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CLASSES OF UNIVERSAL DECISION ELEMENTS USING NEGATIVE SUBSTITUTIONS

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1 *Introduction* In 1953 Sobociński [6] showed that there exists a function of four arguments in two-valued logic which may define any binary function by substitution of the variables p, q or constants 0, 1 into its arguments, this function only being used once in any definition. Such a function is said to generate all the binary functions and is termed a universal decision element. Sobociński also proved that no three-place function can correspond to such a universal decision element. In [3] and [4] the present author considered three-place functions and gave a complete classification of them according to which subsets of the binary functions they generate. In the present paper* we are also concerned with three place functions, but in addition to allowing the substitution of the variables and the constants we also admit the substitution of the negated variables. Under these conditions it is possible for a three-place function to generate all the binary functions, and for the remainder of this paper we term such a function a universal decision element.

For our purpose it is sufficient to divide the three-place functions into 14 distinct classes, the behaviour of the elements of a class being essentially similar. These classes have been discussed in more general terms by several authors (see Harrison [2], p. 148 et. seq. and Ninomaya [5]). We investigate which of the classes consist of functions which are universal decision elements in both the general case and in a restricted case. In section **4** we make the restriction that each variable may be substituted only once (either in true or negated form). It transpires that this implies that exactly one class of functions consists of universal decision elements. The general case, of not restricting the substitutions, is considered in section **5** and two more classes of functions correspond to universal decision elements. In both cases we describe exactly which binary

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functions may be generated by the three-place functions which do not correspond to universal decision elements.

For a binary function Uxy, we define its value sequence to be $\langle k\ell mn\rangle$ where k,ℓ,m,n are specified by the table shown ($k,\ell,m,n \in \{0,1\}$; 0,1 being used for both the logical constants and the truth values they assume.)

x	y	Uxy
0	0	k
0	1	ℓ
1	0	m
1	1	n

Similarly the three-place function Δxyz specified by the table has value sequence $\langle abcdefgh\rangle$. For such a function we shall identify it by its description number, which is the decimal equivalent of the binary (abcdefgh) where a is the most significant bit. Thus 107 is the description number of the function with value sequence $\langle 01101011\rangle$. $\Delta(107)$ will also be used to denote this function.

x	y	z	Δxyz
0	0	0	a
0	0	1	b
0	1	0	c
0	1	1	d
1	0	0	e
1	0	1	f
1	1	0	g
1	1	1	h

Łukasiewicz's notation for the binary functions will be used, as shown in the table.

	notation	value sequence	function
0	0	$\langle 0000\rangle$	
1	K	$\langle 0001\rangle$	conjunction
2	L	$\langle 0010\rangle$	nonimplication
3	I	$\langle 0011\rangle$	
4	M	$\langle 0100\rangle$	nonimplication
5	H	$\langle 0101\rangle$	
6	J	$\langle 0110\rangle$	exclusive or (nonequivalence)
7	A	$\langle 0111\rangle$	disjunction
8	X	$\langle 1000\rangle$	joint denial
9	E	$\langle 1001\rangle$	equivalence
10	G	$\langle 1010\rangle$	
11	B	$\langle 1011\rangle$	implication
12	F	$\langle 1100\rangle$	
13	C	$\langle 1101\rangle$	implication
14	D	$\langle 1110\rangle$	incompatibility
15	V	$\langle 1111\rangle$	

Np will be used for the negation of a variable p.

If in the table representing Δxyz there are i entries of 0 and j entries of 1 then Δxyz is said to be of type $(i|j)$ if $i \geqslant j$; otherwise it is of type $(j|i)$.

In the definitions of some of the following functions Church's conditioned disjunction function [1] is used. This is defined by

$$[x,y,z] = AKxyKNyz\,.$$

When considering the substitutions later we will write x/Nq to indicate that Nq is substituted for x. It will be convenient when considering substitutions for x, y, and z in Δxyz to write the substitution set $(u;v;w)$ to mean x/u, y/v, and z/w in Δxyz.

