left branch is $\sim G$, and the first line under B_0 on the right branch is H, where $\sim G$ and H were obtained by applying rule (2) of **TQC*=** to $G \supset H \in \mathscr{J}_1 \cup \mathscr{J}_2$. Suppose $G \supset H \in \mathscr{J}_1$. Applying the hypothesis of induction to $\mathrm{e}(\mathscr{J}_1 \cup \{\sim G\} \cup \mathscr{J}_2)$, there is a wff I such that $\mathscr{J}_1 \cup \{\sim G\} \cup \{I\}$ and $\mathscr{J}_2 \cup \{\sim I\}$ are inconsistent, $\mathrm{C}(I) \subseteq \mathrm{C}(\mathscr{J}_1 \cup \{\sim G\}) \cap \mathrm{C}(\mathscr{J}_2)$, and every $f \in \mathrm{C}(I)$ occurs the opposite way in $\mathscr{J}_1 \cup \{\sim G\}$ as in I and the same way in $\mathscr{J}_2$ as in I. Since $G \supset H \in \mathscr{J}_1$, $\mathrm{C}(\mathscr{J}_1) = \mathrm{C}(\mathscr{J}_1 \cup \{\sim G\})$ and $\mathrm{C}(I) \subseteq \mathrm{C}(\mathscr{J}_1) \cap \mathrm{C}(\mathscr{J}_2)$. Since $G \supset H \in \mathscr{J}_1$ and every $f \in \mathrm{C}(I)$ occurs the opposite way in $\mathscr{J}_1 \cup \{\sim G\}$ as in I, every $f \in \mathrm{C}(I)$ occurs the opposite way in $\mathscr{J}_1$ as in I. Similarly, applying the induction hypothesis to $\mathrm{e}(\mathscr{J}_1 \cup \{H\} \cup \mathscr{J}_2)$, there is a wff J such that $\mathscr{J}_1 \cup \{H\} \cup \{J\}$ and $\mathscr{J}_2 \cup \{\sim J\}$ are inconsistent, $\mathrm{C}(J) \subseteq \mathrm{C}(\mathscr{J}_1 \cup \{H\}) \cap \mathrm{C}(\mathscr{J}_2)$, and every $f \in \mathrm{C}(J)$ occurs the opposite way in $\mathscr{J}_1 \cup \{H\}$ as in J and the same way in $\mathscr{J}_2$ as in J. So, by essentially the same arguments as above, $\mathrm{C}(J) \subseteq \mathrm{C}(\mathscr{J}_1) \cap \mathrm{C}(\mathscr{J}_2)$ and every $f \in \mathrm{C}(J)$ occurs the opposite way in $\mathscr{J}_1$ as in J. Since $G \supset H \in \mathscr{J}_1$ and $\mathscr{J}_1 \cup \mathscr{J}_2$, $\mathscr{J}_1 \cup \{\sim G\} \cup \{I\}$, $\mathscr{J}_1 \cup \{H\} \cup \{J\}$, $\mathscr{J}_2 \cup \{\sim I\}$, and $\mathscr{J}_2 \cup \{\sim J\}$ are inconsistent, $\mathscr{J}_1 \cup \{\sim(I \supset \sim J)\}$ and $\mathscr{J}_2 \cup \{\sim\sim(I \supset \sim J)\}$ are inconsistent. Since $\mathrm{C}(I) \subseteq \mathrm{C}(\mathscr{J}_1) \cap \mathrm{C}(\mathscr{J}_2)$ and $\mathrm{C}(J) \subseteq \mathrm{C}(\mathscr{J}_1) \cap \mathrm{C}(\mathscr{J}_2)$, $\mathrm{C}(\sim(I \supset \sim J)) \subseteq \mathrm{C}(\mathscr{J}_1) \cap \mathrm{C}(\mathscr{J}_2)$. Since every $f \in \mathrm{C}(I)$ occurs the opposite way in $\mathscr{J}_1$ as in I and the same way in $\mathscr{J}_2$ as in I and every $f \in \mathrm{C}(J)$ occurs the opposite way in $\mathscr{J}_1$ as in J and the same way in $\mathscr{J}_2$ as in J, every $f \in \mathrm{C}(\sim(I \supset \sim J))$ occurs the opposite way in $\mathscr{J}_1$ as in $\sim(I \supset \sim J)$ and the same way in $\mathscr{J}_2$ as in $\sim(I \supset \sim J)$. Take F to be $\sim(I \supset \sim J)$. The case when $G \supset H \in \mathscr{J}_2$ is similar.

