Notre Dame Journal of Formal Logic
Volume 29, Number 3, Summer 1988

An 'Almost Classical' Period-Based Tense Logic

MICHAEL J. WHITE*

Introduction In addition to purely mathematical and logical considerations, there are diverse motivations for the development of tense logics that have intended models with domains containing periods of time rather than temporal "points" or "instants". One such motivation is ontological: a desire to model the conception of temporal points as logical/mathematical constructs from more ontologically fundamental "stretches" of time rather than the converse conception. Another is linguistic: the desire to model certain features of natural language the modeling of which presents difficulties for standard point-based tense logic. One of these features, to quote Burgess, "is *aspect*, the verbal feature which indicates whether we are thinking of an occurrence as an *event* whose temporal stages (if any) do not concern us, or as a protracted *process*, forming, perhaps the backdrop for other occurrences ([1], pp. 124–125).

A typical feature of period-based tense logics has been what might be termed their "intuitionistic flavor." Most period-based semantic models naturally lend themselves to the definition of a strong, intuitionistic, or choice negation operation $\neg$ such that for a "period" or interval x of the model, $x \models \neg A$ just in case for all subintervals $y \subseteq x$, $y \not\models A$. Within the context of a period-based tense logic, it is indeed possible to define a weak, classical, or exclusion negation operation $\sim$, as, for example, was done by Humberstone in the seminal [7]. But, typically, the price to be paid is that $x \models \sim A$ must be interpreted as "it is not the case that A is true throughout x" rather than the nonequivalent "'it is not the case that A' is true throughout x"; for the truth of "it is not the case that A is true throughout x" does not, in such a semantics, preclude the truth of A throughout some y such that $y \subseteq x$ (cf. [1], p. 126).

*Research for this paper was completed during the tenure of a grant awarded by the College of Liberal Arts and Sciences, Arizona State University, for which I am grateful. I am also grateful for the helpful comments by two anonymous referees for *NDJFL*. I should especially like to thank one referee for pointing out a fatal flaw in an earlier attempt at a completeness proof and another mistake—fortunately not fatal and now remedied—in the present completeness proof.

Received August 11, 1986; revised April 20, 1987

The period-based tense logic of the present paper circumvents the intuitionistic character typifying most such logics. The motivation for doing so is, I think, clear. Although there may be nothing inherently unpalatable about a logic with an "intuitionistic flavor", it would be agreeable not to have to force those whose interest in period-based tense logics stems from the ontological or linguistic considerations just mentioned to develop a taste for this flavor if they are not otherwise inclined to do so. Any behavior of the present logic that might, from the classical perspective, seem eccentric is confined to the tense operators.

Further motivation for this paper is a conviction of the author that the considerable resources of point-set topology have not yet been sufficiently utilized in work on period-based tense logic. Part of my purpose is to take some initial steps toward filling in this gap in the literature on period-based tense logic.

1 Semantics for a linear ACP-B tense logic The intuitionistic character typical of period-based tense logic is perhaps not accidental. For the domain of an intended model of such logics is usually (e.g., see [7] and [12]) some subset of the set of open sets of a space homeomorphic with Euclidean 1-space. And it is known that the open sets of any Euclidean space constitute an intuitionistic Heyting algebra, while the closed sets of any such space constitute a Brouwerian algebra, the dual of a Heyting algebra ([4], [9]–[11], and [15]).[1] The semantics of the present period-based logic circumvents this feature, yielding a tense logic with a Boolean propositional basis. It does, however, employ the standard topology of Euclidean 1-space, i.e., the closure, under arbitrary union and finite intersection, of the set of open intervals of real-valued, positive length (under some metrization). A model M of class $\mathfrak{R}$ is a 4-tuple $(X,<,O,V)$, where X is a nondenumerable set, $<$ is a dense and continuous strict linear ordering of the elements of X, O is the set of open sets of the standard Euclidean linear topology on X, and V is a "proto-evaluation function," to be recursively defined, with the set of wffs of propositional tense logic as domain and O as range. Let $B \subset O$ be the following base of this space: B contains all nondegenerate open intervals of X. V may then be inductively defined:

(i) where $A = p^n$ (the n-th propositional variable), $V(A) = \bigcup_{i \in I'} Y_i$ where each $Y_i \in B$ and I' is some subset of a set I indexing B (that is, V assigns to each propositional variable the union of the open sets that are elements of some subset of the base B)[2]

(ii) where $A = \neg B$, $V(A) = -ClV(B)$ (that is, V assigns to each negated wff $\neg B$ the complement of the topological closure of the open set assigned by V to B)

(iii) where $A = (B \vee C)$, $V(A) = V(B) \cup V(C)$

(iv) where $A = (B \wedge C)$, $V(A) = V(B) \cap V(C)$.

In order to complete the inductive definition, I introduce two operators on the set O of open sets of a model M:

$\phi x = (-\infty, \mathrm{Sup}x)$, where $\mathrm{Sup}x$ is the least upper bound in X of x (relative, of course, to $<$) if there is such a lub and $x \neq \varnothing$; if there is no such lub and $x \neq \varnothing$, then $\phi x = (-\infty,\infty) = X$; if $x = \varnothing$, then $\phi x = \varnothing$.

$\pi x = (\mathrm{Inf}x,\infty)$, where Inf$x$ is the greatest lower bound in X of x if there is such a glb and $x \neq \varnothing$; if there is no such glb and $x \neq \varnothing$, then $\pi x = (-\infty,\infty) = \mathrm{X}$; if $x = \varnothing$, then $\pi x = \varnothing$.

The clauses for the tense operations are then straightforward:

(v) where $\mathrm{A} = \mathrm{FB}$, $V(\mathrm{A}) = \phi(V(\mathrm{B}))$
(vi) where $\mathrm{A} = \mathrm{PB}$, $V(\mathrm{A}) = \pi(V(\mathrm{B}))$.

The normal classical definitions for the other propositional connectives are assumed as well as the definitions of the duals of the simple future operator F and the simple past operator P; i.e., G is defined as $\neg\mathrm{F}\neg$ and H as $\neg\mathrm{P}\neg$. The following truth definition can then be given (for nonempty $x \in O$):

$V^+(\mathrm{A},x) = \mathrm{T}$, where A is tense-logic wff and $x \in O$, iff there is no nonempty $y(\in O) \subseteq x$ such that $y \subseteq -ClV(\mathrm{A})$
$V^+(\mathrm{A},x) = \mathrm{F}$ iff, for all $y(\in O) \subseteq x$, $y \subseteq -ClV(\mathrm{A})$.

The *truth* of a wff A in a model M of class $\mathfrak{R}$ is then equated with $V^+(\mathrm{A},\mathrm{X}) = \mathrm{T}$, for X of that model. The $\mathfrak{R}$*-validity* of a wff A may then be defined as the truth of A in all (Euclidean 1-space, period-based) models.

