Notre Dame Journal of Formal Logic
Volume 24, Number 3, July 1983

KM and the Finite Model Property

M. J. CRESSWELL

In [2] Fine proved, among other things, that *KM* has the finite model property.[1] Fine's proof uses normal forms and gives quite a pretty decision procedure for a variety of modal systems. Our aim in this note is to adapt Fine's proof so that it comes closer to a canonical-model type of completeness proof.

KM is *K* with the additional axiom

M $LMp \supset MLp$.

KM is of interest because Goldblatt has proved that its frames are not characterized by any first-order condition on an accessibility relation.[2] Now Segerberg has shown in [7], p. 33, that any logic with the finite model property has the finite frame property. From this it follows that any logic with the finite model property is complete in the sense of being characterized by a class of frames (more specifically by a class of finite frames); we shall prove that *KM* is so characterized. This means that *KM* is a complete logic which does not correspond to a first-order condition.

We begin with a finite set P of propositional variables. Where P is such a set, let Φ_n be the set of all wff (of some langauge of propositional modal logic) made up from P which are of modal degree n or less (see [5], p. 50). Strictly we should indicate that Φ_n depends on P, and write something like Φ_n/P, but we can understand some fixed set to be involved throughout the discussion.

Now Φ_n will be infinite, but it will contain only finitely many non-equivalent formulas ([5], p. 54). In other words any subset Λ of Φ_n will split into a finite number of classes of the form $\{\beta\colon \vdash_K \beta \equiv \alpha\}$ for some $\alpha \in \Phi_n$. Let γ_Λ be the conjunction of all these α. So every wff in Λ is equivalent in K to one of the conjuncts of γ_Λ. Obviously $\gamma_\Lambda \in \Phi_n$.

A set $\Lambda \subseteq \Phi_n$ will be said to be n-maximal iff for every wff $\beta \in \Phi_n$, either $\beta \in \Lambda$ or $\sim\beta \in \Lambda$.

If Δ is a normal propositional modal logic (e.g., see [1], pp. 64f, or [7],

Received April 14, 1982

p. 12), then a set Λ of wff is said to be Δ-consistent iff there is no finite set $\{\beta_1, \ldots, \beta_k\} \subseteq \Lambda$ such that $\vdash_{\Delta} \sim(\beta_1 \wedge \ldots \wedge \beta_k)$.

Given a logic Δ and a finite set P of propositional variables let W_n (relative to Δ and P) be the set of all n-maximal Δ-consistent sets of wff of Φ_n. Obviously W_n is finite, though its members will not be.

Where $x \in W_n$ then γ_x has an important property which we note for future use:

Lemma 1 *If $x, y \in W_n$ then $\gamma_x \in y$ iff $x = y$.*

The proof relies solely on principles of K.[3]

Theorem 2 *If Λ is a Δ-consistent set of wff of Φ_n, then there is some $x \in W_n$ such that $\Lambda \subseteq x$.*

The proof is standard (see, e.g. [5], pp. 151f).

We are now going to be interested in a special kind of finite 'canonical' model. In this model the worlds come in levels depending on the modal degree of the wff in them. Given a finite set P of variables and a logic Δ we define the *$P/m/\Delta$-model* (for each natural number m). The 'worlds' in this model are simply the members of any W_n, for $n \leqslant m$.

R is defined as follows: xRy iff

(1) $x \in W_n$ and $y \in W_{n-1}$, and for every wff $\alpha \in \Phi_{n-1}$: if $L\alpha \in x$ then $\alpha \in y$

or

(2) $x \in W_0$ and $x = y$.[4]

Where $\mathscr{F} = \langle W, R\rangle$ then $\mathscr{F}$ is clearly a finite frame. The canonical model $\mathscr{M}$ is defined on this frame in the usual way by adding an assignment to the variables. For each $p \in P$, and each $x \in W_n$, $x \in V(p)$ iff $p \in x$. (Since $p \in P$, then $p \in \Phi_n$; if $p \notin P$ then the definition may be arbitrary.)