Case 1.3: Rule (3) applied to $\sim(G \supset H)$. The first and second lines under B_0 are G and $\sim H$, respectively, which were obtained by applying rule (3) to $\sim(G \supset H) \in \mathscr{J}_1 \cup \mathscr{J}_2$. Suppose $\sim(G \supset H) \in \mathscr{J}_1$. Applying the hypothesis of induction to $\mathrm{e}(\mathscr{J}_1 \cup \{G, \sim H\} \cup \mathscr{J}_2)$, there is a wff I such that $\mathscr{J}_1 \cup \{G, \sim H\} \cup \{I\}$ and $\mathscr{J}_2 \cup \{\sim I\}$ are inconsistent, $\mathrm{C}(I) \subseteq \mathrm{C}(\mathscr{J}_1 \cup \{G, \sim H\}) \cap \mathrm{C}(\mathscr{J}_2)$, and every $f \in \mathrm{C}(I)$ occurs the opposite way in $\mathscr{J}_1 \cup \{G, \sim H\}$ as in I and the same way in $\mathscr{J}_2$ as in I. Since $\sim(G \supset H) \in \mathscr{J}_1$ and $\mathscr{J}_1 \cup \{G, \sim H\} \cup \{I\}$ is inconsistent, $\mathscr{J}_1 \cup \{I\}$ is inconsistent. Since $\sim(G \supset H) \in \mathscr{J}_1$, $\mathrm{C}(\mathscr{J}_1) = \mathrm{C}(\mathscr{J}_1 \cup \{G, \sim H\})$ and $\mathrm{C}(I) \subseteq \mathrm{C}(\mathscr{J}_1) \cap \mathrm{C}(\mathscr{J}_2)$. Since $\sim(G \supset H) \in \mathscr{J}_1$ and every $f \in \mathrm{C}(I)$ occurs the opposite way in $\mathscr{S}_1 \cup \{G, \sim H\}$ as in I, every $f \in \mathrm{C}(I)$ occurs the opposite way in $\mathscr{S}_1$ as in I. Take F to be I. The case when $\sim(G \supset H) \in \mathscr{S}_2$ is similar.

Case 1.4: Rule (4) applied to $\sim \forall xG$. The first and second lines under B_0 are $\exists x(x = T)$ and $\sim G(T/x)$, respectively, which were obtained by applying rule (4) to $\sim \forall xG \in \mathscr{S}_1 \cup \mathscr{S}_2$, where $T \notin \mathrm{C}(\mathscr{S}_1 \cup \mathscr{S}_2)$. Suppose $\sim \forall xG \in \mathscr{S}_1$. Applying the induction hypothesis to $\mathrm{e}(\mathscr{S}_1 \cup \{\exists x(x = T), \sim G(T/x)\} \cup \mathscr{S}_2)$, there is a wff H such that $\mathscr{S}_1 \cup \{\exists x(x = T), \sim G(T/x)\} \cup \{H\}$ and $\mathscr{S}_2 \cup \{\sim H\}$ are inconsistent, $\mathrm{C}(H) \subseteq \mathrm{C}(\mathscr{S}_1 \cup \{\exists x(x = T), \sim G(T/x)\}) \cap \mathrm{C}(\mathscr{S}_2)$, and every $f \in \mathrm{C}(H)$ occurs the opposite way in $\mathscr{S}_1 \cup \{\exists x(x = T), \sim G(T/x)\}$ as in H and the same way in $\mathscr{S}_2$ as in H. Since $T \notin \mathrm{C}(\mathscr{S}_1) \cup \mathrm{C}(\mathscr{S}_2)$, $T \notin \mathrm{C}(H)$. Since $\mathscr{S}_1 \cup \{\exists x(x = T), \sim G(T/x)\} \cup \{H\}$ is inconsistent, $\mathscr{S}_1 \cup \{\forall y \exists x(x = y), \sim \forall yG(y/x)\} \cup \{H\}$ is inconsistent for some individual variable y foreign to $\exists x(x = T)$ and $\sim G(T/x)$. Since $\sim \forall xG \in \mathscr{S}_1$ and $\forall y \exists x(x = y)$ is an axiom of **QC***=, $\mathscr{S}_1 \cup \{H\}$ is inconsistent. Since $T \notin \mathrm{C}(H)$ and $\mathrm{C}(\mathscr{S}_1) = \mathrm{C}(\mathscr{S}_1 \cup \{\exists x(x = T), \sim G(T/x)\}) - \{T\}$, $\mathrm{C}(H) \subseteq \mathrm{C}(\mathscr{S}_1) \cap \mathrm{C}(\mathscr{S}_2)$. Since $\sim \forall xG \in \mathscr{S}_1$ and every $f \in \mathrm{C}(H)$ occurs the opposite way in $\mathscr{S}_1 \cup \{\exists x(x = T), \sim G(T/x)\}$ as in H, every $f \in \mathrm{C}(H)$ occurs the opposite way in $\mathscr{S}_1$ as in H. Take F to be H. The case when $\sim \forall xG \in \mathscr{S}_2$ is similar.

