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On the Relationship Between One-Point Frames and Degrees of Unsatisfiability of Modal Formulas

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Let L be a normal modal logic and χ_L the class of the frames on which it holds: χ_L determines, in the set $\mathscr{F}$ of all modal formulas, the subset Y of those formulas which are true in every frame of χ_L. From Y we can obtain the set $\neg Y$ of the negation of the formulas of Y, and then $\overline{Y}$ and $\overline{\neg Y}$; i.e., the complements of Y and $\neg Y$ in $\mathscr{F}$.

Up to this point the situation is like that of the Classical Propositional Calculus, where we have the sets T of the tautologies, $\neg T$ (formulas that are false under each valuation), $\overline{T}$ (false under at least one valuation), and $\overline{\neg T}$ (true under at least one valuation). Moreover, the truth-functionality of the classical connectives entails that these sets are the only sets of formulas that can be determined by taking into account the possible truth value of a formula with respect to the models of a given class, when we analyze the situation only by means of the words "for all", "there exists", "true", and "false" referred to the models of the class. In fact we can consider all the models of the Classical Propositional Calculus to be built on a single frame with a single point: so the words "for all" and "there exists" can be referred only to the valuations.

In the case of a modal logic the situation is more involved; a formula ψ is true in a class χ_L of frames if: for each frame $\boldsymbol{A} \in \chi_L$, each valuation V on $\boldsymbol{A}$, and each point a of $\boldsymbol{A}$, $\langle \boldsymbol{A}, V\rangle \models \psi[a]$. By interchanging "for all" with "there exists", or commuting the quantifier referring to the valuations with that referring to the points, or interchanging $\models$ with $\not\models$, we get many different sets of formulas determined by χ_L. These sets, which will be called U-sets (see Definition 2.1), indicate different degrees of unsatisfiability of a formula with respect to χ_L.

The first aim of this paper is to determine necessary and sufficient conditions which a class χ_L must satisfy in order that some U-sets coincide. Through

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this analysis we find that all these conditions concern, roughly speaking, the behaviour of the two frames $\boldsymbol{A}_0 = \langle\{a\}, \phi\rangle$ and $\boldsymbol{A}_1 = \langle\{a\}, \{\langle a, a\rangle\}\rangle$ with respect to the frames of χ_L.

Now, a problem much present in the literature is that of investigating which semantical properties of a class of frames are expressible by sets of modal formulas. On the other side, the properties (expressed in terms of $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$) that we have used in our analysis (see Definition 2.8) seem to be of some interest in themselves. So the second problem we deal with regards the possibility of expressing syntactically these semantical properties.

1 ***Background material*** The modal language considered in this paper has an infinite set of propositional letters $p_0, p_1, p_2, \ldots$, a propositional constant $\perp$ (the *falsum*), the connectives $\wedge$ and $\neg$, and the modal operator $\Box$. We write $\Box^n$, $n < \omega$, instead of $\Box\Box \ldots n$ times, while $\vee$, $\supset$, $\equiv$, and $\Diamond$ are defined as usual. Let ψ be a modal formula: we define the *modal degree* of ψ (in symbols $dg(\psi)$) as follows:

$dg(p) = dg(\perp) = 0$, for propositional letters p
$dg(\neg\psi) = dg(\psi)$
$dg(\psi \wedge \phi) = max\{dg(\psi), dg(\phi)\}$
$dg(\Box\psi) = dg(\psi) + 1$.

A *normal modal logic* is a set of modal formulas that (i) does not contain $\perp$, (ii) contains all classical tautologies, (iii) contains all the formulas of the form $\Box(\psi \supset \phi) \supset (\Box\psi \supset \Box\phi)$, and that is closed under (iv) modus ponens, (v) necessitation, and (vi) the formation of substitution instances. Since in this paper we only deal with normal modal logics, the words "normal" and "modal" are often omitted; moreover we identify a logic with each set of its axioms. The names of logics that aren't new are those of [5].

The semantic structures are *frames* and *models. Frames* are ordered couples $\langle A, R\rangle$ of a nonempty domain $A = \{a_0, a_1, \ldots\}$ (elements of A are called *points*) with a binary relation R on A (frames are denoted by $\boldsymbol{A}, \boldsymbol{B}$, etc.). Two frames $\boldsymbol{A}$ and $\boldsymbol{B}$ are *isomorphic* if they are isomorphic as ordered sets; we shall identify isomorphic frames. *Models* are ordered couples $\langle \boldsymbol{A}, V\rangle$ with $\boldsymbol{A}$ a frame and V a *valuation*; i.e., a function from the set of propositional letters into the power-set of A. We write $V^{-1}(a)$ to represent the set $\{p: a \in V(p)\}$. The well-known Kripke truth-definition defines the notion $\langle \boldsymbol{A}, V\rangle \models \psi[a]$, for a model $\langle \boldsymbol{A}, V\rangle$, $a \in A$, ψ a modal formula. Then $\boldsymbol{A} \models \psi[a]$ means that for every V on $\boldsymbol{A}$ $\langle \boldsymbol{A}, V\rangle \models \psi[a]$; $\langle \boldsymbol{A}, V\rangle \models \psi$ means that for every a of A $\langle \boldsymbol{A}, V\rangle \models \psi[a]$; and $\boldsymbol{A} \models \psi$ means that for every a of A $\boldsymbol{A} \models \psi[a]$. We write $Th(\boldsymbol{A})$ to represent the set $\{\psi: \boldsymbol{A} \models \psi\}$, and, if Γ is a set of formulas, we write $\boldsymbol{A} \models \Gamma$ instead of $\Gamma \subseteq Th(\boldsymbol{A})$.