Remark 1.0: The present semantics does not assign a truth value to each wff relative to *every* open set (arbitrary union of open intervals) of each model. For example, if in a given model an open set x includes open sets y and z included in $V(\mathrm{A})$ and $V(\neg\mathrm{A})$, respectively, neither A nor $\neg$A will be either true or false at x in that model. There is a sense in which this is an entirely classical facet of the semantics. In the standard point-based semantics for tense logic, the range of the valuation function is typically the power set of the set of temporal points or instants. It is certainly not assumed that each wff will be assigned a truth value relative to *every* element of the power set, i.e., relative to every set of instants. A wff that is assigned the value truth relative to some points in a given set and the value falsity relative to other points of that same set is not considered to be either true or false "relative to that set". An analogous situation obtains in the present case. Here the range of the proto-evaluation function V is the set of open sets of the space. And there seems to be no more reason to require that each wff be assigned a truth value relative to each member of this set than there is to require that each wff be assigned a truth value relative to each member of the power set of the set of instants in the standard point-based semantics for tense logic.

Claim 1.0 *Every wff of the form* $(\mathrm{A} \vee \neg\mathrm{A})$ *is valid.*

Proof: Suppose not. Then there is some model M_i in which some wff of this form is not true. Then $V_i^+((\mathrm{A} \vee \neg\mathrm{A}),\mathrm{X}_i) \neq \mathrm{T}$. So there is some nonempty $y(\in O_i) \subseteq \mathrm{X}_i$ such that $y \subseteq -ClV_i(\mathrm{A} \vee \neg\mathrm{A})$. The last expression simplifies to $-(ClV_i(\mathrm{A}) \cup Cl - ClV_i(\mathrm{A}))$, which, in turn, simplifies to $-ClV_i(\mathrm{A}) \cap -Cl - ClV_i(\mathrm{A})$, i.e., to $-ClV_i(\mathrm{A}) \cap (ClV_i(\mathrm{A}))^\circ$ (that is, to the intersection of the complement of the closure of $V_i(\mathrm{A})$ and the *interior* of the closure of $V_i(\mathrm{A})$).[3] But, since $(ClV_i(\mathrm{A}))^\circ \subseteq ClV_i(\mathrm{A})$, and $-ClV_i(\mathrm{A}) \cap ClV_i(\mathrm{A}) = \varnothing$, it follows that $-ClV_i(\mathrm{A}) \cap (ClV_i(\mathrm{A}))^\circ = \varnothing$. So y must be empty, and the *reductio* is complete. As a result, the propositional basis of the present logic loses its intuitionistic flavor and becomes entirely classical.

Remark 1.1: A salient feature of the preceding semantic definitions is that a given open set and the interior of its closure are semantically indistinguishable even in cases where they are not identical. This fact will be further exploited in the metatheoretical considerations of Section 3. As a simple example of this feature, consider the union of the open intervals (p,q), (q,r), and (r,s), where $p < q < r < s$. This union is an open set not including any open sets having as an element either the point q or the point r. The interior of the closure of this open set is another open set, namely the interval (p,s), which obviously does include such open sets. However, the V and V^+ functions that we have defined allow us, in effect, to semantically identify such open sets.[4]

Remark 1.2: The clauses for the tense operators in the inductive definition of V call for some comment since they yield what must initially appear to be some very unintuitive consequences. For example, they result in the validity of the theses $A \supset FA$ and $A \supset PA$ (and their duals, $GA \supset A$ and $HA \supset A$). Here the insightful comment of van Benthem seems apposite: "The 'point view' has become so dominant that it has become internalized to the extent of becoming an 'intuition', whereas the period view has to be reconstructed explicitly from material infected by its rival ([17], p. 59). I believe that intuitions to be represented by the present semantic interpretation of the tense operators can be inculcated. As an aid to this project, I first give the (derived) clauses for G and H in the inductive definition of the proto-evaluation function V:

(vii) where $A = GB$, $V(A) = \cup\{(p,\infty): (p,\infty) \subseteq V(B)\}$ if there are any such $(p,\infty) \subseteq V(B)$, $= \varnothing$ otherwise
(viii) where $A = HB$, $V(A) = \cup\{(-\infty,p): (-\infty,p) \subseteq V(B)\}$ if there are any such $(-\infty,p) \subseteq V(B)$, $= \varnothing$ otherwise.

One interpretation that can be given to the simple future operator F is the following: FA iff it has not yet ceased to be the case that there is a forthcoming stretch of "A-ing". Then GA iff it is or has begun to be the case that there will always be A-ing going on. Analogous interpretations can, of course, be given for the past tense operators.

Consider, then, a wff of the form $A \supset FA$. Suppose that there is currently A-ing in progress but that after this "stretch" of A-ing terminates there will never thereafter be any other stretches of A-ing. According to the preceding informal interpretation of the F operator, FA will remain true until the current stretch of A-ing is terminated. But that means that the first "opportunity" for FA not to be true will be an interval *after* the current stretch of A-ing has terminated, i.e. an interval at which $\neg A$ is the case. So there will not be an interval at which A is instantiated but FA is not.

The supposition implicit in this interpretation of the temporal operators is that there are (or that there can be constructed) temporal points or instants that are the limits of temporal stretches (open intervals). For example, if it becomes true that there will be no more stretches of A-ing (or "states of A-ness"), there is a point p that is the limit (least upper bound) of A-stretches/states, and the limit (greatest lower bound) of the state of its being the case that there will never thereafter be any A-stretches/states. But tense-logic wffs are not evaluated at such limit points. So we cannot say that $\neg FA$ or $G\neg A$ is true at p. The present treatment of the tense operators eschews the assumption—made in [7], for example—that a wff of the form $\neg$FA is true with respect to a stretch or interval which is the "last" interval with respect to which A is true. The preceding informal semantic characterization of the tense operators provides an argument for this "nonstandard" facet of the semantics. Since $\neg$FA is true if it *has* ceased to be the case that there is a forthcoming stretch of A-ing or A-ness, it can plausibly be maintained that $\neg$FA is true with respect to a temporal interval at which A-ing is in progress only if that interval contains the limit (least upper bound) of the "last" interval of A-ing: for it is only "at" this point that it begins to be true that it *has* ceased to be the case that there is a forthcoming stretch of A-ing. But, since the units of semantic evaluation are *open* sets (i.e., arbitrary unions and finite intersections of open intervals), there are no such right-closed units of semantic evaluation. So, to repeat the previous claim, the first opportunity for FA not to be true, and, thus, for $\neg$FA to be true, will be an interval *after* the current ("last") stretch of A-ing has terminated. One might say that temporal points play a sort of ghostly role in the present semantics. They are assumed to be available in the form of limits for the purpose of defining open intervals and, hence, open sets. But that is their *only* manifestation; they have no other semantic presence.

As a result of evaluating wffs only at open sets and of the preceding account of the tense operators, we have the following semantic homogeneity condition satisfied:

Claim 1.1 *For any tense-logic wff* A, *model* M, *and* $x \in O$, $V^+(\mathrm{A},x) = \mathrm{T}$ *if* $x \subseteq V(\mathrm{A})$.

