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Skolem Fragments

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W. V. Quine splits the fundamental question of ontology—What is there?—into two questions: What does a theory say that there is? and What theories ought we adopt? Of these, only the former seems amenable to philosophical treatment. Quine thus attempts to formulate an adequate criterion of ontological commitment. Syntactically, the initially existentially quantified sentences of a theory appear to constitute the locus of its ontological commitments (cf. [5], [2], [3], and [1]). Semantically, however, Quine offers at least three criteria of commitment: a theory is committed to (1) the objects in the domain of its intended model (cf. [4]); (2) the objects in the domain of its intended model that cannot be eliminated by means of proxy functions (cf. [6]); or (3) the objects in the domain of every model of it (or to objects of kinds such that some objects of those kinds are in each of its models) (cf. [7]). In this paper I shall show that Quine's syntactic criterion corresponds to and, indeed, follows from the third semantic criterion.

Any philosopher using a syntactic criterion of ontological commitment such as Quine's that determines commitments according to sentences of the form

$$\exists x_1 \ldots \exists x_n B$$

must hold that the commitments of a theory are exactly those of its fragment consisting of initially existentially quantified sentences. I shall call this portion of a theory its *Skolem fragment.*

What is the semantic relation between a theory and its Skolem fragment? In standard logic, they are equivalent. If our logic allows vacuous quantification, then any formula A is equivalent to $\exists xA$, where 'x' does not occur free in A. Furthermore, any exclusive logic counts $\exists x(x = x)$ valid, thus ruling out a null domain. For any formula A in a theory T, therefore, the equivalent

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$\exists x(x = x \;\&\; A)$, where '$x$' is not free in A, will inhabit T's Skolem fragment. The presence of identity is inessential to this result. Where A is in T and $B(x)$, in which 'x' occurs nonvacuously, has a valid universal closure, $\exists x(B(x) \;\&\; A)$ (again, with 'x' not free in A) also occupies T's Skolem fragment.

I have argued in [1] that we can characterize ontological commitment only by taking empty domains into account. A sentence makes an ontological commitment only if it forces an object into the domain; anything making such a commitment, therefore, must come out false on the null domain. Provided that we employ an inclusive logic and regard vacuous quantification as ill-formed, the Skolem fragment of a theory will not in general be equivalent to the theory itself. But I shall show that, if we measure commitment semantically by criterion (3) above, any theory and its Skolem fragment have identical ontological commitments. A theory's Skolem fragment forces into the domain any object forced into the domain by the theory as a whole.

1 Preliminaries I shall begin by defining a *first-order language L* as having a countable set of individual variables, represented metalinguistically by 'x', with or without subscripts; a countable set of individual constants, represented metalinguistically by 'a', with or without subscripts; and a finite or countable set of n-ary predicate constants for each $n > 0$. Any individual constant or variable is a *term.* An n-ary predicate followed by n terms is a formula; if A and B are formulas, so are $\sim A$ and $(A \;\&\; B)$. If A is a formula with free variable x, then $\exists xA$ and $\forall xA$ for formulas too. Every formula is constructible by a finite number of applications of these rules. If L contains the additional condition that, if t and t' are terms, then $t = t'$ is a formula, L is a *first-order language with identity.* Any formula of L containing no free variable is a *sentence.*

A *model* $M = \langle D, \Diamond\rangle$ of L is an ordered pair consisting of a possibly empty set D (its *domain*) and a function $\Diamond$ such that, if D is nonempty: (a) for any term T of L, $\Diamond(t) \in D$; and (b) for any n-ary predicate constant F of L, $\Diamond(F) \subseteq D^n$; if D is empty, $\Diamond$ is simply the null set. There is only one model, therefore, having a null domain; I shall call it M_0. In general I shall refer to the cardinality of the domain of $M = \langle D, \Diamond\rangle$ by writing '$|D|$'. A model $M = \langle D, \Diamond\rangle$ is a *t-variant* of $M' = \langle D', \Diamond'\rangle$ if and only if: (a) $D = D'$; (b) for all predicate constants F of L, $\Diamond(F) = \Diamond'(F)$; and (c) for all terms $t' \neq t$ of L, $\Diamond(t') = \Diamond'(t')$. $A^{t/x}$ results from substituting t for all and only free occurrences of x throughout A.

I shall define the valuation function from formulas of L into truth values as follows. Each formula of L has as its interpretation a set of models which, intuitively, make it true.