Let $\langle A, R\rangle$ be a frame and $A' \subseteq A$. We define $\bar{\bar{A}}'/\boldsymbol{A}$ to be the frame $\langle B, R\rangle$ in which

$B = \{a \in A$: for some $a' \in A'$ $a'R^*a\} \cup A'$ (where R^* is the transitive closure of R)

and

$R' = R \cap B \times B$.

In such a frame we refer to B as $\bar{\bar{A}}'$. We define a frame to be *undecomposable* if there aren't two nonempty subsets $A', A'' \subseteq A$ such that $\bar{\bar{A}}' \cap \bar{\bar{A}}'' = \phi$ and $\bar{\bar{A}}' \cup \bar{\bar{A}}'' = A$. $\boldsymbol{B}$ is a subframe of $\boldsymbol{A}$, in symbols $\boldsymbol{B} \subseteq \boldsymbol{A}$, if there is $A' \subseteq A$ such that $\boldsymbol{B} = \bar{\bar{A}}'/\boldsymbol{A}$. $\bar{\bar{A}}'/\langle \boldsymbol{A}, V\rangle$ is the model $\langle \bar{\bar{A}}'/\boldsymbol{A}, V'\rangle$ where, for each $a \in A'$, $V'^{-1}(a) = V^{-1}(a)$. We use, and sometimes without mention, the following

Proposition 1.1 (Generation Theorem) *For all $a \in A'$ and all formulas ψ*

$$\bar{\bar{A}}'/\langle \boldsymbol{A}, V\rangle \vDash \psi[a] \text{ iff } \langle \boldsymbol{A}, V\rangle \vDash \psi[a]$$
$$\bar{\bar{A}}'/\boldsymbol{A} \vDash \psi[a] \text{ iff } \boldsymbol{A} \vDash \psi[a].$$

For L a logic, we set $\chi_L = \{\boldsymbol{A}: \boldsymbol{A} \vDash L\}$. A class χ of frames is *axiomatizable* if $\chi = \chi_L$ for some L. We write, for Γ a set of formulas, $\chi \vDash \Gamma$ if, for each $\boldsymbol{A}$ of χ, $\boldsymbol{A} \vDash \Gamma$. L is *complete* (with respect to χ_L) if $L \vdash \psi$ iff $\chi_L \vDash \psi$. Since in our paper we only deal with axiomatizable classes of frames, which are closed under subframes and disjoint unions of frames, then we may consider, without loss of generality, undecomposable frames only.

2 Basic definitions and results

Definition 2.1 We define a *U-set* of a logic L as a set of formulas of one of the following two kinds:

(i) $\{\psi: QA, QV, Qa, \langle \boldsymbol{A}, V\rangle \# \psi[a]\}$
(ii) $\{\psi: QA, Qa, QV, \langle \boldsymbol{A}, V\rangle \# \psi[a]\}$

where Q stands for "for all" or "there exists", $\boldsymbol{A} \in \chi_L$, V is a valuation on $\boldsymbol{A}$, $a \in A$, and # stands for $\vDash$ or $\nvDash$.

Of the 32 sets of formulas we get in this way, only 24 are a priori different from one another, since two equal consecutive quantifiers commute. Furthermore, we can easily get rid of a number of relations among the U-sets of a logic L which are immediate consequences of their respective definitions and do not depend on L. First, let

$$Y = \{\psi: QA, QV, Qa, \langle \boldsymbol{A}, V\rangle \# \psi[a]\}$$

be a U-set of type (i). By reading Q' as "for all" if Q is "there exists" and vice versa, and $\#'$ as $\nvDash$ if # is $\vDash$ and vice versa, we have that

$$\begin{aligned}
\{\psi: QA, QV, Qa, \langle \boldsymbol{A}, V\rangle \#' \psi[a]\} &= \{\psi: \neg\psi \in Y\} = \neg Y\\
\{\psi: Q'A, Q'V, Q'a, \langle \boldsymbol{A}, V\rangle \#' \psi[a]\} &= \{\psi: \psi \notin Y\} = \overline{Y}\\
\{\psi: Q'A, Q'V, Q'a, \langle \boldsymbol{A}, V\rangle \# \psi[a]\} &= \{\psi: \neg\psi \notin Y\} = \overline{\neg Y}.
\end{aligned}$$

The same holds if the U-set is of type (ii). Four our purpose it is therefore enough to consider only six U-sets for each given L, since any other U-set will be related to one of these by some "Boolean" connection like the ones above. The reason for our choice of the six U-sets will soon be clear. The definition is as follows:

Definition 2.2 Let L be a logic. We set:

$$L_\alpha = \{\psi: \forall \boldsymbol{A}, \forall V, \forall a \quad \langle \boldsymbol{A}, V\rangle \vDash \psi[a]\}$$

($L_\alpha = \{\psi: \chi_L \vDash \psi\}$ and $L_\alpha = L$ iff L is complete)

$$
\begin{aligned}
L_\beta &= \{\psi\colon \forall \boldsymbol{A}, \exists a, \ \forall V \ \langle \boldsymbol{A}, V\rangle \models \psi[a]\} \\
L_\gamma &= \{\psi\colon \forall \boldsymbol{A}, \forall V, \exists a \ \ \langle \boldsymbol{A}, V\rangle \models \psi[a]\} \\
L_\delta &= \{\psi\colon \exists \boldsymbol{A}, \forall V, \forall a \ \ \langle \boldsymbol{A}, V\rangle \models \psi[a]\} \\
L_\varepsilon &= \{\psi\colon \exists \boldsymbol{A}, \exists a, \ \forall V \ \langle \boldsymbol{A}, V\rangle \models \psi[a]\} \\
L_\theta &= \{\psi\colon \exists \boldsymbol{A}, \forall V, \exists a \ \ \langle \boldsymbol{A}, V\rangle \models \psi[a]\}
\end{aligned}
$$

(remember that $\boldsymbol{A} \in \chi_L$, V is a valuation on $\boldsymbol{A}$ and $a \in A$).