Proof: For every nonempty open set $y \subseteq x \subseteq V(\mathrm{A})$, $y \subseteq V(\mathrm{A})$ and $y \subseteq ClV(\mathrm{A})$. Therefore, there is no nonempty $y(\in O) \subseteq x$ such that $y \subseteq -ClV(\mathrm{A})$ and, by definition, $V^+(\mathrm{A},x) = \mathrm{T}$. This claim can be strengthened to the following biconditional:

Claim 1.2 *For any tense-logic wff* A, *model* M, *and* $x \in O$, $V^+(\mathrm{A},x) = \mathrm{T}$ *iff* $x \subseteq (ClV(\mathrm{A}))^\circ$.

Proof: L $\Rightarrow$ R: Suppose that $V^+(\mathrm{A},x) = \mathrm{T}$. Then there is no nonempty open set $y \subseteq x$ such that $y \subseteq -ClV(\mathrm{A})$. So $x \subseteq ClV(\mathrm{A})$. (If not, there would be an open subset $y = x \cap -ClV(\mathrm{A})$ of x such that $y \subseteq -ClV(\mathrm{A})$.) But since, for any sets Y and Z, if $\mathrm{Y} \subseteq \mathrm{Z}$, then $\mathrm{Y}^\circ \subseteq \mathrm{Z}^\circ$, and, for any *open set* z, $z = z^\circ$, it follows that $x \subseteq (ClV(\mathrm{A}))^\circ$.

R $\Rightarrow$ L: Suppose that $x \subseteq (ClV(\mathrm{A}))^\circ$. Then, for all nonempty y such that $y \subseteq x$, $y \subseteq (ClV(\mathrm{A}))^\circ \subseteq ClV(\mathrm{A})$. So there is no nonempty $y \subseteq x$ such that $y \subseteq -ClV(\mathrm{A})$. Hence, by definition, $V^+(\mathrm{A},x) = \mathrm{T}$.[5]

2 *Axiomatization of linear ACP-B tense logic*

For the axiomatization of a linear almost classical period-based tense logic, I take Lemmon's "minimal" tense logic, sometimes designated K_t, as a base. This consists of axioms for classical propositional logic (with uniform substitution and *modus ponens* as primitive inference rules) plus the following axioms:

(A1) $G(p \supset q) \supset (Gp \supset Gp)$
(A2) $H(p \supset q) \supset (Hp \supset Hq)$
(A3) $p \supset HFp$
(A4) $p \supset GPp$

and inference rules:

(R3) $\vdash A \Rightarrow \vdash GA$,
(R4) $\vdash A \Rightarrow \vdash HA$.

For our linear ACP-B tense logic (call it 'LinP-B') adjoin the following axioms:

(A5) $p \supset Fp$
(A6) $FFp \supset Fp$[6]
(A7) $Fp \wedge Fq \supset F(p \wedge q) \vee F(p \wedge Fq) \vee F(Fp \wedge q)$
(A8) $Pp \wedge Pq \supset P(p \wedge q) \vee P(p \wedge Pq) \vee P(Pp \wedge q)$.

3 Metatheory for a linear ACP-B tense logic (Part 1) The metatheory for LinP-B utilizes the concept of a *temporal algebra*, as found, for example, in [16]. A temporal algebra is a structure $\mathbf{C} = (C;0,1,+,\cdot,',f,p)$, where $(C;0,1,+,\cdot,')$ is a Boolean algebra[7] and f and p are 1-place operations satisfying the following conditions, for any $a,b \in C$:

(i) $f0 = p0 = 0$
(ii) $f(a + b) = fa + fb$
(iii) $p(a + b) = pa + pb$
(iv) $(fa\cdot b) = 0 \Leftrightarrow (a\cdot pb) = 0$.

For LinP-B we add the following constraints on f and p:

(v) $((fa)'\cdot ffa) = 0 = ((pa)'\cdot ppa)$
(vi) $((fa)'\cdot a) = 0 = ((pa)'\cdot a)$
(vii) $((f(a\cdot b)+f(a\cdot fb)+f(fa\cdot b))'\cdot(fa\cdot fb)) = 0 = ((p(a\cdot b)+p(a\cdot pb)+p(pa\cdot b))'\cdot(pa\cdot pb))$.[8]

Call the class of temporal algebras satisfying these additional contraints (v) through (vii) $\mathfrak{T}$.

A temporal algebraic model $\mathbf{M}$ is then defined as an ordered pair $\mathbf{M} = (\mathbf{C},\mathbf{V})$, where the valuation function $\mathbf{V}$ has as domain the set of tense-logic wffs and as range the set C of elements of $\mathbf{C}$ and is inductively defined as follows:

(1) where $A = p^n$ (i.e., the n-th propositional variable), $\mathbf{V}(A) \in C$
(2) where $A = \neg B$, $\mathbf{V}(A) = (\mathbf{V}(B))'$
(3) where $A = (B \vee C)$, $\mathbf{V}(A) = \mathbf{V}(B) + \mathbf{V}(C)$
(4) where $A = (B \wedge C)$, $\mathbf{V}(A) = \mathbf{V}(B)\cdot\mathbf{V}(C)$
(5) where $A = FB$, $\mathbf{V}(A) = f\mathbf{V}(B)$
(6) where $A = PB$, $\mathbf{V}(A) = p\mathbf{V}(B)$.

A tense-logic wff A is *true* in an algebraic model $\mathbf{M}$ iff, for that model, $\mathbf{V}(A) = 1$ identically. A tense logic L is determined by a given class $\mathfrak{C}$ of temporal algebras just in case: (a) every theorem of L is assigned an element identically equal to 1 in every model based on each algebra in $\mathfrak{C}$, and (b) every tense-logic wff assigned an element identically equal to 1 in every model based on each algebra in $\mathfrak{C}$ is a theorem of L. In view of the general completeness results of algebraic semantics for tense logics containing K_t (see, e.g., [16]), it is a virtually trivial matter to establish

Claim 3.0 LinP-B *is determined by the class* $\mathfrak{T}$ *of temporal algebras.* (Proof omitted)

Claim 3.1 *An epimorphism can be defined from the field of open sets of the "underlying" space of each model M of class* $\mathfrak{M}$ *to a temporal algebra* $\mathbf{C}$ *of class* $\mathfrak{T}$.

Proof: For any $x,y \in O$ of such a model M, let $x \approx y$ iff $(Clx)^\circ = (Cly)^\circ$ and $[x] = \{y: x \approx y\}$. The result obviously is a partitioning of O since $\approx$ is an equivalence relation. Let $\mathrm{C} = \{[x] : x \in O\}$, and define relations on C satisfying the following conditions:

(A) $[x]' = [-Clx]$
(B) $[x] + [y] = [x \cup y]$
(C) $[x] \cdot [y] = ([x]' \cup [y]')'$
(D) $f[x) = [\phi x]$
(E) $p[x] = [\pi x]$.

It is necessary to show that the relations $'$, $+$, $\cdot$, f, and p are (functional) operations on C and that they are the "right" ones, i.e., that such a structure—call it $\mathbf{C}$—is a member of the class $\mathfrak{T}$ of temporal algebras. The first part of this task is to establish that (when the f and p operators are ignored) $\mathbf{C}$ is a Boolean algebra. In order to expedite this rather tedious task, I state as a lemma some useful topological theorems.