2 *The value sequences generated* As substitutions into the variables of Δxyz we allow any of the variables p, q, the negated variables Np, Nq and the constants 0, 1. Two obvious restrictions can be applied:

(a) To generate functions which depend essentially on two variables the substitution set must contain both of p (either as p or Np) and q (either as q or Nq);
(b) The first substitution of p or Np into Δxyz is in a place preceding the first substitution of q or Nq.

Since we may use negated variables in the substitution set it is clear that any function which can generate $\langle k\ell mn\rangle$ can also generate $\langle \ell knm\rangle$, $\langle mnk\ell\rangle$, and $\langle nm\ell k\rangle$. These three functions follow by replacing q/Nq, p/Np, and both q/Nq, p/Np respectively in the substitution set for $\langle k\ell mn\rangle$. For example, if $\Delta pqNp$ gives $\langle k\ell mn\rangle$, then $\Delta pNqNp$ gives $\langle \ell knm\rangle$, $\Delta Npqp$ gives $\langle mnk\ell\rangle$, and $\Delta NpNqp$ gives $\langle nm\ell k\rangle$. Considering the 16 binary functions which are to be generated we may arrange them into classes, such that the generation of one element of the class implies the whole class may be generated using the above replacements.

$$\begin{aligned}\mathbf{Z}_1 &= \{K,L,M,X\},\\ \mathbf{Z}_2 &= \{A,B,C,D\},\\ \mathbf{Z}_3 &= \{J,E\},\\ \mathbf{Z}_4 &= \{I,F,H,G\},\\ \mathbf{Z}_5 &= \{0,V\}.\end{aligned}$$

Strictly $\mathbf{Z}_4$ should be written as two classes $\{I,F\}$ and $\{H,G\}$, but if we can generate an element of one class we can generate the corresponding element of the other class by interchanging the variables p, q. The two constant functions in $\mathbf{Z}_5$ can always be generated by any function that depends essentially on all three variables, and we omit this class from all the following discussion.

We will consider two groups of substitution sets. In the first at least one constant must be contained in each substitution set, this being the restriction applied in section **4**. The second group consists of the further substitution sets which do not include any constants. It transpires that there are effectively six distinct substitution sets, since two substitution sets which differ only in a negated variable will both generate the same $\mathbf{Z}_i$ sets.

	substitution set	resulting value sequence
group 1	$(0;p;q)$	⟨abcd⟩
	$(1;p;q)$	⟨efgh⟩
	$(p;0;q)$	⟨abef⟩
	$(p;1;q)$	⟨cdgh⟩
	$(p;q;0)$	⟨aceg⟩
	$(p;q;1)$	⟨bdfh⟩
group 2	$(p;p;q)$	⟨abgh⟩
	$(p;q;p)$	⟨acfh⟩
	$(p;q;q)$	⟨adeh⟩
	$(p;q;Np)$	⟨bdeg⟩
	$(p;q;Nq)$	⟨bcfg⟩
	$(p;Np;q)$	⟨cdef⟩

3 *The equivalence classes of three-place functions* Since we are allowing negated input variables as substitutions into Δxyz two three-place functions may be regarded as equivalent (in that they generate essentially the same set of binary functions) if and only if they differ by some negation and/or permutation of the variables. In addition to each function $\Delta(i)$, $i \leqslant 127$, there is its negation, $\Delta(j)$ $(j = 255 - i)$, which generates an essentially similar set of binary functions. ($\mathbf{Z}_1$ and $\mathbf{Z}_2$ are interchanged.) These classes have been investigated in detail by several authors and Harrison [2] gives a list of representatives of all such classes for four-place functions. (See also Ninomaya [5].) In the three-place case there are 14 classes which we list below, divided according to type. For each class we list only the description numbers of the functions in the class. We also omit all the functions $\Delta(j)$, $j \geqslant 128$ since their class is easily found by considering $\Delta(255 - j)$. Since we are concerned with generating the binary functions the classes of interest, $\mathbf{Q}_i$, are those which consist of functions that depend essentially on all three variables. The other classes are denoted by $\mathbf{R}_i$. For each class a representative function from the class is given (these are not Harrison's representatives but are more convenient for our purpose). In the later sections we shall only consider the functions generated by the $\mathbf{Q}_i$ classes.

