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$\aleph_0$-Categoricity Over a Predicate

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1 Introduction We are concerned here with a condition on a theory T which says that a model of T is determined in some weak way by its (P,L_0) relativised reduct, namely that whenever M and N are countable models of T with the same relativised reduct M_0 then M and N are isomorphic over M_0. In the case that T says that P is empty this reduces to T being $\aleph_0$-categorical, a situation characterised by Ryll-Nardzewski's theorem. If T says that P is the whole model, then in fact for any models M and N with the same L_0-reduct, M and N will be the same, a situation which is characterised by Beth's definability theorem. Both the Ryll-Nardzewski and Beth characterisations are syntactic in that they say that T must prove a set of sentences of a specified kind. Our syntactic condition for T to be $\aleph_0$-categorical over (P,L_0) is rather difficult to state simply. Essentially there will be, for each $n < \omega$, a fixed collection of L-formulas which serve to link n-tuples in models of T to tuples in the P-part of the models, such that if $\bar{a} \in M$ is so related to $\bar{c}$ in P^M then the type of $\bar{a}$ over P^M depends uniformly on the type of $\bar{c}$ over $M^P \upharpoonright L_0$.

A stronger condition that one could place on T is that for *all* models M, N of T, if $M^P \upharpoonright L_0 = N^P \upharpoonright L_0 = M_0$ then M and N are isomorphic over M_0. This property (which we call strong categoricity over (P, L_0)) corresponds exactly to Gaifman's single-valued definitions [2]. (Note that if, for example, P is always empty, then T can have this property if and only if T is the theory of some finite structure.) To give a syntactic characterisation of strong categoricity would seem much more difficult, although Gaifman has proved, in unpublished work, that if we assume in addition to strong categoricity that every model M of T is rigid over P^M then every model M of T is explicitly definable from $M^P \upharpoonright L_0$ in a uniform manner.

Our notation here is standard. I will not be working in a big saturated model.

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T will denote a theory in a countable language L. We assume that L contains only relation symbols (so any subset of an L-structure is a substructure). P will be a fixed unary predicate in L, and L_0 will be a sublanguage of L. If M is an L-structure, then P^M denotes the *subset* of M consisting of those elements which are in P. Again, if M is an L-structure then $M^P \restriction L_0$ denotes the L_0-structure whose universe is P^M and whose L_0-relations are obtained from those of M by restriction. We often refer to $M^P \restriction L_0$ as M_0.

If $\psi(\bar{x})$ is a formula then $\psi^P(\bar{x})$ denotes as usual the formula obtained from ψ by relativising all quantifiers to P. So if M is an L-structure, $\psi(\bar{x})$ an L_0-formula, and $\bar{c} \in P^M$, then $M_0 \models \psi(\bar{c})$ iff $M \models \psi^P(\bar{c})$.

Definition 1 T is said to be $\aleph_0$-categorical over (P, L_0) if, whenever M and N are countable (including finite) models of T and $M^P \restriction L_0 = N^P \restriction L_0$, then there is an isomorphism f of M with N such that $f \restriction P^M$ is the identity. (Or, as we shall say, M and N are isomorphic over M_0, where $M_0 = M^P \restriction L_0 = N^P \restriction L_0$.)

T is said to be strongly categorical over (P, L_0) if T satisfies Definition 1 but without the restriction that M and N be countable. It is clear that there are T which are $\aleph_0$-categorical over (P, L_0) but not strongly categorical over (P, L_0).

2 The main result In this section I give some preliminary definitions and then state the main result, the easy direction of which can be observed immediately, and the difficult direction of which is proved in later sections.

Definition 2 Let $n < \omega$, and $\bar{y}$ be any tuple of variables. d is said to be an *n-schema in $\bar{y}$ over L_0* if d is a map from L-formulas whose free variables include $x_0, \ldots, x_{n-1}$, to L_0-formulas, where if the L-formula ϕ has free variables $x_0, \ldots, x_{n-1}, \bar{z}$, then $d\phi$ has free variables $\bar{z}, \bar{y}$. (In this case $d\phi$ is written as $(d\phi)(\bar{z}, \bar{y})$.)

Definition 3 Let M be an L-structure, $\bar{a}$ an n-tuple of M, d an n-schema in $\bar{y}$ over L_0, and $\bar{c}$ a tuple from M_0 $(= M^P \restriction L_0)$ with $l(\bar{c}) = l(\bar{y})$. Then we say that *$d(\bar{c})$ defines $tp(\bar{a}/P^M)$* if for each L-formula $\phi(\bar{x}, \bar{z})$ and $\bar{b}$ in P^M, $M \models \phi(\bar{a}, \bar{b})$ if and only if $M_0 \models (d\phi)(\bar{b}, \bar{c})$.

Remark 4: Note that $d(\bar{c})$ defines $tp(\bar{a}/P^M)$ if and only if for each L-formula $\phi(\bar{x}, \bar{z})$ we have $M \models (\forall \bar{z} \in P)(\phi(\bar{a}, \bar{z}) \leftrightarrow (d\phi)^P(\bar{z}, \bar{c}))$.