Lemma 3.1.0 *For open sets* x,y,z, (a) $(Clx)^\circ = (Cly)^\circ$ *iff* $Clx = Cly$; (b) $-Clx = (Cl - Clx)^\circ$; (c) $(Cl((Clx)^\circ \cap (Cly)^\circ))^\circ = (Cl(x \cap y))^\circ$; (d) $(Cl(x \cup (Cly \cap Clz)))^\circ = (Cl(x \cup (y \cap z)))^\circ$.

Subclaim 3.1.0 *The* $'$, $+$, *and* $\cdot$ *relations on* C *are functional. That is, for any open sets* x,y,z,w, (a) *if* $y \in [x]$, *then* $[x]' = [y]'$; (b) *if* $z \in [x]$ *and* $w \in [y]$, *then* $[x] + [y] = [z] + [w]$; (c) *if* $z \in [x]$ *and* $w \in [y]$, *then* $[x] \cdot [y] = [z] \cdot [w] = [x \cap y] = [z \cap w]$.

Proofs: For (a): if the antecedent is satisfied, then by (a) of Lemma $Clx = Cly$, and hence $-Clx = -Cly$, and, by (b) of the Lemma, the consequent follows. For (b): if antecedent is satisfied, by (a) of Lemma $Clz = Clx$ and $Clw = Cly$. Hence, $Clx \cup Cly = Cl(x \cup y) = Clz \cup Clw = Cl(z \cup w)$, and $(Cl(x \cup y))^\circ = (Cl(z \cup w))^\circ$, yielding the consequent. For (c): Use (a) and (b) of this Subclaim to obtain (given the antecedent of (c)) $([x]' + [y]')' = ([z]' + [w]')' = (Cl((Clx)^\circ \cap (Cly)^\circ))^\circ = (Cl((Clz)^\circ \cap (Clw)^\circ))^\circ$. Then use (c) of the Lemma to obtain $(Cl(x \cap y))^\circ = (Cl(z \cap w))^\circ$, yielding the consequent.

Subclaim 3.1.1 *The structure* $\mathbf{C}$ *is a lattice with* [X] *the unit element and* $[\varnothing]$ *the zero element.*

Proof: Consider the induced relation $\leqq$ such that $[x] \leqq [y]$ iff $(\exists z \in [x])(\exists w \in [y])(z \subseteq w)$. It follows that if $z \subseteq w$, then $Clz \subseteq Clw$ and $(Clz)^\circ = (Clx)^\circ \subseteq (Clw)^\circ = (Cly)^\circ$. But $(Clx)^\circ \in [x]$ and $(Cly)^\circ \in [y]$. So $[x] \leqq [y]$ iff $(Clx)^\circ \subseteq$

$(Cly)^{\circ}$. The reflexivity, transitivity, and antisymmetry of $\leqq$ follow. Since, for any open x, $(Clx)^{\circ}$ is the largest element of $[x]$, it follows that $[x] + [y]$ is the supremum (lub) of $[x]$ and $[y]$ with respect to this relation, while $[x] \cdot [y]$ is their infimum (glb). So, by definition, **C** is a lattice. And it is obvious that [X] will be the maximal or unit element, $[\varnothing]$ the minimal or zero element of the lattice.

Subclaim 3.1.2 *The lattice* **C** *is distributive and complemented.*

Proof: To show that $[x] + ([y] \cdot [z]) = ([x] + [y]) \cdot ([x] + [z])$ use (d) of Lemma 3.1.0. The dual distributive principle then follows from the fact that **C** is a lattice. For complementation: $[x] + [x]' = 1$ iff $(Cl(x \cup -Clx))^{\circ} = (Cl\mathrm{X})^{\circ} = \mathrm{X}$. But, by (a) of Lemma 3.1.0, this identity holds iff $Cl(x \cup -Clx) = Cl\mathrm{X} = \mathrm{X}$. But $Cl(x \cup -Clx) = Clx \cup Cl{-Clx}$. Since $Clx \cup -Clx = \mathrm{X}$ and $-Clx \subseteq Cl{-Clx}$, $Clx \cup Cl{-Clx} = \mathrm{X}$. $[x] \cdot [x]' = 0$ iff $(Cl(x \cap -Clx))^{\circ} = (Cl\varnothing)^{\circ} = \varnothing$. By (a) of Lemma 3.1.0, the identity holds iff $Cl(x \cap -Clx) = Cl\varnothing = \varnothing$. Since $-Clx = (-x)^{\circ}$ and $(-x)^{\circ} \subseteq -x$, it follows that $(x \cap -Clx) = \varnothing$. But, then, $Cl(x \cap -Clx) = Cl\varnothing = \varnothing$.

From Subclaims 3.1.0, 3.1.1, and 3.1.2 it follows that a structure **C** is a Boolean algebra (with respect to the $'$, $+$, and $\cdot$ operations). In order to complete the proof of Claim 3.1 it suffices to show that the relations f and p are (functional) operations and that they satisfy constraints (i) through (vii) on the f and p operations of the class $\mathfrak{T}$ of temporal algebras.

Subclaim 3.1.3 *For all* $x, y \in O$, *if* $y \in [x]$, *then* $f[x] = f[y]$ $(p[x] = p[y])$.

Proof: Suppose that the antecedent is satisfied. Then $(Clx)^{\circ} = (Cly)^{\circ}$ and, by (a) of Lemma 3.1.0, it follows that $Clx = Cly$. Then either (1) both x and y are neither left- nor right-bounded, (2) $x = y = \varnothing$, or (3) x and y must be either both left-bounded or both right-bounded. In case (1) $\phi x = \pi x = \phi y = \pi y = \mathrm{X}$. In case (2) $\phi x = \pi x = \phi y = \pi y = \varnothing$. In case (3), first suppose both are left-bounded and right-unbounded. Then $\mathrm{Inf}x = \mathrm{Inf}y$. For suppose not. Then, either $\mathrm{Inf}x < \mathrm{Inf}y$ or $\mathrm{Inf}y < \mathrm{Inf}x$. So, either $\mathrm{Inf}x \in Clx, \notin Cly$ or $\mathrm{Inf}y \in Cly, \notin Clx$, either of which yields a contradiction. So, $\pi y = \pi x = (\mathrm{Inf}x, \infty)$ and $\phi x = \phi y = (-\infty, \infty) = \mathrm{X}$. Similarly, *mutatis mutandis*, when both x and y are both right-bounded and left-unbounded, and when both are both right- and left-bounded.

Subclaim 3.1.4 *The f and p operations of a structure* **C** *satisfy contraints* (i) *through* (vii) *on the f and p operations of the class $\mathfrak{T}$ of temporal algebras.*

Proof: I leave this proof, which is straightforward, to the reader.

From these two additional Subclaims it then follows that beginning with a (Euclidean 1-space) model M of class $\mathfrak{R}$, we have constructed a structure **C** that is an element of the class $\mathfrak{T}$ of temporal algebras such that the map $m(x) = [x]$ is an epimorphism from the underlying field of open sets of M onto **C**. This completes the proof of Claim 3.1.