With this choice the following inclusions hold for every L:

$$L_\alpha \subseteq L_\beta \subseteq L_\gamma \subseteq L_\delta \subseteq L_\varepsilon \subseteq L_\theta.$$

The only nontrivial inclusion is $L_\gamma \subseteq L_\delta$, which will be shown in Corollary 2.7.

Definition 2.3 We set $\boldsymbol{A}_0 = \langle\{a\}, \phi\rangle$ and $\boldsymbol{A}_1 = \langle\{a\},\{\langle a, a\rangle\}\rangle$. Let a be a point of a frame $\boldsymbol{A}$. We define a as *strongly terminal* if $\overline{\overline{\{a\}}}/\boldsymbol{A} = \boldsymbol{A}_0$, as *weakly terminal* if $\overline{\overline{\{a\}}}/\boldsymbol{A} = \boldsymbol{A}_1$, and as *terminal* if it is weakly or strongly terminal.

Lemma 2.4 *Let $\boldsymbol{B}$ be a frame such that $\boldsymbol{A}_0 \not\subseteq \boldsymbol{B}$, let V be a valuation on $\boldsymbol{A}_1$, and let V' be the valuation on $\boldsymbol{B}$ such that, for all $b \in B$, $V'^{-1}(b) = V^{-1}(a)$, where a is the point of $\boldsymbol{A}_1$. Then for all $b \in B$ and all formulas ψ*

$$\langle \boldsymbol{B}, V'\rangle \models \psi[b] \text{ iff } \langle \boldsymbol{A}_1, V\rangle \models \psi.$$

Proof: By induction on the construction of ψ. If $\psi = p$ then the statement holds by definition of V'. The induction steps for $\wedge$ and $\neg$ are trivial. Let $\langle \boldsymbol{A}_1, V\rangle \models \Box\psi$. Since aRa we have $\langle \boldsymbol{A}_1, V\rangle \models \psi$; by hypothesis for each $b \in B$ it holds $\langle \boldsymbol{B}, V'\rangle \models \psi[b]$ and then $\langle \boldsymbol{B}, V'\rangle \models \Box\psi$. On the other side let $\langle \boldsymbol{B}, V'\rangle \models \Box\psi[b]$; from $\boldsymbol{A}_0 \not\subseteq \boldsymbol{B}$ it follows that there is $b' \in B\ bRb'$. Then $\langle \boldsymbol{B}, V'\rangle \models \psi[b']$, $\langle \boldsymbol{A}_1, V\rangle \models \psi$, and $\langle \boldsymbol{A}_1, V\rangle \models \Box\psi$.

Corollary 2.5 (from [5]) *For every frame $\boldsymbol{A}$, $Th(\boldsymbol{A}) \subseteq Th(\boldsymbol{A}_0)$ or $Th(\boldsymbol{A}) \subseteq Th(\boldsymbol{A}_1)$. Then, for every L, $\boldsymbol{A}_0 \in \chi_L$ or $\boldsymbol{A}_1 \in \chi_L$.*

Proof: Trivial.

Corollary 2.6 *For every L, L_δ is one of the following sets of formulas: $Th(\boldsymbol{A}_0)$, $Th(\boldsymbol{A}_1)$, or $Th(\boldsymbol{A}_0) \cup Th(\boldsymbol{A}_1)$.*

Proof: If $\boldsymbol{A}_0 \in \chi_L$ and $\boldsymbol{A}_1 \notin \chi_L$ then, by Corollary 2.5, for each $\boldsymbol{A} \in \chi_L$, $Th(\boldsymbol{A}) \subseteq Th(\boldsymbol{A}_0)$ and then $L_\delta = Th(\boldsymbol{A}_0)$. Analogously, if $\boldsymbol{A}_1 \in \chi_L$ and $\boldsymbol{A}_0 \notin \chi_L$ then $L_\delta = Th(\boldsymbol{A}_1)$, and, if $\boldsymbol{A}_0, \boldsymbol{A}_1 \in \chi_L$, $L_\delta = Th(\boldsymbol{A}_0) \cup Th(\boldsymbol{A}_1)$. By Corollary 2.5 no other cases are possible.

Corollary 2.7 *For every L, $L_\gamma \subseteq L_\delta$.*

Proof: If $\psi \in L_\gamma$ and $\boldsymbol{A}_0[\boldsymbol{A}_1]$ belongs to χ_L, as $Card(\boldsymbol{A}_0)$ $[Card(\boldsymbol{A}_1)] = 1$, we obtain $\boldsymbol{A}_0 \models \psi[\boldsymbol{A}_1 \models \psi]$. Then, from Corollary 2.5, it follows $\psi \in L_\delta$.

As we observed in the introduction, almost all the conditions of identity among L_α–L_θ are expressible in terms of $\boldsymbol{A}_0, \boldsymbol{A}_1$. For the sake of brevity we give the following

Definition 2.8 Let $\boldsymbol{A}$ be a frame. We write:

$\mathrm{Cond}_0(\boldsymbol{A})$ if $\boldsymbol{A}_0 \subseteq \boldsymbol{A}$

$\mathrm{Cond}_1(\boldsymbol{A})$ if $\boldsymbol{A}_1 \subseteq \boldsymbol{A}$
$\mathrm{Cond}_{01}(\boldsymbol{A})$ if $\boldsymbol{A}_0 \subseteq \boldsymbol{A}$ or $\boldsymbol{A}_1 \subseteq \boldsymbol{A}$
$\mathrm{Cond}_2(\boldsymbol{A})$ if $\boldsymbol{A}_0 = \boldsymbol{A}$ or $\boldsymbol{A}_0 \not\subseteq \boldsymbol{A}$.