We will be concerned with formulas which are indexed by members of trees, where these trees are subsets of ${}^{\omega>}\omega$, i.e., subsets of ${}^{\omega>}\omega$ which are closed under initial segments. If $\eta \in {}^{\omega>}\omega$, then by a successor of η we mean something of the form $\eta^\frown\langle i\rangle$ for some $i \in \omega$. The set of successors of η will be denoted by η^+. An endpoint of a tree $S \subset {}^{\omega>}\omega$ is just a member of S, no successor of which is a member of S. Finally, $\eta \in {}^{\omega>}\omega$ will be said to be odd or even depending on whether $l(\eta)$ is odd or even.

Definition 5 Let $S \subset {}^{\omega>}\omega$ be a tree. S will be said to be *good* if

(i) S has no infinite branches
(ii) if $\eta \in S$ is odd then either η is an endpoint of S or for all $i < \omega$, $\eta^\frown\langle i\rangle \in S$
(iii) if $\eta \in S$ is even then there is $k < \omega$ such that $[\eta^\frown\langle i\rangle \in S$ if and only if $i \leqslant k]$ (so in particular η cannot be an endpoint of S).

In what follows $\bar{x}$ denotes the sequence of variables $x_0, \ldots, x_{n-1}$.

Theorem 6 Suppose $T \vdash (\exists x)Px$. $\wedge$ *T is $\aleph_0$-categorical over (P,L_0) if and only if for each n, $1 \leqslant n < \omega$, we have: there is a good tree S_n, and there are for each odd $\eta \in S_n$* (i) *an L-formula $\alpha_\eta(\bar{x},\bar{y}_\eta)$, and* (ii) *$d_\eta$, an n-schema in $\bar{y}_\eta$ over L_0 (where d_η actually depends on α_η); and for each odd $\eta \in S$ which is not an endpoint there are L_0-formulas $\psi_{\eta^\frown\langle i\rangle}(\bar{y}_\eta)$ for $i < \omega$, such that*

I(i) *if η is an endpoint of S_n, then for any $M \models T$, n-tuple $\bar{a} \in M$ and $\bar{c} \in M_0$ such that $M \models \alpha_\eta(\bar{a},\bar{c})$, $d_\eta(\bar{c})$ defines $tp(\bar{a}/P^M)$,*

(ii) *if $\eta \in S_n$ is odd but not an endpoint of S_n, then for any $M \models T$, n-tuple $\bar{a} \in M$, and $\bar{c} \in M_0$ such that $M \models \alpha_\eta(\bar{a},\bar{c})$ and $M_0 \models \neg\psi_{\eta^\frown\langle i\rangle}(\bar{c})$ for all $i < \omega$, $d_\eta(\bar{c})$ defines $tp(\bar{a}/P^M)$,*

(iii) *$\{\neg\psi^P_{\eta^\frown\langle i\rangle}(\bar{y}): i < \omega\}$ is not equivalent modulo $T \cup P\bar{y}$ to any finite subset of itself (for each odd $\eta \in S_n$).*

II(i) $T \vdash (\forall\bar{x}) \bigvee_{\substack{l(\eta)=1 \\ \eta\in S_n}} (\exists\bar{y}_\eta \in P)\alpha_\eta(\bar{x},\bar{y}_\eta)$

(ii) *if $\eta \in S_n$ is odd and not an endpoint of S_n then for each $i < \omega$ we have*

$$T \vdash (\forall\bar{x})(\forall\bar{y}_\eta \in P)(\alpha_\eta(\bar{x},\bar{y}_\eta) \wedge \psi^P_{\eta^\frown\langle i\rangle}(\bar{y}_\eta) \rightarrow \bigvee_{\substack{\tau\in S_n \\ \tau\in(\eta^\frown\langle i\rangle)^+}} (\exists\bar{y}_\tau \in P)(\alpha_\tau(\bar{x},\bar{y}_\tau))).$$

For the case $n = 1$ we also demand that there be, for each odd $\eta \in S_1$ an L_0-formula $\chi_\eta(\bar{y}_\eta)$ such that

$$T \vdash (\forall\bar{y}_\eta \in P)(\chi^P_\eta(\bar{y}_\eta) \leftrightarrow (\exists\bar{x})\alpha_\eta(\bar{x},\bar{y}_\eta)).$$

Note 7: By Definition 5(iii) the disjuncts in the formulas in II(i) and II(ii) are finite. Note also by Remark 4 that I(i) and I(ii) are syntactic properties of T (for each η).

Proof of $\Leftarrow$ of Theorem 6: Let T satisfy the right-hand side conditions. I first assert that:

(*) For any model M of T and n-tuple $\bar{a}$ from M there are an odd $\eta \in S_n$ and $\bar{c} \in M_0$ such that $M \models \alpha_\eta(\bar{a},\bar{c})$ and such that either η is an endpoint of S_n or $M_0 \models \neg\psi_{\eta^\frown\langle i\rangle}(\bar{c})$ for all $i < \omega$.