If, beginning with a model M of class $\mathfrak{R}$, We take the temporal algebra **C** of class $\mathfrak{T}$ thus defined by it and, for each propositional variable p^n, let $\mathbf{V}(\mathrm{p}^n) = [x]$ $(\in \mathrm{C})$ if $V(\mathrm{p}^n) = x (\in O)$, it follows that

Claim 3.2 *For all tense-logic wffs,* A, $\mathbf{V}(A) = [x]$ *iff* $(\exists y \in [x])(V(A) = y)$.

Proof: By induction of the usual sort, e.g., on wff length.

As a corollary of Claims 3.0, 3.1, and 3.2, we have

Claim 3.3 LinP-B *is sound with respect to the class* $\mathfrak{R}$ *of models.*

Proof: From 3.0 we have the soundness of LinP-B with respect to the class of models based on the class $\mathfrak{T}$ of temporal algebras. I.e., every LinP-B theorem is assigned by the **V** function of a model **M** the unit element 1 of the algebra **C** of that model. But, by Claims 3.1 and 3.2, for each such model **M**, there is a model M of class $\mathfrak{R}$ equivalent to it. That is, for every such model M, for every LinP-B theorem A, there will be an $x \in [X] = 1$, such that $V(A) = x$. But it then follows by Claim 1.2 that $V^+(A,X) = T$, i.e., that A is true in model M. Since this is true for all LinP-B theorems and all models M of class $\mathfrak{R}$, the claim follows.

Remark 3.0: There certainly are less circuitous and cumbersome ways of obtaining the preceding soundness result. The main interest of this approach is, I think, technical. It or the essentially equivalent use of the operators mentioned in Note 4 on regular open sets provides methods for "Booleanizing" a model, the underlying structure ("general frame") of which is a Heyting algebra.

3 Metatheory for a linear ACP-B tense logic (Part 2) Fortunately, there is a relatively simple procedure for obtaining the completeness of LinP-B with respect to the class $\mathfrak{R}$ of Euclidean 1-space models. I first outline this procedure and then sketch in some of the details. Standard results yield the completeness of LinP-B with respect to the class of standard or Kripke frames characterized by 'alternative' or 'accessibility' (or temporal ordering) relations that are: (i) transitive, (ii) reflexive, and (iii) comparable or connected. Therefore, any nontheorem of LinP-B will be falsified, with respect to standard point-based semantics, in a model based on a standard frame possessing a transitive, reflexive, and comparable alternative relation. However, it can additionally be shown that, with respect to standard point-based semantics, LinP-B possesses the *finite model property*: any nontheorem is falsified in some *finite* model the underlying frame of which is transitive, reflexive, and comparable. A corollary of Segerberg's "bulldozing" theorem can then be invoked to show that any nontheorem of LinP-B is falsified in a denumerably infinite *linear* standard model, that is, a standard model the underlying frame of which possesses a denumerable number of points and a transitive, reflexive, comparable, and *antisymmetric* temporal ordering or accessibility relation. Finally, it is shown that for any such standard model, there is an equivalent Euclidean 1-space model of class $\mathfrak{R}$. The consequence is that every nontheorem of LinP-B is not true in some model M of class $\mathfrak{R}$ (relative to the period-based semantics of Section 1 of the present paper), a consequence that establishes the completeness of LinP-B with respect to the class $\mathfrak{R}$ of Euclidean 1-space models.

Claim 3.4 LinP-B *is complete with respect to the class* $\mathfrak{R}$ *of Euclidean 1-space models.*

In order to fill in some of the details of the proof, as outlined above, for this Claim, I employ the concepts of standard (Kripke) frames and models for tense logics containing Lemmon's K_t. A standard frame $\Phi = \langle W,R \rangle$ is an ordered pair, where W is a nonempty set (intuitively the set of temporal instants or points) and R is a dyadic relation on W. A standard model $\Upsilon = \langle \Phi,V \rangle$ is an ordered pair of standard frame and evaluation function V, with the set of wffs of propositional tense logic as domain and $\mathcal{P}W$ as range. V is defined in the standard way: that is, where wff A is a sentence letter V(A) is an "arbitrary" member of $\mathcal{P}W$; where A is a truth-functional compound of other wffs B (or B and C), V(A) is defined by the appropriate Boolean operations on V(B) (or V(B) and V(C)); where A is FB, $V(A) = \{w \in W: (\exists u \in W)(Rwu \wedge u \in V(B))\}$; and where A is PB, $V(A) = \{w \in W: (\exists u \in W)(Ruw \wedge u \in V(B))\}$. Wff A is then said to be *true* at a temporal point/instant w in a standard model Υ_i iff $w \in V_i(A)$; *valid* in a model Υ_i iff it is true at all points $w \in W_i$ of that model; *valid* in a frame Φ_i iff it is valid in each model that can be constructed on that frame; *valid* over a class $\mathfrak{C}$ of frames iff it is valid in each frame in that class. A tense logic L containing K_t is determined by a class $\mathfrak{C}$ of frames just in case (a) L is sound with respect to $\mathfrak{C}$ (every theorem of L is valid over $\mathfrak{C}$) and (b) L is complete with respect to $\mathfrak{C}$ (every wff that is valid in every frame in $\mathfrak{C}$ is a theorem of L).

Subclaim 3.4.0 LinP-B *is complete with respect to the class of frames characterized by an accessibility relation that is transitive, reflexive, and comparable (or connected: i.e., for every* w,u $\in$ W *of each such frame,* Ruw *or* Rwu*).*

Proof Sketch: It is known that Axiom (5) implies a reflexivity condition, (6) a transitivity condition, and (7) and (8) a comparability condition on the accessibility relation of the frame of the Henkin (canonical) standard model of a tense-logic extension of K_t having these wffs as additional axioms. It then follows by an appropriate "fundamental theorem" for such Henkin standard models for tense logic that any tense-logic wff valid over this class of frames is a theorem of LinP-B.[9]

Subclaim 3.4.1 LinP-B *possesses the finite model property; more particularly, any nontheorem of* LinP-B *is false in a model the domain of which is finite and the accessibility relation of which is transitive, reflexive, and comparable.*

Proof Sketch: A tense (or modal) logic L has the finite model property (fmp) just in case every nontheorem of L is false in some finite standard model, i.e., is false is some model constructed on a standard frame the domain W of which contains a finite number of elements. It follows from Subclaim 3.4.0 that each nontheorem of LinP-B will be false (and, hence, that its negation will be true) in some model constructed on a transitive, reflexive, and comparable frame. The filtration method can be used to show that the negation $\neg A$ of any such nontheorem will be true in a finite model Υ the frame Φ of which is a frame for LinP-B, that is, a frame characterized by a transitive, reflexive, and comparable accessibility relation. Let Γ be the smallest (and, hence, finite) set of wffs that contains $\neg A$ and is closed under subformulas. Define as follows a finite standard model Υ^* that is a filtration through Γ of the model Υ in which $\neg A$

is true. Define the equivalence class [w] to be the set of all u $\in$ W that agree with w with respect to assignment of truth values to wffs in the set Γ, and let $W^* = \{[w]: w \in W\}$. Let $R^*[w][u]$ iff both, for every wff FB $\in \Gamma$, if either u $\in$ V(B) or u $\in$ V(FB), then w $\in$ V(FB), and, for every PB $\in \Gamma$, if either w $\in$ V(B) or w $\in$ V(PB), then u $\in$ V(PB). If, for each propositional variable p^n, we let $V^*(p^n) = \{[w]: w \in V(p^n)\}$, the result can be shown to be a filtration of Υ through Γ, and a "fundamental theorem" for filtrations entails that for each wff B $\in \Gamma$, $V^*(B) = \{[w]: w \in V(B)\}$.