Let *Cond* be one among $Cond_0$–$Cond_2$. We say that χ_L *satisfies Cond* if, for each $\boldsymbol{A} \in \chi_L$, $Cond(\boldsymbol{A})$, and we say that *Cond* is *syntactically expressible* if there is a set Γ of formulas such that χ_L satisfies *Cond* iff $\chi_L \models \Gamma$.

Throughout the paper we shall determine which cases among $Cond_0$–$Cond_2$ are syntactically expressible.

*3 **Identities among** L_δ, L_ε, L_θ* We show that all the identities among L_δ, L_ε, and L_θ depend on $Cond_2$ (Theorem 3.1) and that $Cond_2$ is expressible by the formula $\Box\neg\Box\bot$ (Theorem 3.2).

Theorem 3.1 *For every L we have:*

(a) *if χ_L satisfies $Cond_2$ then $L_\delta = L_\varepsilon = L_\theta$*
(b) *if χ_L does not satisfy $Cond_2$ then $L_\delta \neq L_\varepsilon \neq L_\theta$.*

Proof: (a) We have seen that $L_\delta \subseteq L_\epsilon \subseteq L_\theta$; so we have only to show that $Cond_2$ implies $L_\theta \subseteq L_\delta$. Suppose $\psi \in L_\theta$; this means that there exists a frame $\boldsymbol{B} = \langle B, R\rangle$ of χ_L such that for each V on B there is a $b \in B$ $\langle \boldsymbol{B}, V\rangle \models \psi[b]$. If $\boldsymbol{B} = \boldsymbol{A}_0$ then, since $Card(\boldsymbol{A}_0) = 1$, we have, for each V, $\langle \boldsymbol{B}, V\rangle \models \psi$ and then $\psi \in L_\delta$. If $\boldsymbol{B} \neq \boldsymbol{A}_0$, from $\boldsymbol{A}_0 \not\subseteq \boldsymbol{B}$ and Lemma 2.4 we obtain that for each V on $\boldsymbol{A}_1$ $\langle \boldsymbol{A}_1, V\rangle \models \psi$. Then, with $\boldsymbol{A}_1$ belonging to χ_L ($Th(\boldsymbol{B}) \subseteq Th(\boldsymbol{A}_1)$ and $\boldsymbol{B} \in \chi_L$) we have $\psi \in L_\delta$.

(b) Suppose χ_L does not satisfy $Cond_2$. First we show $L_\delta \neq L_\varepsilon$. Let $\boldsymbol{B}$ be a frame of χ_L such that $\boldsymbol{B} \neq \boldsymbol{A}_0$ and $\boldsymbol{A}_0 \subseteq \boldsymbol{B}$. Then (we assume $\boldsymbol{B}$ to be undecomposable) there are two points b, b' of $\boldsymbol{B}$ such that b' is strongly terminal and bRb'. So $\Diamond\Box\bot \in L_\varepsilon$. On the other side we have, for each L, $\Diamond\Box\bot \notin L_\delta$ because, for every $\boldsymbol{A}$, $\boldsymbol{A} \not\models \Diamond\Box\bot$. In fact $\boldsymbol{A} \models \Diamond\Box\bot[a]$, for a point a, implies that there is a strongly terminal point a' which belongs to $\boldsymbol{A}$, and then $\boldsymbol{A} \not\models \Diamond\Box\bot[a']$. Finally we show that $L_\varepsilon \neq L_\theta$, showing that $\overline{\neg L_\varepsilon} \neq \overline{\neg L_\theta}$. Let us consider the formula $\psi = \psi_1 \vee \psi_2$ where

$$\psi_1 = \neg p \wedge \Box\bot \quad \text{and} \quad \psi_2 = \neg\Box\bot \wedge (\Diamond\Box\bot \supset \Diamond(p \wedge \Box\bot)).$$

For every L, $\psi \in \overline{\neg L_\epsilon}$, i.e., for each $\boldsymbol{A}$ and a of $\boldsymbol{A}$ there exists V such that $\langle \boldsymbol{A}, V\rangle \models \psi[a]$. In fact: (i) if a is strongly terminal, $a \notin V(p)$ implies $\langle \boldsymbol{A}, V\rangle \models \psi_1[a]$; (ii) if a is not strongly terminal and there aren't strongly terminal points a' such that aRa', then $\boldsymbol{A} \not\models \Diamond\Box\bot[a]$ and then $\boldsymbol{A} \models \psi_2[a]$; (iii) if a is not strongly terminal and there is a strongly terminal point a' such that aRa', $a' \in V(p)$ implies $\langle \boldsymbol{A}, V\rangle \models \psi_2[a]$. Now we show that if χ_L does not satisfy $Cond_2$ then $\psi \notin \overline{\neg L_\theta}$; i.e., there exists $\boldsymbol{A}$ of χ_L such that for each V there is a $\langle \boldsymbol{A}, V\rangle \not\models \psi[a]$. Let $\boldsymbol{A} \in \chi_L$, $\boldsymbol{A} \neq \boldsymbol{A}_0$, $\boldsymbol{A}_0 \subseteq \boldsymbol{A}$ and let a be a point of $\boldsymbol{A}$ such that $\boldsymbol{A} \models \Diamond\Box\bot[a]$. Now let Y be the set of the strongly terminal points of $\boldsymbol{A}$: for each V on $\boldsymbol{A}$, if, for a b of Y, $b \in V(p)$, then $\langle \boldsymbol{A}, V\rangle \not\models \psi_1[b]$, and, from $\boldsymbol{A} \not\models \neg\Box\bot[b]$ it follows that $\langle \boldsymbol{A}, V\rangle \not\models \psi[b]$, while if, for each $b \in Y$, $b \notin V(p)$, then $\langle \boldsymbol{A}, V\rangle \not\models \Diamond(p \wedge \neg\Box\bot)[a]$ and then, from $\boldsymbol{A} \not\models \Box\bot[a]$, we obtain $\langle \boldsymbol{A}, V\rangle \not\models \psi[a]$.