(a) *Functions of type* $(4|4)$. There are 70 functions of this type divided into six classes:

$\mathbf{Q}_1 = \{30,45,54,57,75,86,89,99,101,106,108,120\}$, $[y,x,Eyz]$;
$\mathbf{Q}_2 = \{27,29,39,46,53,58,71,78,83,92,114,116\}$, $[y,z,x]$;
$\mathbf{Q}_3 = \{23,43,77,113\}$, $AAKxyKyzKzx$;
$\mathbf{Q}_4 = \{105\}$, $JJxyz$;
$\mathbf{R}_1 = \{60,90,102\}$, Jxy;
$\mathbf{R}_2 = \{15,51,85\}$, x.

(b) *Functions of type* $(5|3)$. There are 112 functions of this type divided into three classes, all the functions depending essentially on all three variables:

$\mathbf{Q}_5 = \{25,26,28,37,38,44,52,56,61,62,67,70,74,82,88,91,94,98, 100,103,110,118,122,124\}$, $[z,x,Lyz]$;
$\mathbf{Q}_6 = \{7,11,13,14,19,21,31,35,42,47,49,50,55,59,69,76,79,81,84, 87,93,112,115,117\}$, $KyAxz$;
$\mathbf{Q}_7 = \{22,41,73,97,104,107,109,121\}$, $[Kyz,x,Jyz]$.

(c) *Functions of type* (6 | 2). There are 56 functions of this type divided into three classes:

$\mathbf{Q}_8 = \{6,9,18,20,33,40,65,72,96,111,123,125\}$, $KxJyz$;
$\mathbf{Q}_9 = \{24,36,66,126\}$, $[Xyz,x,Kyz]$;
$\mathbf{R}_3 = \{3,5,10,12,17,34,48,63,68,80,95,119\}$, Kxy.

(d) *Functions of type* (7 | 1). There are 16 functions of this type, all in $\mathbf{Q}_{10}$.

$$\mathbf{Q}_{10} = \{1,2,4,8,16,32,64,127\},\ Kxyz.$$

(e) *Functions of type* (8 | 0). There are only the two constant functions of this type, viz:

$$\mathbf{R}_4 = \{0\}.$$

4 *Universal decision elements in the restricted case* In this section, it is shown that exactly one class of functions generate all the binary functions, under the restriction that one element of the substitution set must be a constant. There are six distinct value sequences in group 1 and these must match the four classes $\mathbf{Z}_1$ to $\mathbf{Z}_4$.

Proposition 1 *If Δxyz generates all the binary functions it must be of type* (4 | 4).

Initially Δxyz must be either (4 | 4) or (5 | 3) in order to generate ⟨0111⟩ and ⟨1000⟩. Suppose it were (5 | 3) and, without loss of generality suppose the value sequence of Δxyz contains 5 0's. Consider the six value sequence of group 1: ⟨abcd⟩, ⟨efgh⟩, ⟨abef⟩, ⟨cdgh⟩, ⟨aceg⟩, ⟨bdfh⟩. Since Δxyz must generate $\mathbf{Z}_3$ one of these six value sequences must either equal ⟨0110⟩ or ⟨1001⟩. It follows that ⟨0111⟩ cannot be generated since no value sequence contains a repetition of either the two middle entries or the first and last entries from a distinct value sequence. This would be required since we only have one further entry of 1 available. It follows that Δxyz is (4 | 4).

A detailed investigation of the four classes of this type reveals that only $\mathbf{Q}_1$ will generate all the binary functions, the complete results being as below:

class	binary functions generated
$\mathbf{Q}_1$	$\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$
$\mathbf{Q}_2$	$\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_4$
$\mathbf{Q}_3$	$\mathbf{Z}_1$ and $\mathbf{Z}_2$
$\mathbf{Q}_4$	$\mathbf{Z}_3$
$\mathbf{Q}_5$	$\mathbf{Z}_1$ or $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$
$\mathbf{Q}_6$	$\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_4$

class	binary functions generated
$\mathbf{Q}_7$	$\mathbf{Z}_1$ or $\mathbf{Z}_2$, and $\mathbf{Z}_3$
$\mathbf{Q}_8$	$\mathbf{Z}_1$ or $\mathbf{Z}_2$, and $\mathbf{Z}_3$
$\mathbf{Q}_9$	$\mathbf{Z}_1$ or $\mathbf{Z}_2$, and $\mathbf{Z}_4$
$\mathbf{Q}_{10}$	$\mathbf{Z}_1$ or $\mathbf{Z}_2$