Suppose not and let $M \models T$ and $\bar{a} \in M$ be a counterexample. We will define $\eta_r \in S_n$ and $\bar{c}_r \in M_0$ for $1 \leqslant r < \omega$ such that $l(\eta_r) = 2r - 1$, η_{r+1} is an extension of η_r, and $M \models \alpha_{\eta_r}(\bar{a},\bar{c}_r)$ for all r. η_1 and $\bar{c}_1$ are given by II(i). Suppose we have η_r and $\bar{c}_r$ with $M \models \alpha_{\eta_r}(\bar{a},\bar{c}_r)$. Then η_r is not an endpoint of S_n, and moreover for some $i < \omega$, $M_0 \models \psi_{\eta^\frown\langle i\rangle}(\bar{c}_r)$. Thus by II(ii) there is $\tau \in S_n$ which is a successor of $\eta_r{}^\frown\langle i\rangle$, and also $\bar{c} \in M_0$ such that $M \models \alpha_\tau(\bar{a},\bar{c})$. Put $\bar{c}_{r+1} = \bar{c}$ and $\eta_{r+1} = \tau$. Thus such η_r can be defined. But they then define an infinite branch of S_n, which is impossible, as S_n was good. So (*) is established.

Now let M and N be countable models of T such that $M^P \upharpoonright L_0 = N^P \upharpoonright L_0 = M_0$. We will obtain an isomorphism of M and N over M_0 by the standard back-and-forth argument. First let a be an element of M. Let $\eta \in S_1$ and $\bar{c} \in M_0$ be as given by (*). It then follows from I(i) and (ii), that $d_\eta(\bar{c})$ defines $tp(a/P^M)$. Moreover $M_0 \models \chi_\eta(c)$. Thus there is $b \in N$ such that $N \models \alpha_\eta(b,\bar{c})$ (as $M_0 = N^P \restriction L_0$). If η is not an endpoint of S_1 then by (*), $M_0 \models \neg\psi_{\eta^\wedge\langle i\rangle}(\bar{c})$ for all $i < \omega$. So by I(i) and (ii) again, $d_\eta(\bar{c})$ defines $tp(b/P^N)$. So for any L-formula $\phi(x,\bar{z})$ and $\bar{d}$ in P^M (= P^N) we have

$$M \models \phi(a,\bar{d}) \text{ iff } M_0 \models (d_\eta\phi)(\bar{d},\bar{c}) \text{ iff } N \models \phi(b,\bar{d}).$$

So a and b have the same types over $P^M = P^N$ in M and N, respectively. Now suppose that $\bar{a}$, $\bar{b}$ are n-tuples from M, N respectively with the same types over P^M. Choose any $a \in M$. Again by (*) we find $\eta \in S_{n+1}$ and $\bar{c} \in M_0$ with $M \models \alpha_\eta(\bar{a}^\wedge a,\bar{c})$ and $tp(\bar{a}^\wedge a/P^M)$ defined by $d_\eta(\bar{c})$. As $\bar{a}$ and $\bar{b}$ have the same types over $P^M = P^N$ there is $b \in N$ such that $N \models \alpha_\eta(\bar{b}^\wedge b,\bar{c})$, and so $d_\eta(\bar{c})$ defines $tp(\bar{b}^\wedge b/P^N)$. Thus, as above, $\bar{a}^\wedge a$ and $\bar{b}^\wedge b$ have the same types over P^M in M and N, respectively. This argument shows that M and N are isomorphic over M_0, completing the proof.

In the next two sections we develop the material allowing us to prove the other direction of Theorem 6.

3 Uniform reduction and completeness over (P, L_0)

Definition 7 T is said to be *complete over* (P, L_0) if whenever M, N are models of T such that $M^P \restriction L_0 = N^P \restriction L_0 = M_0$ then

$$(M,a)_{a \in M_0} \equiv (N,a)_{a \in M_0}.$$

Note that if P is always empty then this just says that T is complete, and if P is always the whole model this says that T implicitly defines the relations of $L - L_0$ in terms of L_0.

Proposition 8 *Let T be $\aleph_0$-categorical over (P, L_0). Then T is complete over (P, L_0).*

Proof: So let M, N be models of T with $M^P \restriction L_0 = N^P \restriction L_0 = M_0$. We have to show that $(M,a)_{a \in M_0} \equiv (N,a)_{a \in M_0}$. If M_0 is countable then this follows immediately from the $\aleph_0$-categoricity of T over (P, L_0). So we may assume that M_0 is uncountable. Suppose by way of contradiction that there is an (L-) formula $\phi(\bar{x})$ and tuple $\bar{a} \in M_0$ such that

$$M \models \phi(\bar{a}) \quad \text{and} \quad N \models \neg\phi(\bar{a}).$$

Now we define *countable* models M^i and N^i for $i < \omega$, such that

(i) $(M^i: i < \omega)$ is an ascending chain of elementary substructures of M
(ii) $(N^i: i < \omega)$ is a chain of elementary substructures of N
(iii) $\bar{a} \in M^0$
(iv) for each $i < \omega$, $P^{M^i} \subset P^{N^i}$ and $P^{N^i} \subset P^{M^{i+1}}$.

This is easily obtained.

It is then clear that $((M^i)_0: i < \omega)$ and $((N^i)_0: i < \omega)$ are both chains of elementary substructures of M_0. (Remember that we write $(M^i)_0$ for $(M^i)^P \restriction L_0$, etc.) Moreover, by (iv) if $M^\omega = \bigcup_{i<\omega} M^i$ and $N^\omega = \bigcup_{i<\omega} N^i$, then

$$(M^{\omega})_0 = \bigcup_{i<\omega} (M^i)_0 = \bigcup_{i<\omega} (N^i)_0 = (N^{\omega})_0, \quad \text{and also} \quad \bar{a} \in (M^{\omega})_0.$$

Also M^{ω}, N^{ω} are countable and $M^{\omega} \models \phi(\bar{a})$ and $N^{\omega} \models \neg\phi(\bar{a})$. But this is impossible, for M^{ω} and N^{ω} are isomorphic over $(M^{\omega})_0$ (by $\aleph_0$-categoricity over (P, L_0)).