Theorem 3.2 *$Cond_2$ is expressible by the formula $\Box\neg\Box\bot$.*

Proof: For each $\boldsymbol{A}$, $Cond_2(\boldsymbol{A})$ iff $\boldsymbol{A} \models \Box\neg\Box\bot$. In fact $\boldsymbol{A}_0 \models \Box\neg\Box\bot$; let then $\boldsymbol{A} \neq \boldsymbol{A}_0$. $\boldsymbol{A}_0 \not\subseteq \boldsymbol{A}$ iff $\boldsymbol{A} \models \neg\Box\bot$ and, as $\boldsymbol{A} \neq \boldsymbol{A}_0$, $\boldsymbol{A} \models \neg\Box\bot$ iff $\boldsymbol{A} \models \Box\neg\Box\bot$.

Corollary 3.3 *For every complete L, χ_L satisfies $Cond_2$ iff $L \vdash \Box\neg\Box\bot$.*

Proof: Trivial.

4 Identities among L_β, L_γ, L_δ The identities among L_β, L_γ, and L_δ are related to $Cond_0$, $Cond_1$, and $Cond_{01}$ (Theorems 4.1–4.3). Contrary to $Cond_2$, we show that these conditions aren't syntactically expressible (Theorem 4.4), while they become expressible if referred to classes of transitive frames.

Theorem 4.1 *For every L, $L_\beta = L_\gamma$ iff χ_L satisfies $Cond_{01}$.*

Proof: ($\Rightarrow$) Let

$$\psi = p \supset \Box p \ .$$

We have, for every L, $\psi \in L_\gamma$; this is obvious by considering ψ in the form $\neg p \vee \Box p$. By hypothesis $L_\gamma = L_\beta$, then $\psi \in L_\beta$, but, for every $\boldsymbol{A}$ and a, $\boldsymbol{A} \models \psi[a]$ iff $\overline{\overline{\{a\}}}/\boldsymbol{A}$ is $\boldsymbol{A}_0$ or $\boldsymbol{A}_1$.

($\Leftarrow$) Let $\psi \in L_\gamma$, $\boldsymbol{A} \in \chi_L$, and a be a terminal point of $\boldsymbol{A}$. $\overline{\overline{\{a\}}}/\boldsymbol{A}$ is $\boldsymbol{A}_0$ or $\boldsymbol{A}_1$ and, since $Card(\overline{\overline{\{a\}}}) = 1$, $\psi \in L_\gamma$ implies $\overline{\overline{\{a\}}}/\boldsymbol{A} \models \psi$, i.e., $\boldsymbol{A} \models \psi[a]$. Then $\psi \in L_\beta$.

Theorem 4.2 *For every L, $L_\beta = L_\delta$ iff χ_L satisfies $Cond_0$ or it satisfies $Cond_1$.*

Proof: ($\Rightarrow$) By Corollary 2.5 either $\boldsymbol{A}_0$ or $\boldsymbol{A}_1$ belongs to χ_L. Let us suppose $\boldsymbol{A}_0 \in \chi_L$: we show that χ_L satisfies $Cond_0$. Obviously $\Box\bot \in L_\delta$ and if, for *reductio*, there is a frame $\boldsymbol{A}$ of χ_L such that $\boldsymbol{A}_0 \not\subseteq \boldsymbol{A}$, then $\boldsymbol{A} \models \neg\Box\bot$ and $\Box\bot \notin L_\beta$. Let us now suppose $\boldsymbol{A}_1 \in \chi_L$: we show that χ_L satisfies $Cond_1$. Set:

$$\psi = (p \supset \Box p) \wedge \neg\Box\bot.$$

Since $\boldsymbol{A}_1 \models \psi$ we have $\psi \in L_\delta$; but, if there is a frame $\boldsymbol{A}$ of χ_L such that $\boldsymbol{A}_1 \not\subseteq \boldsymbol{A}$ then, for each a of $\boldsymbol{A}$, either a is strongly terminal, and in such a case $\boldsymbol{A} \not\models \neg\Box\bot[a]$, or it is not terminal and then there exists a' of $\boldsymbol{A}$, $a' \neq a$ and aRa'; in such a case if we choose a V such that $a \in V(p)$ and $a' \notin V(p)$, we obtain $\langle \boldsymbol{A}, V\rangle \not\models p \supset \Box p[a]$. So we have that for every a of $\boldsymbol{A}$, $\boldsymbol{A} \not\models \psi[a]$ and then $\psi \notin L_\beta$.

($\Leftarrow$) If χ_L satisfies $Cond_0$ then $\boldsymbol{A}_1 \notin \chi_L$. Therefore, by Corollary 2.6, $L_\delta = Th(\boldsymbol{A}_0)$ and then $Cond_0$ implies $L_\delta = L_\beta$. Analogously if χ_L satisfies $Cond_1$.