For $\Delta xyz = [y,x,Eyz]$, our representative function from $\mathbf{Q}_1$, the following substitution sets generate the 16 binary functions:

value sequence generated	substitution set x	y	z
⟨0000⟩	0	0	1
⟨0001⟩	Np	0	Nq
⟨0010⟩	Np	0	q
⟨0011⟩	1	p	q
⟨0100⟩	p	0	Nq
⟨0101⟩	p	q	1
⟨0110⟩	0	p	Nq
⟨0111⟩	p	1	q
⟨1000⟩	p	0	q
⟨1001⟩	0	p	q
⟨1010⟩	p	Nq	1
⟨1011⟩	p	1	Nq
⟨1100⟩	1	Np	q
⟨1101⟩	Np	1	q
⟨1110⟩	Np	1	Nq
⟨1111⟩	1	1	1

5 *The general case* We now have available the six value sequences of group 2 in addition to those of group 1. As might be expected more classes can now generate all the binary functions, $\mathbf{Q}_2$ and $\mathbf{Q}_5$ now being adequate. It is no longer true that such functions have to be $(4|4)$ in order to generate all the binary functions. A detailed investigation of the functions gives the results in the following table:

class	binary functions generated
$\mathbf{Q}_1$	$\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$
$\mathbf{Q}_2$	$\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$
$\mathbf{Q}_3$	$\mathbf{Z}_1$, $\mathbf{Z}_2$ and $\mathbf{Z}_4$
$\mathbf{Q}_4$	$\mathbf{Z}_3$ and $\mathbf{Z}_4$
$\mathbf{Q}_5$	$\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$
$\mathbf{Q}_6$	$\mathbf{Z}_1$, $\mathbf{Z}_2$ and $\mathbf{Z}_4$
$\mathbf{Q}_7$	$\mathbf{Z}_3$ and $\mathbf{Z}_4$
$\mathbf{Q}_8$	$\mathbf{Z}_1$ or $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$
$\mathbf{Q}_9$	$\mathbf{Z}_1$ or $\mathbf{Z}_2$ and $\mathbf{Z}_4$
$\mathbf{Q}_{10}$	$\mathbf{Z}_1$ or $\mathbf{Z}_2$

For a function in $\mathbf{Q}_1$ the example substitutions given above will still suffice while for the representatives of $\mathbf{Q}_2$ and $\mathbf{Q}_5$, $\Delta xyz = [y,z,x]$ and $\Delta xyz = [z,x,Lyz]$, the following substitution sets generate the 16 binary functions. For substitution sets from group 1 and group 2 there is a much wider choice of possible substitution sets for the various binary functions.

value sequence generated	$\Delta xyz = [y,z,x]$ substitution set			$\Delta xyz = [z,x,Lyz]$ substitution set		
	x	y	z	x	y	z
⟨0000⟩	0	0	0	0	0	0
⟨0001⟩	0	p	q	p	0	q
⟨0010⟩	p	0	q	0	p	q
⟨0011⟩	p	q	0	p	q	1
⟨0100⟩	0	Np	q	p	q	0
⟨0101⟩	p	q	1	1	p	q
⟨0110⟩	p	Np	q	p	1	Nq
⟨0111⟩	p	1	q	p	q	p
⟨1000⟩	Np	0	q	p	Nq	0
⟨1001⟩	p	Np	Nq	p	1	q
⟨1010⟩	p	Nq	1	1	p	Nq
⟨1011⟩	1	p	q	p	Nq	p
⟨1100⟩	Np	q	0	Np	q	1
⟨1101⟩	Np	1	q	Np	q	Np
⟨1110⟩	1	Np	q	Np	Nq	Np
⟨1111⟩	1	1	1	1	1	1

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