Case 1.5: Rule (5) applied to $\forall xG$ and $\exists y(y = T)$. The first line under B_0 is $G(T/x)$ which was obtained by applying rule (5) to $\{\forall xG, \exists y(y = T)\} \subseteq \mathscr{S}_1 \cup \mathscr{S}_2$. There are three cases.

Case 1.5.1: $\forall xG \in \mathscr{S}_1$ and $\exists y(y = T) \in \mathscr{S}_1$. Applying the hypothesis of induction to $\mathrm{e}(\mathscr{S}_1 \cup \{G(T/x)\} \cup \mathscr{S}_2)$, there is a wff H such that $\mathscr{S}_1 \cup \{G(T/x)\} \cup \{H\}$ and $\mathscr{S}_2 \cup \{\sim H\}$ are inconsistent, $\mathrm{C}(H) \subseteq \mathrm{C}(\mathscr{S}_1 \cup \{G(T/x)\} \cap \mathrm{C}(\mathscr{S}_2)$, and every $f \in \mathrm{C}(H)$ occurs the opposite way in $\mathscr{S}_1 \cup \{G(T/x)\}$ as in H and the same way in $\mathscr{S}_2$ as in H. Since $\{\forall xG, \exists y(y = T)\} \subseteq \mathscr{S}_1$, $\forall xG \supset (\exists y(y = T) \supset G(T/x))$ is a theorem of **QC***=, and $\mathscr{S}_1 \cup \{G(T/x)\} \cup \{H\}$ is inconsistent, $\mathscr{S}_1 \cup \{H\}$ is inconsistent. Since $\exists y(y = T) \in \mathscr{S}_1$, $\mathrm{C}(\mathscr{S}_1) = \mathrm{C}(\mathscr{S}_1 \cup \{G(T/x)\})$ and $\mathrm{C}(H) \subseteq \mathrm{C}(\mathscr{S}_1) \cap \mathrm{C}(\mathscr{S}_2)$. Since $\forall xG \in \mathscr{S}_1$ and every $f \in \mathrm{C}(H)$ occurs the opposite way in $\mathscr{S}_1 \cup \{G(T/x)\}$ as in H, every $f \in \mathrm{C}(H)$ occurs the opposite way in $\mathscr{S}_1$ as in H. Take F to be H. The case when $\forall xG \in \mathscr{S}_2$ and $\exists y(y = T) \in \mathscr{S}_2$ is similar.