Theorem 4.3 *Let L be a logic such that $L \vdash \Box p \supset \Box\Box p$. $L_\gamma = L_\delta$ iff χ_L satisfies $Cond_0$ or it satisfies $Cond_1$.*

Proof: ($\Rightarrow$) If $\boldsymbol{A}_0 \in \chi_L$ then the proof is the same of that of Theorem 4.2. Let $\boldsymbol{A}_1 \in \chi_L$. We show that $L_\gamma = L_\delta$ implies that χ_L satisfies $Cond_1$. Suppose, for *reductio*, that there is a frame $\boldsymbol{A} = \langle A, R_A\rangle$ of χ_L such that $\boldsymbol{A}_1 \not\subseteq \boldsymbol{A}$. If $\boldsymbol{A}$ contains strongly terminal points, then $\boldsymbol{A}_0 \in \chi_L$. In such a case, since $\boldsymbol{A}_0 \models \Box\bot$ and $\boldsymbol{A}_1 \models \neg\Box\bot$, $\Box\bot \in L_\delta$ and $\Box\bot \notin L_\gamma$ and then $L_\delta \neq L_\gamma$. Suppose $\boldsymbol{A}$ to be without strongly terminal points; then $\boldsymbol{A}$ is without terminal points. We show that in such a case the frame $\boldsymbol{B} = \langle B, R_B\rangle$, where $B = \{b_0, b_1\}$ and $R_B = B \times B$, is a p-morphic image of $\boldsymbol{A}$.

Let w.o. be a well ordering on A. We define a function f from A into B as follows: let a_0 be the first point of A (following w.o.) and $a_{0,n}$, $n < \omega$, a chain of points such that $a_{0,0} = a_0$, $a_{0,n}\, R_A\, a_{0,n+1}$ and $a_{0,n} \neq a_{0,n+1}$ (since $\boldsymbol{A}$ is without terminal points such a chain exists). We set $f(a_{0,n}) = b_0$ if n is even, b_1 otherwise. Let now a_α be the first point for which f has not been defined. If there is a point a' such that $a_\alpha R_A\, a'$ and $f(a')$ have been defined, then set $f(a_\alpha) = b_0$ ($f(a_\alpha)$ in such a case is not essential); if not, proceed as in the case of a_0. It is easy to see that f is a p-morphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. In fact f is onto, $a\, R_A a'$ implies $f(a)\, R_B\, f(a')$; moreover $L \vdash \Box p \supset \Box\Box p$ implies that R_A is transitive and then, for each $a \in A$, there exist a', a'' such that $a\, R_A a'$, $a\, R_A a''$, $f(a') = b_0$, and $f(a'') = b_1$. Therefore, as χ_L is closed under p-morphic images, $\boldsymbol{B} \in \chi_L$.

Let us consider now the formula

$$\psi = (p \supset \Box p) \wedge (\neg p \supset \Box \neg p).$$

Obviously $\boldsymbol{A}_1 \models \psi$ and then $\psi \in L_\delta$. But, if we consider the model $\langle \boldsymbol{B}, V\rangle$, where $V(p) = \{b_0\}$, we have that, for each $b \in B$, $\langle \boldsymbol{B}, V\rangle \not\models \psi[b]$, and then $\psi \notin L_\gamma$.

($\Leftarrow$) Obvious from Theorem 4.2.[1]

Theorem 4.4 *$Cond_0$, $Cond_1$, and $Cond_{01}$ aren't syntactically expressible.*

Proof: First we show the theorem for $Cond_0$. Let us consider the two frames

$$\boldsymbol{B} = \langle \omega, \{\langle n, n+1\rangle, n < \omega\}\rangle$$
$$\boldsymbol{B}' = \langle \omega, \{\langle n+1, n\rangle, n < \omega\}\rangle$$

and a formula ψ such that $dg(\psi) = m$. $\boldsymbol{B}' \models \psi$ implies $\boldsymbol{B}' \models \psi[m+1]$ and then $\boldsymbol{B} \models \psi$. So we have $Th(\boldsymbol{B}') \subseteq Th(\boldsymbol{B})$. Let us now suppose that there exists Γ such that $\chi_L \models \Gamma$ iff χ_L satisfies $Cond_0$, and consider χ_Γ. From $Th(\boldsymbol{B}') \subseteq Th(\boldsymbol{B})$ and $\boldsymbol{B} \notin \chi_\Gamma$ it follows $\boldsymbol{B}' \not\models \Gamma$. Let n be a point of $\boldsymbol{B}'$ such that there is a $\gamma \in \Gamma\, \boldsymbol{B}' \not\models \gamma[n]$: we have $\overline{\overline{\{n\}}}/\boldsymbol{B}' \not\models \gamma$. Let now

$$\psi = \bigvee_{i \leqslant n} \Diamond^i \Box \bot \ .$$

Since $\overline{\overline{\{n\}}} = \{n, n-1, \ldots 0\}$, we have $\overline{\overline{\{n\}}}/\boldsymbol{B}' \models \psi$. Moreover, $\boldsymbol{A} \models \psi$ implies $\boldsymbol{A}_0 \subseteq \boldsymbol{A}$; then $\chi_{\{\psi\}} \models \Gamma$ and $\overline{\overline{\{n\}}}/\boldsymbol{B}' \in \chi_{\{\psi\}} \subseteq \chi_\Gamma$, which contradicts $\overline{\overline{\{n\}}}/\boldsymbol{B}' \not\models \gamma$.