In order to complete the proof of the Subclaim, it is necessary to show that this filtration frame Φ^* is a frame validating each LinP-B theorem; i.e., it is necessary to show that R^* is a transitive, reflexive, and comparable relation on W^*. A simple tense-logic modification of known results for modal logics (cf. [2], Theorems 3.19 and 3.20, pp. 105–106) entails that the accessibility relation R^* for the filtration frame, as defined above, is both reflexive and transitive. It is also comparable. For suppose that for any [w], [u] $\in W^*$, it is not the case that $R^*[w][u]$. But, then, by the definition of R^*, either (a) there is some FB $\in$ Γ such that (u $\in$ V(B) $\vee$ u $\in$ V(FB)) and w $\notin$ V(FB) or (b) there is a PB $\in \Gamma$ such that (w $\in$ V(B) $\vee$ w $\in$ V(PB)) and u $\notin$ V(PB). However, due to the transitivity of the R-relation on W in Φ, it follows that *in any of these cases*, it is not the case that Rwu. But, by the comparability of the R-relation, it follows that Ruw. It can then easily be shown (and is, in fact, a necessary condition on the R^*-relation of a filtration) that $R^*[u][w]$. So the R^*-relation on the frame of the filtration model Υ^* is comparable (connected). This completes the proof sketch of the Subclaim. The consequence is that any nontheorem A of LinP-B is false at some point in a finite model the frame of which validates all theorems of LinP-B, i.e., the frame of which possesses a transitive, reflexive, and comparable temporal accessibility relation (cf. [13], p. 314).

Subclaim 3.4.2 *Any nontheorem* A *of* LinP-B *is false in some denumerable, linear (standard) model.*

Proof Sketch: Proof of this subclaim depends on showing that, for any standard model the underlying frame of which is transitive, reflexive, and comparable, there is an equivalent model (that is, a model that validates exactly the same wffs) the underlying frame of which is *linear*, i.e. possesses an accessibility relation that is transitive, reflexive, comparable, *and antisymmetric* (($\forall$w,$\forall$u $\in$ W)(Rwu $\wedge$ Ruw $\Rightarrow$ w = u)). This latter result is a corollary of the "Bulldozer" theorem of Segerberg ([14], Theorem 1.2, p. 81; [13], pp. 304–305). The colorful terminology derives from the fact that the domain W of a transitive and reflexive (but not necessarily antisymmetric) frame can be partitioned into equivalence classes, which Segerberg calls *clusters*, such that for elements w and u of such a cluster either (both Rwu and Ruw) or w = u. The Bulldozer theorem guarantees that an equivalent model can be produced by "bulldozing" or flattening out these clusters. In the particular case of a "bimodal logic" (a modal logic such as tense logic with two primitive modal operators the accessibility relations for which are the converse of each other) this can be done by, in effect, replacing each point in each cluster with a linearly ordered set of points without first or last members and, then, linearly ordering these linearly ordered sets (see [13], pp. 304–305). In the present case, by Subclaim 3.4.1, we have each nontheorem of LinP-B falsified in some model the underlying frame of which is transitive, reflexive, and comparable and which can be partitioned into a finite number of clusters each containing a finite number of points. Using the Bulldozer theorem we can construct an equivalent linear model by replacing each point in this model by a linearly ordered set of points having the order type of the signed integers ($\omega^* + \omega$) and linearly ordering these resulting sets. Since there will be a finite number of "copies" of the signed integers in the result, the cardinality of this model will be denumerable. Consequently, each nontheorem of LinP-B will not be valid in (i.e., will be false in) a denumerable model the underlying frame of which is linear (transitive, reflexive, antisymmetric, and comparable).

Subclaim 3.4.3 *For each denumerable linear (standard) model of Subclaim 3.4.2, there is an equivalent model of class* $\mathfrak{R}$*; that is, for any such denumerable, linear standard model* $\Upsilon = \langle\langle \mathrm{W},\mathrm{R}\rangle,\mathrm{V}\rangle$ *there is a (Euclidean 1-space) model* $M = (\mathrm{X},<,O,V)$ *of class* $\mathfrak{R}$ *such that, for any tense-logic wff* A, $\mathrm{V}(\mathrm{A}) = \mathrm{W}$ *(wff* A *is true at all* $\mathrm{w} \in \mathrm{W}$ *of* Υ*) iff (with respect to model M)* $V^+(\mathrm{A},\mathrm{X}) = \mathrm{T}$.

Proof Sketch: We begin with a denumerable, linear, and reflexive standard model Υ of the sort constructed in Subclaim 3.4.2 and define, for each point $\mathrm{w}_i \in \mathrm{W}$, an open interval x_i, where x_i can be identified with a nondenumerable set of points characterized by a dense and continuous strict (irreflexive) linear order, without any maximum or minimum elements. We further stipulate that the order type of each such x_i is that of Euclidean 1-space or of the (signed) real numbers, customarily designated 'λ'. Extend this ordering $<$ to the union of the x_i's: for any w_i, $\mathrm{w}_j \in \mathrm{W}$, if $\mathrm{Rw}_i\mathrm{w}_j$ and $\mathrm{w}_i \neq \mathrm{w}_j$, then $(\forall \mathrm{p}, \forall \mathrm{p}' \in \cup x_i)(\mathrm{p} \in x_i \wedge p' \in x_j \Rightarrow \mathrm{p} < \mathrm{p}')$. Because of the ordering of w_i's in the standard model Υ, each w_i has a *unique* successor and a unique predecessor with respect to the reflexive linear ordering R of that model. Consequently, each interval x_i will have a unique successor (predecessor) x_j. Continuity can be restored to the union of the x_i's in two stages: (a) Add, for each x_i that has as its immediate successor interval x_j, a point p such that for all $\mathrm{q} \in x_i$, $\mathrm{q} < \mathrm{p}$, and for all $\mathrm{r} \in x_j$, $\mathrm{p} < \mathrm{r}$. For such a p, $\mathrm{p} \notin (x_i \cup x_j)$, $\mathrm{p} \in Cl(x_i \cup x_j)$, and p is the supremum of x_i and the infimum of x_j. (b) There will now be a finite number of linearly ordered collections of x_i's, each such collection being of order type ($\omega^* + \omega$) with respect to the x_i's that it contains. These collections will themselves be linearly ordered and, if such a collection has a successor (predecessor), that successor (predecessor) will be unique. So, add a further point between any "adjacent" pairs of such collections. (i) The resulting set of x_i's together with added limit points is obviously dense and continuous with respect to strict linear order $<$ as we have defined it. (ii) It also obviously has neither a first nor a last point. (iii) Finally, it contains a denumerable subset dense in it. For each x_i, by stipulation of order type λ, contains such a set; and since there are but a denumerable number of x_i's, the union of the denumerable subsets contained in these will be a denumerable subset contained in the union of the x_i's together with the added limit points. Since conditions (i) through (iii) are individually necessary and jointly sufficient for a totally ordered set's having order type λ, we have the strict linearly and continuously ordered domain X (the union of all the x_i's with the added limit points) of order type λ characterizing a (Euclidean 1-space) model M of class $\mathfrak{R}$.