Case 1.5.2: $\forall xG \in \mathscr{S}_1$, $\exists y(y = T) \notin \mathscr{S}_1$, and $T \notin \mathrm{C}(\mathscr{S}_1)$. Applying the induction hypothesis as in Case 1.5.1, there is a wff H with the properties listed above. Since $T \notin \mathrm{C}(\mathscr{S}_1)$, $\forall xG \in \mathscr{S}_1$ and $\mathscr{S}_1 \cup \{G(T/x)\} \cup \{H\}$ is inconsistent, $\mathscr{S}_1 \cup \{\forall yG(y/x)\} \cup \{\exists yH(y/T)\}$ is inconsistent for some individual variable y foreign to $G(T/x)$ and H, and hence $\mathscr{S}_1 \cup \{\exists yH(y/T)\}$ is inconsistent. Since $\mathrm{C}(\exists yH(y/T)) = \mathrm{C}(H) - \{T\}$ and $\mathrm{C}(\mathscr{S}_1) = \mathrm{C}(\mathscr{S}_1 \cup \{G(T/x)\}) - \{T\}$, $\mathrm{C}(\exists yH(y/T)) \subseteq \mathrm{C}(\mathscr{S}_1) \cap \mathrm{C}(\mathscr{S}_2)$. Since $\forall xG \in \mathscr{S}_1$ and every $f \in \mathrm{C}(H)$ occurs the opposite way in $\mathscr{S}_1 \cup \{G(T/x)\}$ as in H, every $f \in \mathrm{C}(\exists yH(y/T))$ occurs the opposite way in $\mathscr{S}_1$ as in $\exists yH(y/T)$. Since $\exists y(y = T) \in \mathscr{S}_2$ and $\mathscr{S}_2 \cup \{\sim H\}$ is inconsistent, $\mathscr{S}_2 \cup \{\sim \exists y(H(y/T)\}$ is inconsistent. Since every $f \in \mathrm{C}(H)$ occurs the same way in $\mathscr{S}_2$ as in H, every $f \in \mathrm{C}(\exists yH(y/T))$ occurs the same way in $\mathscr{S}_2$ as in $\exists yH(y/T)$. Take F to be $\exists yH(y/T)$. The case when $\forall xG \in \mathscr{S}_2$, $\exists y(y = T) \notin \mathscr{S}_2$, and $T \notin \mathrm{C}(\mathscr{S}_2)$ is similar.

Case 1.5.3: $\forall xG \in \mathscr{S}_1$, $\exists y(y = T) \notin \mathscr{S}_1$, and $T \in \mathrm{C}(\mathscr{S}_1)$. Applying the induction hypothesis again as above, there is a wff H as in Case 1.5.1. Since $\forall xG \in \mathscr{I}_1$, $\forall xG \supset (\exists y(y = T) \supset G(T/x))$ is a theorem of **QC***=, and $\mathscr{I}_1 \cup \{G(T/x)\} \cup \{H\}$ is inconsistent, $\mathscr{I}_1 \cup \{\sim(\exists y(y = T) \supset \sim H)\}$ is inconsistent. Since $\forall xG \in \mathscr{I}_1$, $T \in C(\mathscr{I}_1)$, and $\exists y(y = T) \in \mathscr{I}_2$, $C(\sim(\exists y(y = T) \supset \sim H)) \subseteq C(\mathscr{I}_1) \cap C(\mathscr{I}_2)$. Since $\forall xG \in \mathscr{I}_1$ and every $f \in C(H)$ occurs the opposite way in $\mathscr{I}_1 \cup \{G(T/x)\}$ as in H, every $f \in C(\sim(\exists y(y = T) \supset \sim H))$ occurs the opposite way in $\mathscr{I}_1$ as in $\sim(\exists y(y = T) \supset \sim H)$. Since $\exists y(y = T) \in \mathscr{I}_2$ and $\mathscr{I}_2 \cup \{\sim H\}$ is inconsistent, $\mathscr{I}_2 \cup \{\sim\sim(\exists y(y = T) \supset \sim H)\}$ is inconsistent. Since every $f \in C(H)$ occurs the same way in $\mathscr{I}_2$ as in H, every $f \in C(\sim(\exists y(y = T) \supset \sim H))$ occurs the same way in $\mathscr{I}_2$ as in $\sim(\exists y(y = T) \supset \sim H)$. Take F to be $\sim(\exists y(y = T) \supset \sim H)$.[18] The case when $\forall xG \in \mathscr{I}_2$, $\exists y(y = T) \notin \mathscr{I}_2$, and $T \in C(\mathscr{I}_2)$ is similar.

Case 1.6: Rule (6) applied to G and $T_1 = T_2$. The first line under B_0 is $G(T_2/\!/T_1)$ which was obtained by applying rule (6) to $\{G, T_1 = T_2\} \subseteq \mathscr{I}_1 \cup \mathscr{I}_2$, where G is an atomic wff and T_2 occurs at least one time more in $G(T_2/\!/T_1)$ than in G. There are three cases.