So the proposition is proved.

If M is a model of T and $\bar{a} \in P^M$, then $tp_{M_0}(\bar{a})$ will denote the type of $\bar{a}$ over ϕ in the model $M_0 = M^P \restriction L_0$. (So $tp_{M_0}(\bar{a})$ is a set of L_0-formulas.) $tp_M(\bar{a})$ will just denote the type of $\bar{a}$ (over ϕ) in M.

Lemma 9 *Let T be complete over (P, L_0). Let M be a model of T, $\bar{a}$ and $\bar{b}$ n-tuples in P^M, and suppose that $tp_{M_0}(\bar{a}) = tp_{M_0}(\bar{b})$. Then $tp_M(\bar{a}) = tp_M(\bar{b})$.*

Proof: Let N be an elementary extension of M such that N_0 is sufficiently homogeneous. Clearly $M_0 \prec N_0$, and so $tp_{N_0}(\bar{a}) = tp_{N_0}(\bar{b})$, whereby there will be an automorphism f of N_0 such that $f(\bar{a}) = \bar{b}$. Let N' be an L-structure such that $(N')_0 = N_0$ and moreover

$$(N', f(c))_{c \in N_0} \cong (N, c)_{c \in N_0}.$$

Thus clearly

$$tp_{N'}(\bar{b}) = tp_N(\bar{a}).$$

On the other hand, by the completeness of T over (P, L_0) and the facts that $N' \models T$ and $(N')_0 = N_0$ it follows that

$$tp_{N'}(\bar{b}) = tp_N(\bar{b}).$$

Thus $tp_N(\bar{a}) = tp_N(\bar{b})$, and so $tp_M(\bar{a}) = tp_M(\bar{b})$.

Proposition 10 *Let T be complete over (P, L_0). Then for any L-formula $\phi(\bar{x})$ there is an L_0-formula $\psi(\bar{x})$ such that for any model M of T and tuple $\bar{a}$ in P^M we have*

$$M \models \phi(\bar{a}) \quad \textit{if and only if} \quad M_0 \models \psi(\bar{a}).$$

Equivalently we could say $T \vdash (\forall \bar{x} \in P)(\phi(\bar{x}) \leftrightarrow \psi^P(\bar{x}))$.

Proof: This is a standard application of completeness. Given $\phi(\bar{x})$, an L-formula, we put $\Gamma = \{\psi(\bar{x}) \in L_0 \colon T \vdash \phi(\bar{x}) \wedge P(\bar{x}) \rightarrow \psi^P(\bar{x})\}$. Then one shows that $T \cup \{\psi^P(\bar{x}) \colon \psi(\bar{x}) \in \Gamma\} \cup \{P(\bar{x})\} \vdash \phi(\bar{x})$, using Lemma 9. Then by compactness one finds a formula $\psi(\bar{x}) \in \Gamma$ such that $T \vdash (\forall \bar{x})(\phi(\bar{x}) \leftrightarrow \psi^P(\bar{x}))$.

Proposition 10 is called the uniform reduction theorem and a variant of it is proved in a more general setting in [1].

Corollary 11 *Let T be $\aleph_0$-categorical over (P, L_0). Then for every formula $\phi(\bar{x})$ of L there is a formula $\psi(\bar{x})$ of L_0 such that*

$$T \vdash (\forall \bar{x} \in P)(\phi(\bar{x}) \leftrightarrow \psi^P(\bar{x})).$$

4 Atomicity over P

Definition 12 The L-structure M is said to be atomic over P if for every tuple $\bar{a} \in M$, $tp_M(\bar{a}/P^M)$ is isolated (where $tp_M(\bar{a}/A)$ is said to be isolated, if this type can be a finitely axiomatised modulo $Th(M, b)_{b \in A}$).

Lemma 13 *Let $M \prec N$ and $\bar{a} \in M$. Suppose that $tp_M(\bar{a}/P^M)$ is isolated. Then $tp_N(\bar{a}/P^N)$ is also isolated.*

Proof: Suppose that $\bar{c} \in P^M$ and the formula $\alpha(\bar{x}, \bar{c})$ isolates (i.e., axiomatises) $tp_M(\bar{a}/P^M)$. It is then clear that for every formula $\phi(\bar{x}, \bar{z})$ the sentence $(\forall \bar{z} \in P)((\forall \bar{x})(\alpha(\bar{x}, \bar{c}) \rightarrow \phi(\bar{x}, \bar{z})) \vee (\forall \bar{x})(\alpha(\bar{x}, \bar{c}) \rightarrow \neg\phi(\bar{x}, \bar{z})))$ is true in M, and thus also in N. Thus $\alpha(\bar{x}, \bar{c})$ isolates $tp_N(\bar{a}/P^N)$.

