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# Some Notes on Iterated Forcing With $2^{\aleph_0} > \aleph_2$

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***Introduction*** By Solovay and Tenenbaum ([7]) and Martin and Solovay ([3]) we can iterate c.c.c. forcing with finite support. There have been many works on iterating more general kinds of forcings adding reals (e.g., [4]), getting generalizations of $MA$, and so on, but we were usually restricted to $2^{\aleph_0} = \aleph_2$. Note only this is a defect per se, but there are statements that we think are independent but which follow from $2^{\aleph_0} \leq \aleph_2$.

Some time ago Groszek and Jech (in [2]) got $2^{\aleph_0} > \aleph_2 + MA$ for a family of forcing wider than c.c.c. but for $\aleph_1$ dense sets only.

In Section 1 we generalize RCS iteration to $\kappa$-RS iteration.

In Section 2 we combine from [4], X, XII (i.e., RS iteration and some properness and semicompleteness) with Gitik's definition of order ([1]). (He uses Easton support, each $Q$ ($\{2\}$, $\kappa_i$) -complete where for important $i$, $\kappa_i = i$. His main aim was properties of the club filter on inaccessible: precipitousness and approximation to saturation.)

In Section 3 we get $MA$-like consequences (strongest-from supercompact). In Section 4 we get that, e.g., for Sacks forcing (though not included), and in the models we naturally get, for every $\aleph_1$ dense subset there is a directed set intersecting all of them.

In Section 5 we solve the second Abraham problem.

The main result was announced (somewhat inaccurately) in [6].

***1 On $\kappa$-revised support iteration*** We redo [4], Ch. X, Section 1, with "$< \kappa$" instead countable.

Remarks 1.0:

(1) Now if $P_1 = P_0 * \underset{\sim}{Q}_0$, $\underset{\sim}{q}_1$ a $P_1$-name, $G_0 \subseteq P_0$ generic over $V$, then in $V[G_0]$, $\underset{\sim}{q}_1$ can be naturally interpreted as a $Q_0$-name, called $\underset{\sim}{q}_1/G_0$,

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which has a $P_0$-name $\underset{\sim}{q_1}/G_0$, *or* $q_1/P_0$; but usually we do not care to make those fine distinctions.

(2) Using $\bar{Q} = \langle P_i, \underset{\sim}{Q_i}: i < \alpha\rangle$, $P_\alpha$ will mean $R\mathrm{Lim}\ \bar{Q}$ (see Definition 1.2).

(3) If $D$ is a filter on a set $J$, $D \in V$, $V \subseteq V^\dagger$ (e.g., $V^\dagger = V[G]$) then in an abuse of notation, $D$ will denote also the filter it generates (on $J$) in $V^\dagger$.

(4) $D_\kappa$ is the closed unbounded filter on $\kappa$.

**Definition 1.1** We define the following notions by *simultaneous induction on* $\alpha$:

(A) $\bar{Q} = \langle P_i, \underset{\sim}{Q_i}: i < \alpha\rangle$ is a $\kappa$-RS iteration (RS stands for revised support)

(B) a $\bar{Q}$-named ordinal (or $[j,\alpha)$-ordinal)

(C) a $\bar{Q}$-named atomic condition (or $[j,\alpha)$-condition), and we define $\underset{\sim}{q} \restriction \xi$, $\underset{\sim}{q} \restriction \{\xi\}$ for a $\bar{Q}$-named atomic $[j,\alpha)$-condition $\underset{\sim}{q}$ and ordinal $\xi$.

(D) the $\kappa$-RS limit of $\bar{Q}$, $R\mathrm{Lim}_\kappa \bar{Q}$ which satisfies $P_i <\circ R\mathrm{Lim}_\kappa \bar{Q}$ for every $i < \kappa$ and we define $p \restriction \beta$ for $p \in R\mathrm{Lim}_\kappa \bar{Q}$, $\beta < \alpha$. (We may omit $\kappa$.)

(A) We *define* "*$\bar{Q}$ is a $\kappa$-*RS *iteration*"

$\alpha = 0$: no condition.