Case 1.6.1: $G \in \mathscr{I}_1$ and $T_1 = T_2 \in \mathscr{I}_1$. Applying the hypothesis of induction to $e(\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\} \cup \mathscr{I}_2)$, there is a wff H such that $\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\} \cup \{H\}$ and $\mathscr{I}_2 \cup \{\sim H\}$ are inconsistent, $C(H) \subseteq C(\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\}) \cap C(\mathscr{I}_2)$, and every $f \in C(H)$ occurs the opposite way in $\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\}$ as in H and the same way in $\mathscr{I}_2$ as in H. Since $\{G, T_1 = T_2\} \subseteq \mathscr{I}_1$, $G \supset (T_1 = T_2 \supset G(T_2/\!/T_1))$ is a theorem of **QC***=, and $\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\} \cup \{H\}$ is inconsistent, $\mathscr{I}_1 \cup \{H\}$ is inconsistent. Since $T_1 = T_2 \in \mathscr{I}_1$, $C(\mathscr{I}_1) = C(\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\})$ and $C(H) \subseteq C(\mathscr{I}_1) \cap C(\mathscr{I}_2)$. Since $G \in \mathscr{I}_1$ and every $f \in C(H)$ occurs the opposite way in $\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\}$ as in H, every $f \in C(H)$ occurs the opposite way in $\mathscr{I}_1$ as in H. Take F to be H. The case when $G \in \mathscr{I}_2$ and $T_1 = T_2 \in \mathscr{I}_2$ is similar.

Case 1.6.2: $G \in \mathscr{I}_1$, $T_1 = T_2 \notin \mathscr{I}_1$, and $T_2 \notin C(\mathscr{I}_1)$. Applying the induction hypothesis as in Case 1.6.1, there is a wff H with the properties listed above. Since $T_2 \notin C(\mathscr{I}_1)$, $G \in \mathscr{I}_1$, and $\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\} \cup \{H\}$ is inconsistent, $\mathscr{I}_1 \cup \{G\} \cup \{H(T_1/T_2)\} = \mathscr{I}_1 \cup \{H(T_1/T_2)\}$ is inconsistent. Since $C(H(T_1/T_2)) = C(H) - \{T_2\}$ and $C(\mathscr{I}_1) = C(\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\} - \{T_2\}$, $C(H(T_1/T_2)) \subseteq C(\mathscr{I}_1) \cap C(\mathscr{I}_2)$. Since $G \in \mathscr{I}_1$ and every $f \in C(H)$ occurs the opposite way in $\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\}$ as in H, every $f \in C(H(T_1/T_2))$ occurs the opposite way in $\mathscr{I}_1$ as in $H(T_1/T_2)$. Since $T_1 = T_2 \in \mathscr{I}_2$ and $\mathscr{I}_2 \cup \{\sim H\}$ is inconsistent, $\mathscr{I}_2 \cup \{\sim H(T_1/T_2)\}$ is inconsistent. Since every $f \in C(H)$ occurs the same way in $\mathscr{I}_2$ as in H, every $f \in C(H(T_1/T_2))$ occurs the same way in $\mathscr{I}_2$ as in $H(T_1/T_2)$. Take F to be $H(T_1/T_2)$. The case when $G \in \mathscr{I}_2$, $T_1 = T_2 \notin \mathscr{I}_2$, and $T_2 \notin C(\mathscr{I}_2)$ is similar.

Case 1.6.3: $G \in \mathscr{I}_1$, $T_1 = T_2 \notin \mathscr{I}_1$, and $T_2 \in C(\mathscr{I}_1)$. Applying the induction hypothesis as above, there is a wff H as in Case 1.6.1. Since $G \in \mathscr{I}_1$, $G \supset (T_1 = T_2 \supset G(T_2/\!/T_1))$ is a theorem of **QC***=, and $\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\} \cup \{H\}$ is inconsistent, $\mathscr{I}_1 \cup \{\sim(T_1 = T_2 \supset \sim H)\}$ is inconsistent. Since $G \in \mathscr{I}_1$, $T_2 \in C(\mathscr{I}_1)$, and $T_1 = T_2 \in \mathscr{I}_2$, $C(\sim(T_1 = T_2 \supset \sim H)) \subseteq C(\mathscr{I}_1) \cap C(\mathscr{I}_2)$. Since $G \in \mathscr{I}_1$ and every $f \in C(H)$ occurs the same way in $\mathscr{I}_1 \cup \{G(T_2/\!/T_1)\}$ as in H, every $f \in C(\sim(T_1 = T_2 \supset \sim H))$ occurs the opposite way in $\mathscr{I}_1$ as in $\sim(T_1 = T_2 \supset \sim H)$. Since $T_1 = T_2 \in \mathscr{I}_2$ and $\mathscr{I}_2 \cup \{\sim H\}$ is inconsistent, $\mathscr{I}_2 \cup \{\sim\sim(T_1 = T_2 \supset \sim H)\}$ is inconsistent. Since every $f \in C(H)$ occurs the same way in $\mathscr{J}_2$ as in H, every $f \in C(\sim(T_1 = T_2 \supset \sim H))$ occurs the same way in $\mathscr{J}_2$ as in $\sim(T_1 = T_2 \supset \sim H)$. Take F to be $\sim(T_1 = T_2 \supset \sim H)$. The case when $G \in \mathscr{J}_2$, $T_1 = T_2 \notin \mathscr{J}_2$, and $T_2 \in C(\mathscr{J}_2)$ is similar.