Theorem 3 *For any wff $\beta \in \Phi_n$, where $n \leqslant m$, and any $x \in W_n$:*

$$\mathscr{M} \models_x \beta \text{ iff } \beta \in x.$$

Proof: The proof is standard, by induction on the construction of β; the only thing to remember is the level of β at each stage. Obviously the theorem holds for $p \in P$. For the truth functors we simply note that if $\sim\beta \in \Phi_n$ then so is β, and if $\alpha \vee \beta \in \Phi_n$ then so are α and β. The induction for L is perhaps worth doing in full:

Suppose $\mathscr{M} \mathrel{=\!\!|}_x L\beta$, where $x \in W_n$ and $L\beta \in \Phi_n$. If $L\beta \in \Phi_n$ then $n \geqslant 1$, so there is some $y \in W_{n-1}$ such that xRy and $\mathscr{M} \mathrel{=\!\!|}_y \beta$. Now $\beta \in \Phi_{n-1}$ and, by the induction hypothesis, $\beta \notin y$. So, by definition of R, $L\beta \notin x$.

Suppose $L\beta \notin x$, where $x \in W_n$ and $L\beta \in \Phi_n$. As before if $L\beta \in \Phi_n$ then $n \geqslant 1$. Consider the set $L^-(x)$ of all δ such that $L\delta \in x$. Clearly $L^-(x) \cup \{\sim\beta\} \subseteq \Phi_{n-1}$ and, since Δ is a normal modal logic and $\sim L\beta \in x$, $L^-(x) \cup \{\sim\beta\}$ is Δ-consistent. So, by Theorem 2, $L^-(x) \cup \{\sim\beta\} \subseteq y$, for some $y \in W_{n-1}$. Since $\sim\beta \in y$ then $\beta \notin y$ and so, by the induction hypothesis, $\mathscr{M} \mathrel{=\!\!|}_y \beta$. But xRy and so $\mathscr{M} \mathrel{=\!\!|}_x L\beta$. This completes the proof of the theorem.

Using Lemma 1 we have the following corollary of Theorem 3:

Corollary 4 *For any $n \leqslant m$, if $x, y \in W_n$, then $\mathcal{M} \models_x \gamma_y$ iff $x = y$.*

Consider now the logic D which is K with the addition of $M(p \supset p)$. Suppose $p \in P$.

Lemma 5 *If Δ is a normal modal logic which contains D then, for any P and m, if $\mathcal{F} (= \langle W, R \rangle)$ is the frame of the $P/m/\Delta$-model then, for every $x \in W$, there is some $y \in W$ such that xRy.*

Proof: Suppose $x \in W_n$, where $n \geqslant 1$. Then, if Δ contains D, $M(p \supset p) \in x$ (since $M(p \supset p) \in \Phi_n$ for all $n \geqslant 1$). So by Theorem 3, $\mathcal{M} \models_x M(p \supset p)$. So there is some y such that $\mathcal{M} \models_y p \supset p$ and xRy. If $x \in W_0$ then xRx. In either case there is some $y \in W$ such that xRy.

Lemma 6 *KM contains D.*

Proof: M is equivalent in K to $M(Mp \supset Lp)$ and $\vdash_K Mq \supset M(p \supset p)$. So $KM \vdash D$.

The next theorem is the crucial theorem of the paper.

Theorem 7 *If Δ is KM then, for every number m and every finite set P of variables, the frame $\mathcal{F}$ of the $P/m/\Delta$-model is a KM frame.*

Proof: Suppose $\mathcal{F} (= \langle W, R \rangle)$ is not a KM frame. Let $\mathcal{M} (= \langle \mathcal{F}, V \rangle)$ be the $P/m/\Delta$-model. Then there is some model $\mathcal{M}^* = \langle \mathcal{F}, V^* \rangle$, such that for some $w^* \in W_n$ (for some $n \leqslant m$): $\mathcal{M}^* \dashv_{w^*} LMp \supset MLp$. (Obviously if any instance of M fails on $\mathcal{F}$ then this one does.)

We first note that $n > 1$; for suppose that $n = 0$ or $n = 1$. In either case, if w^*Rx then $x \in W_0$. So suppose $\mathcal{M}^* \models_{w^*} LMp$. Then $\mathcal{M}^* \models_x Mp$ for every $x \in W_0$ such that w^*Rx. By Lemmas 5 and 6 there is at least one such x, and, by the definition of R, for $x \in W_0$, if xRy then $x = y$. So since $\mathcal{M}^* \models_x Mp$, then $\mathcal{M}^* \models_x Lp$. So $\mathcal{M}^* \models_{w^*} MLp$. In other words M cannot fail where $w^* \in W_0$ or W_1.