Proposition 14 *Let T be $\aleph_0$-categorical over (P, L_0). Then every model of T is atomic over P.*

Proof: By Lemma 13 it is clearly enough to show that every countable model of T is atomic over P. So let M be a countable model of T. Suppose by way of contradiction that there is a tuple $\bar{a} \in M$ such that $tp_M(\bar{a}/P^M)$ is not isolated. Let $p = tp_M(\bar{a}/P^M)$. Let $q(x) = \{P(x)\} \cup \{x \neq c : c \in P^M\}$. Then by the Omitting types theorem, $T' = Th((M, c)_{c \in P}M)$ has a countable model $(N, c)_{c \in P}M$ which omits p and q. As this model omits q, we have $N_0 = M_0$. But N and M cannot be isomorphic over M_0, as $(N, c)_{c \in M_0}$ omits p. This contradicts the $\aleph_0$-categoricity of T over (P, L_0), proving the proposition.

Note 15: Let T be $\aleph_0$-categorical over (P, L_0). Fix a formula $\alpha(\bar{x}, \bar{y})$. Given an (L-)formula $\phi(\bar{x}, \bar{z})$, let $\phi_\alpha(\bar{z}, \bar{y})$ denote the formula

$$(\forall \bar{x})(\alpha(\bar{x}, \bar{y}) \rightarrow \phi(\bar{x}, \bar{z})).$$

Let $(d_\alpha\phi)(\bar{z}, \bar{y})$ denote an L_0-formula corresponding to $\phi_\alpha(\bar{z}, \bar{y})$ as given by Corollary 11. So d_α is an n-schema in $\bar{y}$ over L_0 (in the sense of Definition 2), where $n = l(\bar{x})$.

Now let $M \models T$, $\bar{a} \in M$ and suppose that the formula $\alpha(\bar{x}, \bar{c})$ isolates $tp(\bar{a}/P^M)(\bar{c} \in P^M)$. It is then easy to see that $d_\alpha(\bar{c})$ defines $tp(\bar{a}/P^M)$. We will proceed to show that we can choose such α's "uniformly in T" as asserted in Theorem 6.

Given T and $n < \omega$, we will construct a tree of formulas such that any infinite branch of this tree gives rise to a model M of T and n-tuple $\bar{a} \in M$ such that $tp_M(\bar{a}/P^M)$ is not isolated. It will follow (from Proposition 14) that if T is $\aleph_0$-categorical over (P, L_0) then this tree has no infinite branches. This, together with Note 15 will allow us to prove the left to right direction of Theorem 6.

Let us now fix $n < \omega$. $\bar{x}$ will denote the n-tuple of variables $(x_0, \ldots, x_{n-1})$.

Definition 16 Let $\alpha(\bar{x}, \bar{y})$ and $\phi(\bar{x}, \bar{z})$ be L-formulas. By "$\alpha(\bar{x}, \bar{y})$ is a $\phi(\bar{x}, \bar{z})$-atom" we mean the formula "$(\forall \bar{z})((\forall \bar{x})(\alpha(\bar{x}, \bar{y}) \rightarrow \phi(\bar{x}, \bar{z})) \vee (\forall \bar{x})(\alpha(\bar{x}, \bar{y}) \rightarrow \neg\phi(\bar{x}, \bar{z})))$". This is clearly a formula in $\bar{y}$; i.e., a statement about $\bar{y}$.

Let $a_0, \ldots, a_{n-1}$, and c_i for $i < \omega$ be new constants, and let us write $\bar{a}$ for $(a_0, \ldots, a_{n-1})$. Let L' be the expansion of L obtained by adjoining these constants. Let T_1 be $T \cup \{Pc_i : i < \omega\}$. Let us list all L'-sentences as $\{\chi_r : r < \omega\}$.

Now we will define, for certain $\eta \in {}^{\omega>}\omega$, L'-sentences Θ_η so as to satisfy the following:

(i) $\Theta_{\langle\,\rangle}$ is '$\bar{a} = \bar{a}$'.
(ii) If Θ_η is defined and η extends τ then also Θ_τ is defined.
(iii) If η is odd, Θ_η is defined and η extends τ then $\vdash \Theta_\eta \to \Theta_\tau$.
(iv) If Θ_η is defined then $\{\Theta_{\eta\restriction r} : r \leqslant l(\eta)\}$ is consistent with T_1.
(v) Suppose that $l(\eta) = 2r$ and Θ_η is defined. Then $\Theta_{\eta^\wedge\langle 1\rangle}$ is $\bigwedge \{\Theta_{\eta\restriction s} : s \leqslant 2r\} \wedge \neg\chi_r$ if the latter is consistent with T_1. Also $\Theta_{\eta^\wedge\langle 0\rangle}$ is $\bigwedge \{\Theta_{\eta\restriction s} : s \leqslant 2r\} \wedge \chi_r$ if the latter is consistent with T_1, unless χ_r is of the form "$(\exists z \in P)\chi'(z)$" for some $\chi' \in L'$ in which case, for some c_i which does not appear in $\{\Theta_{\eta\restriction s} : s \leqslant 2r\}$, $\Theta_{\eta^\wedge\langle 0\rangle}$ is $\bigwedge \{\Theta_{\eta\restriction s} : s \leqslant 2r\} \wedge \chi'(c_i)$. $\Theta_{\eta^\wedge\langle j\rangle}$ is undefined otherwise.
(vi) Suppose that η is odd, and Θ_η is defined, and so of the form $\alpha(\bar{a}, \bar{c})$ ($\bar{c}$ a tuple of the c_i's). If $\{$"$\alpha(\bar{x}, \bar{c})$ is a $\phi(\bar{x}, \bar{z})$-atom"$: \phi(\bar{x}, \bar{z}) \in L\}$ is not consistent with $T_1 \cup \Theta_\eta$ then $\Theta_{\eta^\wedge\langle 0\rangle}$ is '$\bar{c} = \bar{c}$'. If not, then for some $\kappa \leqslant \omega$, $\Theta_{\eta^\wedge\langle i\rangle}$ is defined iff $i < \kappa$ and moreover $\{\Theta_{\eta^\wedge\langle i\rangle} : i < \kappa\} = \{$"$\alpha(\bar{x}, \bar{c})$ is not a $\phi(\bar{x}, \bar{z})$-atom"$: \phi(\bar{x}, \bar{z}) \in L$ and "$\alpha(\bar{x}, \bar{c})$ is not a $\phi(\bar{x}, \bar{z})$-atom" is consistent with $T_1\}$ and is not equivalent mod T_1 to any proper finite subset of itself. $\Theta_{\eta^\wedge\langle i\rangle}$ is undefined otherwise. (Note in the second case $\Theta_{\eta^\wedge\langle i\rangle}$ can be undefined for all $i < \omega$.)