The proof of the theorem for $Cond_{01}$ is the same as that for $Cond_0$; and the proof for $Cond_1$ can be obtained from that given by replacing $\boldsymbol{B}'$ with $\boldsymbol{B}'' = \langle \omega, \{\langle 0, 0\rangle, \langle n+1, n\rangle, n < \omega\}\rangle$ and ψ with $\psi' = \bigvee_{i \leqslant n} \Diamond^i((p \supset \Box p) \wedge \neg\Box\bot)$.

Theorem 4.6 *Suppose $L \vdash \Box p \supset \Box\Box p$. Then*

(a) *χ_L satisfies $Cond_0$ iff $\chi_L \models \Box\bot \vee \Diamond\Box\bot$*
(b) *χ_L satisfies $Cond_{01}$ iff $\chi_L \models (p \supset \Box p) \vee \Diamond(p \supset \Box p)$*
(c) *χ_L satisfies $Cond_1$ iff $\chi_L \models \Diamond((p \supset \Box p) \wedge \neg\Box\bot)$.*

Proof: (a) Suppose $\boldsymbol{A} \in \chi_L$ and $\boldsymbol{A} \not\models \Box\bot \vee \Diamond\Box\bot[a]$. Then, as R is transitive, $\boldsymbol{A}_0 \not\subseteq \overline{\{a\}}/\boldsymbol{A}$, while $\overline{\{a\}}/\boldsymbol{A} \in \chi_L$. The converse is trivial.

The proofs for (b) and (c) are similar to that of (a); it is sufficient to consider that $\boldsymbol{A} \models p \supset \Box p[a]$ iff a is terminal and $\boldsymbol{A} \models (p \supset \Box p) \wedge \neg\Box\bot[a]$ iff a is weakly terminal.

Remark: Let $Cond_x$ be one among $Cond_0$, $Cond_1$, and $Cond_{01}$. Theorem 4.6 does not imply that the class of the transitive frames satisfying $Cond_x$ is axiomatizable. Obviously this class isn't axiomatizable since it is not closed under subframes. Theorem 4.6 only implies that χ_L (where L is $K4 \cup \{\psi_x\}$ and ψ_x the formula of Theorem 4.6 corresponding to $Cond_x$) contains every axiomatizable class of transitive frames satisfying $Cond_x$. By contrast, Theorem 4.5 implies that there isn't a maximum class among the axiomatizable classes which satisfy $Cond_x$. On the other side the connection between $Cond_2$ and the formula $\Box\neg\Box\bot$ is stronger: in fact $\{\boldsymbol{A}: Cond_2(\boldsymbol{A})\} = \chi_{K\cup\{\Box\neg\Box\bot\}}$. That's why $Cond_2(\boldsymbol{A})$ iff $\boldsymbol{A} \models \Box\neg\Box\bot$, while, for $\boldsymbol{A}$ transitive, $Cond_x(\boldsymbol{A})$ iff for each $\boldsymbol{B} \subseteq \boldsymbol{A}$ $\boldsymbol{B} \models \psi_x$.

Corollary 4.7 *If L is a complete transitive logic, then*

(a) $L_\beta = L_\gamma$ *iff* $L \vdash (p \supset \Box p) \vee \Diamond(p \supset \Box p)$
(b) $L_\beta = L_\delta$ *iff* $L_\gamma = L_\delta$ *iff* $L \vdash \Box\bot \vee \Diamond\Box\bot$ *or* $L \vdash \Diamond(p \supset \Box p) \wedge \neg\Box\bot$.

Proof: Trivial.

5 Conditions of identity between L_α and the other U-sets

Theorem 5.1 $L_\alpha = L_\delta$ *iff the only (undecomposable) frame of* χ_L *is* $\boldsymbol{A}_0[\boldsymbol{A}_1]$.

Proof: $L_\alpha = L_\delta$ iff, for all $\boldsymbol{A}, \boldsymbol{B} \in \chi_L$, $Th(\boldsymbol{A}) = Th(\boldsymbol{B})$. Via Corollary 2.5, it is equivalent to say that, for all $\boldsymbol{A} \in \chi_L$, $Th(\boldsymbol{A}) = Th(\boldsymbol{A}_0)$ or, for all $\boldsymbol{A} \in \chi_L$, $Th(\boldsymbol{A}) = Th(\boldsymbol{A}_1)$. But $Th(\boldsymbol{A}) = Th(\boldsymbol{A}_0)[Th(\boldsymbol{A}_1)]$ iff each point of $\boldsymbol{A}$ is strongly [weakly] terminal.

Theorem 5.2 $L_\alpha = L_\gamma$ *iff the only (undecomposable) frames of* χ_L *are* $\boldsymbol{A}_0$ *or* $\boldsymbol{A}_1$.

Proof: ($\Rightarrow$) As shown in the proof of Theorem 4.1, for each L, $p \supset \Box p \in L_\gamma$, while $\boldsymbol{A} \models p \supset \Box p$ iff each point of $\boldsymbol{A}$ is terminal.
($\Leftarrow$) Trivial.

From Theorem 5.1 it follows that the only complete logics which satisfy $L_\alpha = L_\delta$ (and then, via Theorem 3.2, $L_\alpha = L_\theta$) are $Q_0 = K \cup \{\Box\bot\}$ and $DZ = K\{(p \supset \Box p) \wedge \neg\Box\bot\}$, while Theorem 5.2 implies that the only complete logics for which $L_\alpha = L_\gamma$ are Q_0, DZ, and $Z = K \cup \{p \supset \Box p\}$. The word "complete" is essential: in fact (see [7] and [2]) there are logics strictly included in DZ that have its class of frames. The only result about the identity between L_α and L_β that we have obtained is the following:

Theorem 5.3 *If* $L_\alpha = L_\beta$ *then* χ_L *satisfies* $Cond_2$.