So we may suppose that $n > 1$. We show how to define a wff δ_p of Φ_{n-2} such that $\mathcal{M} \dashv_{w^*} LM\delta_p \supset ML\delta_p$.

We define δ_p as follows. δ_p is the disjunction

$$(\gamma_{x_1} \vee \ldots \vee \gamma_{x_k})$$

where $\{x_1, \ldots, x_k\} = V^*(p) \cap W_{n-2}$. If this set is empty then let δ_p be $\sim(q \supset q)$ for some $q \in P$. Clearly $\delta_p \in \Phi_{n-2}$.

Lemma 8 *For any propositional variable p and any $x \in W_{n-2}$,*

$$\mathcal{M}^* \models_x p \text{ iff } \mathcal{M} \models_x \delta_p.$$

Proof: If $\mathcal{M}^* \models_x p$ then $x \in V^*(p)$, but $x \in W_{n-2}$ and so γ_x is one of the disjuncts of δ_p. By Corollary 4, $\mathcal{M} \models_x \gamma_x$, and so $\mathcal{M} \models_x \delta_p$.

If $\mathcal{M}^* \dashv_x p$ then γ_x is not one of the disjuncts of δ_p. So either (1) δ_p is $\sim(q \supset q)$, or (2) δ_p is a disjunction, each disjunct of which is some γ_y for $y \in W_{n-2}$ and $x \neq y$. If (1) then clearly $\mathcal{M} \dashv_x \delta_p$. If (2) then, since x and $y \in W_{n-2}$ and $x \neq y$, by Corollary 4, $\mathcal{M} \dashv_x \gamma_y$. But this is so for every disjunct of δ_p. So $\mathcal{M} \dashv_x \delta_p$. This proves the lemma.

We want to show that

$$\mathcal{M} \dashv_{w^*} LM\delta_p \supset ML\delta_p.^5$$

Now $\mathcal{M}^* \dashv_{w^*} LMp \supset MLp$ and so (1) $\mathcal{M}^* \models_{w^*} LMp$ and (2) $\mathcal{M}^* \dashv_{w^*} MLp$. From (1) we have $\mathcal{M}^* \models_x Mp$ for every x such that w^*Rx. (Obviously $x \in W_{n-1}$.) This means that for every such x we have $\mathcal{M}^* \models_y p$ for some y such that xRy. Obviously $y \in W_{n-2}$ and so, by Lemma 8, $\mathcal{M} \models_y \delta_p$. And so $\mathcal{M} \models_x M\delta_p$ for every x such that w^*Rx. So $\mathcal{M} \models_{w^*} LM\delta_p$.

From (2) we have that $\mathcal{M}^* \dashv_x Lp$ for every x such that w^*Rx. (Obviously $x \in W_{n-1}$.) So $\mathcal{M}^* \dashv_y p$ for some y such that xRy. Obviously $y \in W_{n-2}$ and so, by Lemma 8, $\mathcal{M} \dashv_y \delta_p$, so $\mathcal{M} \dashv_x L\delta_p$, for every x such that w^*Rx. So $\mathcal{M} \dashv_{w^*} ML\delta_p$. So $\mathcal{M} \dashv_{w^*} LM\delta_p \supset ML\delta_p$.

But $\delta_p \in \Phi_{n-2}$ and so $LM\delta_p \supset ML\delta_p \in \Phi_n$, and so, by Theorem 3, $LM\delta_p \supset ML\delta_p \notin w^*$. But w^* is n-maximal and so $\sim(LM\delta_p \supset ML\delta_p) \in w^*$.

But this contradicts the fact that w^* is *KM*-consistent. So $LMp \supset MLp$ cannot fail at any point on $\mathcal{F}$. So no instance of *M* can fail on $\mathcal{F}$; i.e., $\mathcal{F}$ is a *KM*-frame. This establishes Theorem 7.

Theorem 9 *KM has the finite model property.*

For proof, suppose α is not a theorem of *KM*. Then, where *P* is the set of all the variables in α and m is the modal degree of α, by Theorem 2, there is an m-maximal *KM*-consistent set x such that $\sim\alpha \in x$. Obviously x is a point in the $P/m/KM$-model, $\mathcal{M} = \langle \mathcal{F}, V \rangle$, and so, by Theorem 3, $\mathcal{M} \dashv_x \alpha$. Now $\mathcal{F}$ is finite and, by Theorem 7, $\mathcal{F}$ is a *KM*-frame. So if α is not a theorem of *KM* then α fails on a finite *KM* frame. So any wff is a theorem of *KM* iff it is valid on all finite *KM* frames; i.e., *KM* has the finite model property.