Case 2: $\mathscr{J}_1 \cup \mathscr{J}_2$ is not infinitely extendible. Suppose $\mathscr{J}_1 \cup \mathscr{J}_2$ is inconsistent in **QC***=, but $\mathscr{J}_1 \cup \mathscr{J}_2$ is not infinitely extendible. Let g and $\mathscr{P}$ be as described in the preceding section. Since $\mathscr{J}_1 \cup \mathscr{J}_2$ is inconsistent in **QC***=, as noted in the preceding section, $g(\mathscr{J}_1 \cup \mathscr{J}_2) = g(\mathscr{J}_1) \cup g(\mathscr{J}_2)$ is also. Since $g(\mathscr{J}_1) \cup g(\mathscr{J}_2)$ is infinitely extendible, by Case 1 there is a wff G such that $g(\mathscr{J}_1) \cup \{G\}$ and $g(\mathscr{J}_2) \cup \{\sim G\}$ are inconsistent, $C(G) \subseteq C(g(\mathscr{J}_1)) \cap C(g(\mathscr{J}_2))$, and, for every predicate or sentence parameter $f \in C(G)$, f occurs the opposite way in $g(\mathscr{J}_1)$ as in G and f occurs the same way in $g(\mathscr{J}_2)$ as in G.

Let $C(G) \cap \mathscr{P} = \{T_1, T_2, \ldots, T_n\}$, where $n \geq 0$. Since g is one-one the wff $g^{-1}(G) = G(g^{-1}(T_1), g^{-1}(T_2), \ldots, g^{-1}(T_n)/T_1, T_2, \ldots, T_n)$ is such that $\mathscr{J}_1 \cup \{g^{-1}(G)\}$ and $\mathscr{J}_2 \cup \{\sim g^{-1}(G)\}$ are inconsistent, $C(g^{-1}(G)) \subseteq C(\mathscr{J}_1) \cap C(\mathscr{J}_2)$, and every $f \in C(g^{-1}(G))$ occurs the opposite way in $\mathscr{J}_1$ as in $g^{-1}(G)$ and the same way in $\mathscr{J}_2$ as in $g^{-1}(G)$. Take F to be $g^{-1}(G)$.

NOTES

1. See [1], pp. 13-14.
2. See Case 1.5.3 in the proof of the extended joint consistency theorem, section **4** of this paper, and note 18, below.
3. See [4], pp. 47-68 and 174-197. I am indebted to Hugues Leblanc for many of the basic concepts and techniques employed in this paper, and I have benefitted greatly from his comments on an earlier version of this paper.
4. For a description of **QC***=, see [3].
5. See [4], pp. 55 and 180.
6. Rule (4) is not applied to wffs of the form $\exists x(x = T_1)$, so that $\exists x(x = T)$ and $\sim\sim(T = T_1)$ are not added to the branch. If $\exists x(x = T)$ were to be added, then rule (4) would be, in turn, applicable to it, and the routine would never leave step 3.
7. See [4], p. 188.
8. If a rule were to be added to the rules of **TQC***= specifying that, if a wff of the form $\exists x(x = T)$ occurs on a branch, then a wff of the form $\sim\sim(T = T)$ is to be added to the branch, the routine for **TQC***= would generate free logic model sets satisfying a set of conditions similar to one set presented in [2]. See also the somewhat different conditions for **QC***= model sets presented in [3].
9. See [4], p. 189.
10. Strictly speaking, the routine for **TQC***= presented below is not an algorithm. First, the routine is not "definite" in the sense that it does not specify, e.g., which rule to apply first when two or more truth-functional rules are applicable. However, by appropriately expanding the specification of the steps in the routine, the routine can be made completely definite. This can be done, for example, by requiring in step 3 and step 4 that branches be processed from left to right, that each branch be processed from the bottom up, and that in step 3