Proof: It is easy to see that, for each L, $\Box\neg\Box\bot \in L_\beta$. Then the result follows from Theorem 3.2.

In Example 6.2 we shall show that the converse of Theorem 5.3 is not true, while in Example 6.3 we shall show that $L_\alpha = L_\beta$, which via Theorem 5.3 and Theorem 3.1 implies $L_\delta = L_\varepsilon = L_\theta$, does not imply $L_\beta = L_\gamma$ or $L_\gamma = L_\delta$.

6 Examples We have seen that Q_0 and DZ are the only complete logics satisfying all the identities among U-sets. On the other side, as is obvious a priori, K does not satisfy any identity. In fact, from all the theorems above, we have that a complete logic L does not satisfy any identity among U-sets iff $L \nvdash \Box\neg\Box\bot$ and there is a frame of χ_L without any terminal point. From that we can observe that K is not the unique logic having this property: also $K4$, for instance, has it.

Example 6.1: An interesting example (see [1], [3], and [4]) is given by the logic GL (also called $K4W$, in the notation of [6], or G in [3]). GL is $K \cup \{\Box(\Box p \supset p) \supset \Box p\}$. A frame $\boldsymbol{A}$ belongs to χ_{GL} iff it is transitive and reverse well-founded, i.e., without infinite ascending chains. χ_{GL} satisfies $Cond_0$, and therefore $GL_\beta = GL_\gamma = GL_\delta$, while from $GL \nvdash \Box\neg\Box\bot$ we obtain $GL_\alpha \neq GL_\beta$ and $GL_\delta \neq GL_\varepsilon \neq GL_\theta$. Moreover, via Corollary 2.6, GL_δ, and therefore GL_β, is the logic Q_0. It is known that the operator $\Box$ of GL can be "interpreted", under suitable conditions, as the predicate *Theor* of Peano Arithmetic (PA); under this interpretation $\Box\bot$, i.e., the axiom of Q_0, is the formula which express the inconsistency of PA; so $GL_\beta = Q_0$ says the formulas "near" to being theorems are implied by the inconsistency and that $\neg\Box\bot$, even if it is not a negation of a theorem, is false in each frame of GL.

Example 6.2: $S4Grz$ is $S4 \cup \{\Box(\Box(p \supset \Box p) \supset p) \supset p\}$. A frame $\boldsymbol{A}$ of $S4$ is a frame of $S4Grz$ iff, for each ascending chain $a_0 R\, a_1 R\, a_2 R \ldots$, there exists n such that, for every $n', n'' \geqslant n$, $a_{n'} = a_{n''}$. Then $p \supset \Box p \in S4Grz_\beta$ and $p \supset \Box p \notin S4Grz_\alpha$, while, as χ_{S4Grz} satisfies $Cond_1$ and $Cond_2$, all the identities among the other U-sets are satisfied. This shows that the converse of Theorem 5.3 does not hold.

Example 6.3: Let us consider the logic $S5$. Since the relation of each frame of $S5$ is an equivalence relation, obviously $S5_\alpha = S5_\beta$, and then $S5_\delta = S5_\varepsilon = S5_\theta$. Moreover, since $S5$ has frames without terminal points, $S5_\beta \neq S5_\gamma \neq S5_\delta$. For the well-known relationship existing between $S5$ and Classical Propositional Calculus, we thought that logics satisfying the same identities among U-sets satisfied by $S5$ were "similar" to it. But we have found a logic very different from $S5$ which does it. Consider in fact

$$L = K \cup \{\Diamond p \supset \Box p, \neg\Box\bot\} .$$

$\boldsymbol{A} \in \chi_L$ iff for each a of $\boldsymbol{A}$ there is exactly one point a' $a\,R\,a'$. χ_L satisfies $Cond_2$ but it does not satisfy $Cond_{01}$ and so $L_\beta \neq L_\gamma \neq L_\delta = L_\varepsilon = L_\theta$. Suppose now $\psi \notin L_\alpha$: there exists a point a of a model $\langle \boldsymbol{A}, V\rangle$ such that $\langle \boldsymbol{A}, V\rangle \nvDash \psi[a]$. Let $dg(\psi) = n$ and $a_0, a_1, \ldots, a_n$ be the (not necessarily distinct) points of $\boldsymbol{A}$ such that $a_0 R, a_1 R \ldots a_n$. Let now m be a point of the frame $\boldsymbol{B}$ of Theorem 4.4 and let V' be a valuation on $\boldsymbol{B}$ such that, for each $s \leqslant n$, $V'^{-1}(m+s) = V^{-1}(a_s)$; then we have $\langle \boldsymbol{B}, V'\rangle \nvDash \psi[m]$. In this way we can find, for every b of $\boldsymbol{B}$, a V such that $\langle \boldsymbol{B}, V\rangle \nvDash \psi[b]$. Then, as $\boldsymbol{B} \in \chi_L$, $\psi \notin L_\beta$ and $L_\beta = L_\alpha$.

NOTE

1. We have used the hypothesis $L \vdash \Box p \supset \Box\Box p$ only to show that $L_\gamma = L_\delta$ and $\boldsymbol{A}_1 \in \chi_L$ imply that χ_L satisfies $Cond_1$. Without such a hypothesis we can only show that $L_\gamma = L_\delta$ and $\boldsymbol{A}_1 \in \chi_L$ imply that, for each finite $\boldsymbol{A} \in \chi_L$, $Cond_1(\boldsymbol{A})$.

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