Corollary 10 *KM is complete.*

NOTES

1. *KM* is discussed in [6] on pp. 74-76. In this paper we use the terminology of frames, models, etc. as developed by Segerberg (see [7]) on the basis of [6]. (An account of this is given in [1].)

2. Goldblatt in [3] and van Bentham in [8] give (different) proofs that the class of all *KM* frames is not first-order definable. In Part II of [4], pp. 40-42, Goldblatt proves the stronger result that no class of frames which characterizes *KM* is first-order definable.

3. Lemma 1 need not hold where x and y are not in the same W_n. For suppose $x \in W_n$ and consider any $m > n$. x will be a consistent set of wff of Φ_m, and so there will be a $y \in W_m$ such that $x \subseteq y$. Obviously $\gamma_x \in y$ and yet $x \neq y$.

4. This means that each bottom-level world is related to itself and itself alone. Alternatively we could follow Fine ([2], p. 232) and add a world (we could call it ω) to which each member of W_0 is related. ω would be related to itself alone. This procedure enables Fine to generalize the proof to other systems besides *KM* as described in Note 5 below.

5. We note that $LMp \supset MLp$ is a formula of the kind Fine calls 'uniform' ([2], p. 232). That is to say each propositional variable occurs within the scope of exactly the same number

of modal operators, in this case two. Because $LMp \supset MLp$ is uniform its truth at $w^* \epsilon W_n$ depends on the truth of p only in worlds in W_{n-2}, a fact which is crucial in the proof of Lemma 8. Fine defines a logic to be uniform iff all its axioms are uniform and it is able to prove an analogue of Theorem 7 for all uniform logics (using the variation described in Note 4 above).

Examples of nonuniform modal logics are the 'standard' systems such as *T, S4, B,* and *S5*. The proof given in this paper does not apply to them without adaptation, although in these particular cases the adaptations actually make for simpler structures, because we only need one level of worlds. (Fine [2], p. 235, calls such models 'ungraded'.) We do however need a more complicated definition of R. Suppose Δ is one of *T, S4, B,* or *S5*. Then let the $\Delta/m/P$-canonical model have as its set of worlds just W_m. For T, let xRy be defined as: For every α such that $L\alpha \epsilon \Phi_m$, $L\alpha \epsilon x \Rightarrow \alpha \epsilon y$. For *S4*: $L\alpha \epsilon x \Rightarrow L\alpha \epsilon y$, for B: $L\alpha \epsilon x \Rightarrow \alpha \epsilon y$ and $L\alpha \epsilon y \Rightarrow \alpha \epsilon x$, and for *S5*: $L\alpha \epsilon x \Longleftrightarrow L\alpha \epsilon y$. Obviously the frames have the right properties, and the analogue of Theorem 3 is not hard to prove. The method can obviously be generalized to many other systems, but the definitions of R will need to be tailored to each particular case.

REFERENCES

[1] Cresswell, M. J., "Frames and models in modal logic," pp. 63-86 in *Algebra and Logic,* ed., J. N. Crossley, Springer Lecture Notes in Mathematics, No. 450, 1975.

[2] Fine, K., "Normal forms in modal logic," *Notre Dame Journal of Formal Logic,* vol. 16 (1975), pp. 35-40.

[3] Goldblatt, R. I., "First-order definability in modal logic," *The Journal of Symbolic Logic,* vol. 40 (1975), pp. 35-40.

[4] Goldblatt, R. I., "Metamathematics of modal logic, Part I," *Reports on Mathematical Logic*, no. 6 (1976), pp. 41-77, Part II, *ibid.* no. 7, pp. 21-52.

[5] Hughes, G. E. and M. J. Cresswell, *An Introduction to Modal Logic,* London, Methuen, 1968.

[6] Lemmon, E. J., *The "Lemmon Notes," An Introduction to Modal Logic,* ed. K. Segerberg, Oxford, Blackwell, 1977.

[7] Segerberg, K., *An Essay in Classical Modal Logic,* Uppsala, Sweden, 1971.

[8] van Bentham, J. A. F. K., "A note on modal formulae and relational properties," *The Journal of Symbolic Logic,* vol. 40 (1975), pp. 55-58.

Department of Philosophy
Victoria University of Wellington
Private Bag
Wellington, New Zealand