1. $[[Ft_1 \ldots t_n]] = \{M \neq M_0\colon \langle\Diamond(t_1), \ldots, \Diamond(t_n)\rangle \in \Diamond(F)\} \cup \{M_0\}$.
2. $[[t_1 = t_2]] = \{M \neq M_0\colon \Diamond(t_1) = \Diamond(t_2)\} \cup \{M_0\}$.
3. $[[\sim A]] = \sim[[A]]$.
4. $[[(A \;\&B)]] = [[A]] \cap [[B]]$.
5. $[[\exists xA]] = \{M \neq M_0\colon$ for some term t and some t-variant M' of M, $M' \in [[A^{t/x}]]\}$.

Note that I have called all atomic sentences true in the null domain; nothing hereafter depends on this choice. In the remainder of what follows I shall focus on interpretations of sentences, so I shall construe the valuation function as the restriction of the above function to sentences of L. The result is a function mapping sentences of L into sets of models of L. It is convenient to extend the valuation function to sets of sentences in the following way: where $S = \{A_1, \ldots, A_n, \ldots\}$ is a set of sentences of L, $[[S]] = \bigcap_i [[A_i]]$. A model M of L *satisfies* (or *is a model of*) a set S of sentences of L just in case $M \in [[S]]$.

A set of sentences T of L is a *theory* if and only if $T = \{A$: for every $M \in [[T]], M \in [[A]]\}$ and $[[T]] \neq \varnothing$. The *Skolem fragment* S of a theory T is T's subset containing just those members of T of the form

$$\exists x_1 \ldots \exists x_n B$$

where B may or may not contain additional quantifier occurrences. Say that S is *k-satisfiable* just in case S has a model with a domain of cardinality k. The least cardinal k such that S is k-satisfiable is the *spectrum number* of S (symbolically, $\$S\$$). If S lacks identity as a logical primitive, then S will have models of cardinality k for every $k \geq \$S\$$. In these terms, the Löwenheim-Skolem theorem says that there is no set S of sentences of L such that $\$S\$ > \aleph_0$.

Quine's third semantic criterion specifies that, for any formula A with a single free variable x, and any set of sentences S, S is *ontologically committed to As* if and only if $[[S]] \subseteq [[\exists xA]]$. That is, S makes an ontological commitment to As just in case there are As in every model of S. Quine does not speak of commitments to a certain cardinality of objects in a domain, but the criterion extrapolates easily. If, in every model of S, there is at least one A, then S is ontologically committed to As. S is *ontologically committed to n As* if and only if every $M \in [[S]]$ contains at least n As. More generally, S is *ontologically committed to n objects* (or, *to a domain of cardinality n*) if and only if, for every $M \in [[S]]$, $|D| \geq n$. Note that, if S is committed to n objects or As, S is committed to m objects or As for any $m \leq n$. I shall say that n is *the cardinality of S's commitments* (*to As*) just in case S is ontologically committed to n objects (or As) but not to m objects (or As) for any $m > n$. It follows, of course, that the cardinality of S's commitments is $\$S\$$. S is *free from ontological commitment* if and only if $\$S\$ = 0$.

Where $M = \langle D, \Diamond \rangle$, let D_A be the subset of D consisting of just the objects satisfying A under $\Diamond$. $|D_A|$, the cardinality of this subset, thus represents the number of As in M. I shall say that the *A-spectrum number* of S (symbolically, $\$S/A\$$) is the cardinality of S's commitments to As. Finally, I shall say that $M = \langle D, \Diamond \rangle$ is a *minimal model* of S if and only if $M \in [[S]]$ and $|D| = \$S\$$, and an *A-minimal model* of S if and only if $M \in [[S]]$ and $|D_A| = \$S/A\$$.

2 Lemmas Let S be the Skolem fragment of a theory T. Obviously $S \subseteq T$, so $[[T]] \subseteq [[S]]$. Thus, if S has no models with domains of cardinality $n < k$, T has no such models. Similarly, if S has no models with fewer than n As, neither does T. So $\$T\$ \geq \$S\$$ and $\$T/A\$ \geq \$S/A\$$.

Lemma 1 *For any theory T such that $1 \leq \$T\$ \leq \aleph_0$, there is a set of* *sentences S^* such that*: (a) $\$S^*\$ = \$T\$$, (b) $S^* \subseteq T$, *and* (c) *each $A \in S^*$ has the form $\exists x_1 \ldots \exists x_n B$.*