The Θ_η can clearly be defined so as to satisfy (i)-(vi) above.

Let S_n' be the set of $\eta \in {}^{\omega>}\omega$ such that Θ_η is defined.

Lemma 17 *Suppose that S_n' has an infinite branch. Then T has a countable model M containing an n-tuple $\bar{a}$ such that $tp_M(\bar{a}/P^M)$ is not isolated.*

Proof: Let B be an infinite branch of S_n'. Let $T' = \{\Theta_\eta : \eta \in S\}$. By condition (v) above, T' is complete (in L'). Moreover, by (iv), T' is consistent and contains $T \cup \{Pc_i : i < \omega\}$. Suppose that the L'-sentence $(\exists z \in P)\gamma(z)$ is consistent with T'. $(\exists z \in P)\gamma(z)$ will be χ_r for some $r < \omega$. It is then clear from (v) that for some $i < \omega$, $T' \vdash \gamma(c_i)$. It follows from this that T' has a countable model which omits the type $\{Py\} \cup \{y \neq c_i : i < \omega\}$. Let M' be such a model. We use $\bar{a}$ and c_i to denote the interpretations in M' of these constants. Thus $M = M' \restriction L$ is a model of T and moreover $P^M = \{c_i : i < \omega\}$.

I assert that *$tp_M(\bar{a}/P^M)$ is not isolated.* To see this, suppose that $M \models \beta(\bar{a}, \bar{c})$ where $\beta(\bar{x}, \bar{y}) \in L$ and $\bar{c} \in P^M$. Thus $\beta(\bar{a}, \bar{c}) \in T'$ and so for some odd $\eta \in B$ we have $\vdash \Theta_\eta \to \beta(\bar{a}, \bar{c})$. We can assume that Θ_η is of the form $\alpha(\bar{a}, \bar{c}')$ where $\bar{c}'$ is a tuple of c_i's which includes c. As B is infinite $\Theta_{\eta^\wedge\langle i\rangle}$ is defined for some $i < \omega$. Thus by condition (vi) above, "$\alpha(\bar{x}, \bar{c}')$ is not a $\phi(\bar{x}, \bar{z})$-atom" $\in T'$ for some $\phi(\bar{x}, \bar{z}) \in L$. It is clear from this and Definition 16 that $\alpha(\bar{x}, \bar{c}')$ does not isolate $tp_M(\bar{a}/P^M)$. Thus neither can $\beta(\bar{x}, \bar{c})$ isolate $tp_M(\bar{a}/P^M)$. As $\beta(\bar{x}, \bar{c})$ was an arbitrary formula over P^M satisfied by $\bar{a}$ in M, it follows that $tp_M(\bar{a}/P^M)$ is not isolated. Thus the assertion is proved, and so also the lemma.

Proposition 18 *Let T be $\aleph_0$-categorical over (P, L_0). Then for each n, S_n' has no infinite branch.*

Proof: By Proposition 14 and Lemma 17.

*5 **Proof of the left to right direction of Theorem 6*** Here I point out how the left to right direction of Theorem 6 can be deduced from Proposition 18. So we assume that $T \vdash (\exists x)Px$ and that T is $\aleph_0$-categorical over (P, L_0). Let us fix $n < \omega$ $(n \geqslant 1)$. We know from Proposition 18 that S_n' has no infinite branch. We will construct from S_n' and the attached formulas a tree S_n and attached formulas satisfying the required conditions. In fact I will just show how to construct the first two 'levels' of S_n, the rest of the construction proceeding in the same way.

First let X be the smallest subset of S_n' satisfying (i) $\langle\rangle \in X$ and (ii) if $w \in X$ and $w^+ \cap S_n'$ is finite then $w^+ \cap S_n' \subset X$.

Lemma 19 *X is finite.*

Proof: By Konig's Lemma and the fact that S_n' has no infinite branches.