In order to complete the definition of M, let O be the standard Euclidean topology, formed by arbitrary union and finite intersection from the base of *all* open intervals on X. Each x_i will be a (nonempty) member of O. Finally, define a function m the domain of which the power set $\mathcal{P}$W of the domain of the standard model Υ and the range of which is O such that, for any U $\in \mathcal{P}$W, m(U) = $\bigcup\{x_i\colon \mathrm{w}_i \in \mathrm{U}\}$. Then, define the proto-evaluation function V of M for each propositional variable p^n be letting $V(\mathrm{p}^n) = m(\mathrm{V}(\mathrm{p}^n))$ (where, of course, V is the valuation function of the standard model Υ with which we began). The definition of V is then recursively extended to other tense-logic wffs in the manner described in Section 1.

It can then be proved inductively, although the details of the proof are a bit tedious, that for any tense-logic wff A, $(ClV(\mathrm{A}))^\circ = (Clm(\mathrm{V(A)}))^\circ$.[10] Suppose, then, that some wff A is valid in the standard model Υ. It follows that V(A) = W, that m(A) is the union of all the open intervals x_i with which we began the construction, that $(Clm(\mathrm{V(A)}))^\circ = \mathrm{X}$, that $(ClV(\mathrm{A}))^\circ = \mathrm{X}$, that $V^+(\mathrm{A,X}) = \mathrm{T}$ (by Claim 1.2 of Section 1), and that A is true in model M. Conversely, if A is true in model M ($V^+(\mathrm{A,X}) = \mathrm{T}$), it follows (by Claim 1.2 again) that $(ClV(\mathrm{A}))^\circ = \mathrm{X}$ and that $(Clm(\mathrm{V(A)}))^\circ = \mathrm{X}$. But, unless m(V(A)) is the union of *all* the x_i's, this will not be the case. But, then, for *every* $\mathrm{w}_i \in \mathrm{W}$, $\mathrm{w}_i \in \mathrm{V(A)}$, and A is valid in the standard model Υ. This completes the proof sketch of Subclaim 3.4.3.

The proof of Subclaim 3.4 can now be easily completed. By Subclaim 3.4.2, any nontheorem A of LinP-B is false in some denumerable, linear standard model consisting of a finite number of collections of points each collection being of order type $(\omega^* + \omega)$; so A is not valid in such a model. Then, by Subclaim 3.4.3, A is not true in some (Euclidean 1-space) model of class $\mathfrak{R}$, and A is not $\mathfrak{R}$-valid. Claim 3.4, the completeness of LinP-B with respect to the class $\mathfrak{R}$ of Euclidean 1-space models immediately follows.

Claim 3.5 LinP-B *is determined by the class* $\mathfrak{R}$ *of Euclidean 1-space models.*

Proof: This Claim follows directly from Claims 3.3 and 3.4.

4 Conclusion In this concluding section, I reiterate two points, which pertain directly to the present period-based tense logic, and make a programmatic remark. The first point pertains to the behavior of the tense operators in LinP-B, which, as I earlier indicated, must initially seem eccentric. In particular, the validity of the "reflexive" theses A $\supset$ FA and A $\supset$ PA might suggest a nonstandard view of time. However, as I hope I successfully argued in Section 1, the validity of these theses does not derive from any "funny" view of time. Rather, it is a corollary of a fanatically faithful interpretation of a salient feature of this paper's period-based models: the units of semantic evaluation of these models are exclusively (arbitrary unions and finite intersections of) *open* intervals. In fact, I should maintain that a period-based semantics *not* validating these theses must either explicitly or tacitly assume that periods are right-closed in its treatment of the future operator F and left-closed in its treatment of the past operator P.

Second, I would emphasize the earlier claim that the fact that the period-based models of this paper do not assign a truth value to each wff relative to *every* open interval in no way mitigates the classical character of LinP-B, so interpreted. The metatheory of Section 3 confirms, I believe, my claim that the presence of these truth-value gaps is no more "nonclassical" than is the presence of subsets of points in standard, point-based models for tense and modal logics relative to which some wffs are not assigned a single truth value. In his excellent survey article ([1]) on tense-logic Burgess poses a central problem of interpretation for period-based tense logic, which can be paraphrased as follows: "how to make intuitive sense of the notion $x \subseteq V(\mathrm{A})$ of a sentence A being true *with respect to* a time-period x" (p. 126). The treatment of the tense operators in the present period-based logic along with the fact that not all wffs are assigned a truth value relative to *every* time-period allow us to adopt what is perhaps the most "intuitive" answer to this problem of interpretation: for every tense-logic wff A, '$x \subseteq V(\mathrm{A})$' means that A is true throughout x. In the alternative semantics based on regular open sets alluded to in Note 4, there would be no need for the further valuation function V^+. However, in the semantics of this paper, which utilizes *all* open sets, it follows from Claim 1.2 that $V^+(\mathrm{A},x) = \mathrm{T} \Leftrightarrow x \subseteq (ClV(\mathrm{A}))^\circ$ even though it may be the case that $x \not\subseteq V(\mathrm{A})$. Either way, the upshot is that any isolated points of $V(\mathrm{A})$ or of $-V(\mathrm{A})$ are semantically ignored. This feature seems entirely appropriate for a *period*-based semantics.

Finally, the programmatic remark. Period-based models are now typically presented as sets of "primitive" objects with two dyadic relations defined on the set, intuitively, an "inclusion" relation and a temporal precedence relation. Although this approach has been generally fruitful, it can leave one aspect of the "nature" of the periods or intervals ambiguous—whether they are to be conceived as open, closed, semi-open, etc. This feature of intervals, it seems, can be quite tense-logically significant. It is perhaps particularly important in relation to the development of period-based tense logics intended as alternatives to those now "standard" logics whose point-based models capture various structural properties of time. The topological embedding of intervals in point sets should be particularly useful in this enterprise.

NOTES

1. Of course, more directly relevant to this intuitionistic character is the common assumption—*non*classical, I shall argue—that each wff should be assigned a truth value relative to every period.

2. I could here have substituted "countable subset" for "subset" in view of the fact that every open set on the real line can be represented as the union of a finite or denumerable number of disjoint open intervals. See, e.g., [5], pp. 14f.