Proof: Suppose that L is a first-order language with identity. Define the following sets of sentences of L:

$$S_1 = \{\exists x(x = x)\}$$

$$S_2 = \{\exists x_1 \exists x_2 (x_1 \neq x_2)\}$$

$$S_n = \{\exists x_1 \ldots \exists x_n (x_1 \neq x_2 \,\&\, x_1 \neq x_3 \,\&\ldots\&\, x_{n-1} \neq x_n)\}$$

$$S^* = \bigcup_{n=1}^{\infty} S_n.$$

If L lacks identity, we can define the sets similarly:

$$S_1 = \{\exists x_1 {\sim} (Ax_1 \,\&\, {\sim} Ax_1)\}$$

$$S_2 = \{\exists x_1 \exists x_2 (Ax_1 \,\&\, {\sim} Ax_2)\}$$

$$S_n = \{\exists x_1 \ldots \exists x_n (Ax_1 \,\&\, {\sim} Ax_2 \,\&\, Bx_1 \,\&\, {\sim} Bx_3 \,\&\ldots\&\, Cx_{n-1} \,\&\, {\sim} Cx_n)\}$$

$$S^* = \bigcup_{n=1}^{\infty} S_n,$$

where A, B, C, etc., are expressions of L with one free variable. It is easy to demonstrate the existence of expressions fulfilling the appropriate role. Say that two objects x and y are *weakly discriminable* in language L just in case there is an expression A of L in one free variable such that $A(x) \;\&\; {\sim} A(y)$. Since this is the weakest possible grade of discriminability (cf. [8]), any theory committed to the existence of n objects must be able to discriminate them, pairwise, in this sense. But that requires $(n^2 - n)/2$ expressions, having the characteristics that the above definition of S_n requires. Clearly $\$S^*\$ = \$T\$$, $S^* \subseteq T$, and each member of S^* has the correct form. By obvious alterations of these definitions, we can derive:

Lemma 2 *For any theory T such that $1 \leq \$T/A\$ \leq \aleph_0$, there is a set of sentences S_A^* such that*: (a) $\$S_A^*/A\$ = \$T/A\$$, (b) $S_A^* \subseteq T$, *and each member of S_A^* has the form $\exists x_1 \ldots \exists x_n B$.*

Lemma 3 *If M is an A-minimal model of T, then M is an A-minimal model of S.*

Proof: Assume that M is an A-minimal model of T but not of S. Then, since $M \in [[S]]$, there is a model $M' \in [[S]]$ such that $|D_A'| < |D_A|$. Since M is an A-minimal model of T, $|D_A| = \$T/A\$$. Suppose $|D_A| = 0$; then $|D_A'| < 0$, which is absurd. So suppose $|D_A| > 0$. By Lemma 2, there is a set of initially existentially quantified sentences $S_A^* \subseteq T$ such that $\$S_A^*\$ = \$T/A\$$. Thus, $S_A^* \subseteq S$. It follows that $\$S/A\$ \geq \$S_A^*\$ = |D_A| > |D_A'|$. But then $M' \notin [[S]]$, which is a contradiction.

Using parallel reasoning, we can show

Lemma 4 *If M is a minimal model of T, then M is a minimal model of S.*

3 Theorems I shall show that, for any theory T, the Skolem fragment S of T determines the cardinality of T's commitments. I shall proceed by showing that $\$S\$ = \$T\$$ and that $\$S/A\$ = \$T/A\$$. Suppose that M is an A-minimal model of T. Then $|D_A| = \$T/A\$$. By Lemma 4, M is also an A-minimal model *of* S, so $|D_A| = \$S/A\$$. Assume that T is a theory with Skolem fragment S.

Theorem 5 $\$T\$ = \$S\$$.

Theorem 6 $\$T/A\$ = \$S/A\$$.

Theorem 7 *The cardinality of a theory's commitments (to As) is identical to its Skolem fragment's commitments (to As).*

Theorem 8 *For any cardinal k, a theory is k-satisfiable just in case its Skolem fragment is k-satisfiable.*

Theorem 9 *For any cardinal k, a theory is ontologically committed to k objects or As if and only if its Skolem fragment is ontologically committed to k objects (or As).*

Theorem 10 *A theory T is free from commitment to As if and only if* $\exists xA \notin T$.

Theorem 11 *For any theory T, the following are equivalent:* (a) *T is free from ontological commitment;* (b) *T is* 0*-satisfiable*; *and* (c) *T's Skolem fragment is empty.*

If we adopt Quine's third semantic criterion of ontological commitment, then the ontological commitments of a theory are determined by the theory's Skolem fragment, the set of its initially existentially quantified sentences. The theory's Skolem fragment determines not only whether the theory is committed to, say, *As*, but also to how many *As* the theory is committed. The semantic approach to commitment I have outlined thus accords especially well with a syntactic emphasis on initial existential quantification. We can see, nevertheless, why Quine has tended increasingly toward semantic criteria involving intended interpretations, background languages, etc. The approach of this paper implies that no first-order theory is ontologically committed to more than countably many objects. Since Quine holds both that first-order languages are the canonical notation of science and that uncountable collections constitute the chief source of ontology's interest, he needs an account of ontological commitment more complex than the rather direct analysis I have tried to explicate.

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