Now let $X' = \{w \in X: w^+ \cap X = \phi\}$. Then we have immediately:

Lemma 20 *If $w \in X'$ then w is odd, and either w is an endpoint of S_n' or $w^+ \cap S_n'$ is infinite.*

Now let $Y = \{w \in X$: w is odd and $w^+ \cap S_n'$ is finite and nonempty, and $\Theta_{w^\wedge\langle 0\rangle}$ is not of the form '$\bar{c} = \bar{c}$'$\}$. Then clearly $Y \cap X' = \phi$ and $Y \cup X'$ is finite. Let us enumerate $Y \cup X'$ as $\langle w_i: i < k\rangle$ for some $k < \omega$. Then the set of elements of S_n which have length 1 will be precisely $\{\langle i\rangle: i < k\}$. Now we define the formulas $\alpha_{\langle i\rangle}(\bar{x}, \bar{y}_{\langle i\rangle})$ for $i < k$. First suppose that $w_i = w \in X'$. So Θ_w is a formula of the form $\alpha(\bar{a}, \bar{c})$. Let $\bar{y}_{\langle i\rangle}$ be a sequence of variables which has the same length as $\bar{c}$. Then we put $\alpha_{\langle i\rangle}(\bar{x}, \bar{y}_{\langle i\rangle})$ to be $\alpha(\bar{x}, \bar{y}_{\langle i\rangle})$ which is clearly an L-formula. If $w_i = w$ and $w \in Y$, then let the formula $\Theta_w \wedge \{\neg\Theta_{w^\wedge\langle j\rangle}: w^\wedge\langle j\rangle \in S_n'\}$ be written as $\alpha(\bar{a}, \bar{c})$. We put $\alpha_{\langle i\rangle}(\bar{x}, \bar{y}_{\langle i\rangle})$ to be $\alpha(\bar{x}, \bar{y}_{\langle i\rangle})$ for some suitable sequence $\bar{y}_{\langle i\rangle}$. (Let us also assume that $\vdash \alpha_{\langle i\rangle}(\bar{x}, \bar{y}_{\langle i\rangle}) \rightarrow Py$ for each y in $\bar{y}_{\langle i\rangle}$ and each $i < k$.)

Lemma 21

(i) $T \vdash (\forall\bar{x}) \bigvee_{i<k} (\exists\bar{y}_{\langle i\rangle} \in P)\alpha_{\langle i\rangle}(\bar{x}, \bar{y}_{\langle i\rangle})$.

(ii) *Let $i < k$, $w = w_i$ and either $w \in Y$ or w is an endpoint of S_n'. Then $T \vdash$ "$\alpha_{\langle i\rangle}(\bar{x}, \bar{y}_{\langle i\rangle})$ is a $\phi(\bar{x}, \bar{z})$-atom" for all $\phi(\bar{x}, \bar{z}) \in L$.*

Proof: (i) follows easily from properties (v) and (vi) of the Θ_η, together with the fact that $T \vdash (\exists x)Px$. For (ii), suppose first that $w = w_i$ and w is an endpoint of S_n'. If Θ_w is written as $\alpha(\bar{a}, \bar{c})$, we must have that "$\alpha(\bar{x}, \bar{c})$ is not a $\phi(\bar{x}, \bar{z})$-atom" is inconsistent with T_1 for each $\phi(\bar{x}, \bar{z})$ (by property (vi)) of the Θ_η). But $\alpha_{\langle i\rangle}(\bar{x}, \bar{y}_{\langle i\rangle})$ is $\alpha(\bar{x}, \bar{y}_{\langle i\rangle})$, and thus $T \vdash$ "$\alpha_{\langle i\rangle}(\bar{x}, \bar{y}_{\langle i\rangle})$ is a $\phi(\bar{x}, \bar{z})$-atom" for each $\phi(\bar{x}, \bar{z}) \in L$. Now suppose that $w = w_i$ is in Y. Again if we write $\alpha(\bar{a}, \bar{c})$ for Θ_w then we have by property vi of the Θ_η and the definition of $\alpha_{\langle i\rangle}$ that $\vdash \alpha_{\langle i\rangle}(\bar{x}, \bar{y}_{\langle i\rangle}) \rightarrow$ "$\alpha(\bar{x}, \bar{y}_{\langle i\rangle})$ is a $\phi(\bar{x}, \bar{z})$-atom" whenever "$\alpha(\bar{x}, \bar{y}_{\langle i\rangle})$ is not a $\phi(\bar{x}, \bar{z})$-atom" is consistent with T. As we also have $\vdash \alpha_{\langle i\rangle}(\bar{x}, \bar{y}_{\langle i\rangle}) \rightarrow \alpha(\bar{x}, \bar{y}_{\langle i\rangle})$, it follows that $T \vdash$ "$\alpha_{\langle i\rangle}(\bar{x}, \bar{y}_{\langle i\rangle})$ is a $\phi(\bar{x}, \bar{z})$-atom" for all $\phi(\bar{x}, \bar{z}) \in L$. Thus part ii of the lemma is proved.

Now I define level two of S_n and the attached formulas. Let $i < k$ be such that for $w = w_i$, $w^+ \cap S'_n$ is infinite. Note that $w^+ \subset S'_n$ in this case. We then stipulate that $\langle i,j \rangle \in S_n$ for every $j < \omega$. For any other $i < k$ (i.e., for i satisfying the hypotheses of Lemma 21(ii)) we stipulate that $\langle i,j \rangle \notin S_n$, for all $j < \omega$. Thus if i satisfies the hypotheses of Lemma 21(ii) then $\langle i \rangle$ will be an endpoint of S_n. Now suppose that $\langle i,j \rangle \in S_n$. So clearly $\Theta_{\langle i,j \rangle}$ is defined, and is of the form $\psi(\bar{c})$ where $\bar{c}$ is the tuple of c-constants occurring in $\Theta_{\langle i \rangle}$. Then we define $\psi'_{\langle i,j \rangle}(\bar{y}_{\langle i \rangle})$ to be $\psi(\bar{y}_{\langle i \rangle})$.