3. For basic topological concepts see [8], the more compendious [3], or the yet more compendious [19].

4. An anonymous reader of this paper suggests that the semantic models of this paper and the soundness result could be simplified by limiting the range of the proto-evaluation functions in models of class $\mathfrak{R}$ to a proper subset of the opens sets of the standard Euclidean topology, *viz.*, the set of *regular open sets* (the set of open sets x such that $x = (Clx)^\circ$). This is quite correct, for it is known that the set of regular open sets of a space becomes a complete Boolean algebra when the meet

operation is defined as set-theoretic intersection, the complementation operation on r.o. set x is defined as $-Clx$, and the join operation on r.o. sets x and y is defined as $(Cl(x \cup y))^{\circ}$ (see [18], pp. 506–507). Had these operations been assigned to the appropriate propositional connectives, the proto-evaluation and evaluation functions could have been conflated and the soundness proof could have utilized this established result. Perhaps somewhat perversely, then, I have retained the set of *all* open sets as range of the vaulation function and the concomitantly more complicated definition of truth/validity and soundness proof. On either alternative—which are, for practical purposes, equivalent—isolated points of a valuation set and its set-theoretic complement get semantically ignored. The choice offered by the two variants is whether this gets done "lower" in the semantics (at the level of the assignment of operations to the propositional connectives, as in the semantics that would utilize only regular open sets) or "higher" (at the level of the definition of truth and validity by means of separate proto-evaluation and evaluation functions as is done in the present paper).

5. It should perhaps be emphasized that it does not follow from Claim 1.2 that $V^{+}(\mathrm{A},x) = \mathrm{F}$ iff $x \not\subseteq (ClV(\mathrm{A}))^{\circ}$. It does, of course, follow that $V^{+}(\mathrm{A},x) = \mathrm{F}$ iff $x \subseteq -ClV(\mathrm{A})$.

6. I am reminded by an anonymous reader that the past-tense analogues of Axioms 5 and 6, i.e., p $\supset$ Pp and PPp $\supset$ Pp, are derivable as theorems. Thus, of course, they need not be added as axioms to LinP-B.

7. 0, 1, $+$, $\cdot$, and $'$ designate the zero and unit elements and the join, meet, and complementation operations, respectively.

8. This condition is equivalent to the conjunction of the following, considerably simpler conditions: (vii)$'$ $f\mathrm{a}\cdot(f\mathrm{b})' \neq 0 \Rightarrow f\mathrm{b}\cdot(f\mathrm{a})' = 0$; (vii)$''$ $p\mathrm{a}\cdot(p\mathrm{b})' \neq 0 \Rightarrow p\mathrm{b}\cdot(p\mathrm{a})' = 0$.

9. There are consistent tense logics that are not determined, in the sense of this term defined in the text, by any class of standard frames. See, e.g., [16], Section 4. In such a case the underlying (standard) frame of the logic's Henkin model is not a member of the class of standard frames over which all theorems of the logic are valid. Fortunately, LinP-B is not such a logic.

10. The core of the proof is the verification of the following identities: For any U, V $\in \mathcal{P}$W, (a) $(Clm(\mathrm{W} - \mathrm{U}))^{\circ} = (-m(\mathrm{U}))^{\circ} = (Cl{-}Clm(\mathrm{U}))^{\circ}$; (b) $(Clm(\mathrm{U} \cup \mathrm{V}))^{\circ} = (Cl(m(\mathrm{U}) \cup m(\mathrm{V})))^{\circ}$; (c) $(Clm(\mathrm{U} \cap \mathrm{V}))^{\circ} = (Cl(m(\mathrm{U}) \cap m(\mathrm{V})))^{\circ}$; (d) $(Clm(\{\mathrm{w}_i\colon (\exists \mathrm{w}_j \in \mathrm{U})\mathrm{Rw}_i\mathrm{w}_j\}))^{\circ} = \phi m(\mathrm{U}) = (Cl\phi m(\mathrm{U}))^{\circ}$; (e) $(Clm(\{w_i\colon (\exists \mathrm{w}_j \in \mathrm{U})\mathrm{Rw}_j\mathrm{w}_i\}))^{\circ} = \pi m(\mathrm{U}) = (Cl\pi m(\mathrm{U}))^{\circ}$.

REFERENCES

[1] Burgess, J. P., "Basic tense logic," pp. 89–133 in *Handbook of Philosophical Logic*, Vol. II, Extensions of Classical Logic, Syntheses Library, Vol. 165, Reidel, Dordrecht, 1983.

[2] Chellas, B. F., *Modal Logic: An Introduction*, Cambridge University Press, Cambridge, 1980.

[3] Dugundji, J., *Topology*, Allyn and Bacon, Boston, 1966.

[4] Goldblatt, R., *Topoi: The Categorical Analysis of Logic*, Studies in Logic and the Foundations of Mathematics, Vol. 98, North-Holland, Amsterdam, 1979.

[5] Hartman, S. and J. Mikusinski, *The Theory of Lebesgue Measure and Integration*, trans. L. F. Boron, Pergamon Press, New York, 1961.

[6] Hughes, G. E. and M. J. Cresswell, *A Companion to Modal Logic*, Methuen, London and New York, 1984.

[7] Humberstone, I. L., "Interval semantics for tense logics," *Journal of Philosophical Logic*, vol. 8 (1979), pp. 171–196.

[8] Kuratowski, K., *Topology*, 2 Vols., trans. J. Jaworowski, Academic Press, New York and London, 1966.

[9] McKinsey, J. C. C. and A. Tarski, "The algebra of topology," *Annals of Mathematics*, vol. 45 (1944), pp. 141–191.

[10] McKinsey, J. C. C. and A. Tarski, "On closed elements in closure algebras," *Annals of Mathematics*, vol. 47 (1946), pp. 122–162.

[11] McKinsey, J. C. C. and A. Tarski, "Some theorems about the sentential calculi of Lewis and Heyting," *The Journal of Symbolic Logic*, vol. 13 (1948), pp. 1–15.

[12] Röper, P., "Intervals and tenses," *Journal of Philosophical Logic*, vol. 9 (1980), pp. 451–469.

[13] Segerberg, K., "Modal logics with linear alternative relations," *Theoria*, vol. 36 (1970), pp. 301–322.

[14] Segerberg, K., *An Essay in Classical Modal Logic*, 3 Vols., Uppsala University, Uppsala, 1971.

[15] Tarski, A., "Der Aussagenkalkuel und die Topologie," *Fundamenta Mathematicae*, vol. 31 (1938), pp. 103–134.

[16] Thomason, S. K., "Semantic analysis of tense logics," *The Journal of Symbolic Logic*, vol. 37 (1972), pp. 150–158.

[17] van Benthem, J. F. A. K., *The Logic of Time*, Synthese Library, Vol. 156, Reidel, Dordrecht, 1983.

[18] van Mill, J., "An introduction to $\beta\omega$," pp. 503–567 in *Handbook of Set-Theoretic Topology*, eds. K. Kunen and J. E. Vaughan, North-Holland, Amsterdam, 1984.

[19] Whyburn, G. and E. Duda, *Dynamic Topology*, Springer-Verlag, New York, Heidelberg, and Berlin, 1979.

Department of Philosophy
Arizona State University
Tempe, Arizona 85287