rules (1)-(4) have precedence over rule (6). The routine presented by Leblanc and Wisdom in [4] has been given definite form in a program which I have written in the programming language SNOBOL and run on a CDC 6400 computer. A similar program could be written for the routine for **TQC***=. The routines as presented in [4] and in this paper lack definiteness because they were intended for application by hand in a pedagogical role. The student applying one of these routines employs insight in determining the order in which to apply some of the rules, attempting to close a branch as soon as possible. A second characteristic of algorithms which the routine for **TQC***= lacks is "finiteness". That is, the routine for **TQC***= does not terminate in a finite number of steps when applied to an infinite set of wffs which is consistent, to a set of wffs which has only infinite models, etc. Algorithms and routines which have all of the characteristics of algorithms except for the property of being finite are called "computational methods". So, the routine for **TQC***=, appropriately expanded to become completely definite, is a computational method.

11. Proof that the derived ground rule adds nothing to the deductive power of **TQC***= is similar to those proofs to be found in the literature.

12. See [3].

13. Where T_i is the i-th ($i \geqslant 0$) individual parameter in an enumeration $\mathrm{e}(\mathcal{P})$ of $\mathcal{P}$, $g(T_i) = T_{2i}$ is such a function. See [4], p. 319.

14. See [3].

15. In the literature, n is sometimes called the height of the tree. B_{-j} is the wff at the top of the closed tree and was the last wff in $\mathcal{A}$ activated and added to the tree by the routine for **TQC***= before the routine halted in step 6 during iteration $k \geqslant j + 1$. More intuitively, the length of a closed tree for $\mathcal{A}$ under the enumeration $\mathrm{e}(\mathcal{A})$ is the largest integer used in numbering the lines of the proof. For example, in the closed tree corresponding to the first quantificational axiom, $l(\sim(\forall x(A \supset B) \supset (\forall xA \supset \forall xB))) = 9$ when $A(T/x)$ and $B(T/x)$ are distinct atomic wffs which are not identities. A wff B_{-i} ($i \leqslant j$) in the enumeration $\mathrm{e}(\mathcal{A})$ is conveniently given the line number $-i$ when it is activated and placed at the top of the tree during the $i + 1$-th iteration in step 2 of the routine for **TQC***=. Note that, in each of the proofs of the quantificational axioms of **QC***= given in the preceding section, $j = 0$.

16. The basic technique of the proof is due to Hintikka [1]. As Hintikka points out, given a closed tree for $\mathcal{A}_1 \cup \mathcal{A}_2$, the proof provides an effective means of constructing F.

17. G is the first line under B_0 in the closed tree for $\mathcal{A}_1 \cup \mathcal{A}_2$ under the enumeration $\mathrm{e}(\mathcal{A}_1 \cup \mathcal{A}_2)$. By putting G as the first wff in the enumeration $\mathrm{e}(\mathcal{A}_1 \cup \{G\} \cup \mathcal{A}_2)$, the routine for **TQC***= will activate G first and hence not add it on a line underneath. We presume here that the routine for **TQC***= is applied to $\mathrm{e}(\mathcal{A}_1 \cup \{G\} \cup \mathcal{A}_2)$ in the same way it is to $\mathrm{e}(\mathcal{A}_1 \cup \mathcal{A}_2)$. That is, we take the routine for **TQC***= to be "definite" in the sense discussed in note 10—at least for the generation of the two trees under consideration.

18. Note the crucial presence of = in F in one case of the basis and in this case. The following clause which could be added to the statement of the theorem in standard first order logic does *not* hold in free logic: if neither $\mathcal{A}_1$ nor $\mathcal{A}_2$ is inconsistent and = occurs in neither $\mathcal{A}_1$ nor $\mathcal{A}_2$, = does not occur in F. Consider $\mathcal{A}_1 = \{\forall x \sim (A \supset A)\}$ and $\mathcal{A}_2 = \{\exists xB\}$.

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