Now for each $i < k$, let $d_{\alpha_{\langle i \rangle}}$ be the n-schema in $\bar{y}_{\langle i \rangle}$ as defined in Note 15. Let us rebaptise $d_{\alpha_{\langle i \rangle}}$ as $d_{\langle i \rangle}$.

Lemma 22

(i) *Let $i < k$ and $\langle i \rangle$ be an endpoint of S_n. Then for any $M \models T$, n-tuple $\bar{a} \in M$ and $\bar{c} \in M_0$ such that $M \models \alpha_{\langle i \rangle}(\bar{a},\bar{c})$, $d_{\langle i \rangle}(\bar{c})$ defines $tp(\bar{a}/P^M)$.*
(ii) *Let $i < k$ and $\langle i \rangle$ not be an endpoint of S_n. Then for any $M \models T$, n-tuple $\bar{a} \in M$ and $\bar{c} \in M_0$ such that $M \models \alpha_{\langle i \rangle}(\bar{a},\bar{c})$ and $M \models \neg\psi'_{\langle i,j \rangle}(\bar{c})$ for all $j < \omega$, $d_{\langle i \rangle}(\bar{c})$ defines $tp(\bar{a}/P^M)$.*

Proof: As already mentioned, if $\langle i \rangle$ is an endpoint of S_n then i satisfies the hypothesis of Lemma 21(ii). Now part i of the lemma follows from Lemma 21(ii) and Note 15. For part ii let us first note that if $\langle i \rangle$ is not an endpoint of S_n then $\{\psi'_{\langle i,j \rangle}(\bar{y}_{\langle i \rangle}) : j < \omega\} = \{$"$\alpha_{\langle i \rangle}(\bar{x},\bar{y}_{\langle i \rangle})$ is not a $\phi(\bar{x},\bar{z})$-atom"$: \phi(\bar{x},\bar{z}) \in L$, "$\alpha_{\langle i \rangle}(\bar{x},\bar{y}_{\langle i \rangle})$ is not a $\phi(\bar{x},\bar{z})$-atom" is consistent with $T\}$. Thus if $M \models T$, $\bar{a}$ is an n-tuple of M, $\bar{c} \in M_0$, $M \models \alpha_{\langle i \rangle}(\bar{a},\bar{c})$ and $M \models \neg\psi'_{\langle i,j \rangle}(\bar{c})$ for all $j < \omega$, then clearly $\alpha_{\langle i \rangle}(\bar{x},\bar{c})$ isolates $tp(\bar{a}/P^M)$. Part ii of the lemma now follows from Note 15.

For $\langle i,j \rangle \in S_n$, let $\psi_{\langle i,j \rangle}(\bar{y}_{\langle i \rangle})$ be an L_0-formula corresponding to $\psi'_{\langle i,j \rangle}(\bar{y}_{\langle i \rangle})$ as given by Corollary 11. I now assert that with this definition of the first two levels of S_n, and with the above choice of $\alpha_{\langle i \rangle}$ and $d_{\langle i \rangle}$ for $\langle i \rangle \in S_n$ and of $\psi_{\langle i,j \rangle}$ for $\langle i,j \rangle \in S_n$, that II(i) of Theorem 6 is satisfied, as is I of Theorem 6 (for η of length 1). The satisfaction of II(i) is given by Lemma 21(i), and the satisfaction of I by Lemma 22.

This above construction can be repeated to obtain levels 3 and 4, etc., of S_n and the attached formulas, so as to satisfy the required conditions. The L_0-formulas $\chi_\eta(\bar{y}_\eta)$ mentioned in the last part of Theorem 6 can be obtained from Corollary 11. Thus Theorem 6 is proved.

The problem of characterising theories which are strongly categorical over (P, L_0) would seem to be much more difficult. In this connection we conjecture:

Conjecture 23 T is strongly categorical over (P, L_0) if and only if: (i) T satisfies the uniform reduction theorem (the conclusion of Proposition 10) and (ii) if $M \models T$, $A \subset M$, $A \supset P^M$ and $a \in M$ then $tp_M(a/A)$ is isolated.

Of course even if this were true, there would still remain the task of obtaining from it a syntactic characterisation of strong categoricity.

Finally I will mention some past literature and work on the subject matter of this paper. Strongly categorical theories were introduced, in the form of "single-valued operations" by Gaifman in [2], where he stated the uniform reduction theorem as well as a uniform definability theorem for such theories. Gaifman has also shown, in as yet unpublished work, that if we assume in addition that each model M of T is rigid over P^M, then for each $M \models T$, M is "explicitly definable" from M_0, uniformly in T. Wilfrid Hodges pointed out to me several years ago that if T is strongly categorical over (P, L_0), then every countable model M of T is atomic over P^M. Strongly categorical theories also figure in the author's thesis, where some strengthenings of results mentioned in *this* paragraph were proved.

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