$\alpha$ is limit: $\bar{Q} = \langle P_i, \underset{\sim}{Q_i}: i < \alpha\rangle$ is a $\kappa$-RS iteration iff for every $\beta < \alpha$, $\bar{Q} \restriction \beta$ is one.

$\alpha = \beta + 1$: $\bar{Q}$ is an RCS iteration iff $\bar{Q} \restriction \beta$ is one, $P_\beta = R\mathrm{Lim}_\kappa(\bar{Q} \restriction \beta)$, and $\underset{\sim}{Q_\beta}$ is a $P_\beta$-name of a forcing notion.

(B) We define: $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j,\beta)$-ordinal above $r$. It means $r \in \bigcup_{i<\gamma} P_i$ (where $\gamma = \mathrm{Min}\{\beta, l(\bar{Q})\}$) and $\underset{\sim}{\zeta}$ is a function such that:

(1) $\mathrm{Dom}(\underset{\sim}{\zeta})$ is a subset of $\bigcup \{P_i: i < \gamma\}$

(2) for every $q \in \mathrm{Dom}(\underset{\sim}{\zeta})$ for some $i$, $\{q, r\} \subseteq P_i$ and $P_i \models r \leq q$.

(3) for every $q_1, q_2 \in \mathrm{Dom}(\underset{\sim}{\zeta})$, *if* for some $i < \alpha \{q_1, q_2\} \subseteq P_i$ and in $P_i$ they are compatible *then* $\underset{\sim}{\zeta}(q_1) = \underset{\sim}{\zeta}(q_2)$.

(4) *if* $q \in \mathrm{Dom}(\underset{\sim}{\zeta})$, $q \in \bigcup_{i<\alpha} P_i$ and $i = \tilde{i}(q)$ is the minimal $i$ such that $q \in P_i$ *then* $\underset{\sim}{\zeta}(q)$ is an ordinal $\geq i, j$ but $< \gamma, \beta$.

We define "$\underset{\sim}{\zeta}$ is a $\bar{Q}$-named ordinal above $r$" as "$\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[0, l(\bar{Q}))$ ordinal above $r$". We omit "above $r$" when $r = \varnothing$ (i.e., we omit demand (2)).

(C) We say "$\underset{\sim}{q}$ is a $\bar{Q}$-named atomic $[j,\alpha)$-condition above $r$" if

(1) $\underset{\sim}{q}$ is a pair of functions $(\underset{\sim}{\zeta_q}, \underset{\sim}{cnd_q})$ with a common domain $D = \underset{\sim}{D_q}$:

(2) $\underset{\sim}{cnd_q}$ satisfies (1) and (3) above and:

(3) $\underset{\sim}{\xi_q}$ is a $(\bar{Q} \restriction \alpha)-$named $[j,\alpha)-$ordinal above $r$

(4) for $p \in \underset{\sim}{D_q}$, $cnd_q(p)$ is a $P_{\underset{\sim}{\zeta_q}(p)}$-name of a member of $Q_{\underset{\sim}{\zeta_q}(p)}$. We omit "$[j,\alpha)$-" when $j = 0$, $\alpha = \ell(\bar{Q})$ and we omit "above $r$" when $r = \varnothing$. If $l(\bar{Q}) > \alpha$ we mean $\bar{Q} \restriction \alpha$. We define $\underset{\sim}{q} \restriction \xi$ as $(\underset{\sim}{\zeta_q} \restriction D_1, \underset{\sim}{cnd_q} \restriction D_1)$ where $D_1 = \{p \in \underset{\sim}{D_q}: \underset{\sim}{\zeta_q}(p) < \xi\}$. We define $\underset{\sim}{q} \restriction \{\xi\}$ as $(\underset{\sim}{\zeta_q} \restriction D_2, \underset{\sim}{cnd_q} \restriction D_2)$ where $D_2 = \{p \in \underset{\sim}{D_q}: \underset{\sim}{\zeta_q}(p) = \xi\}$.

(D) We define $R\mathrm{Lim}_\kappa\ \bar{Q}$ as follows:

if $\alpha = 0$: $R\mathrm{Lim}_\kappa\ \bar{Q}$ is trivial forcing with just one condition, $\varnothing$.

if $\alpha > 0$: we call $\underset{\sim}{q}$ an atomic condition of $R\mathrm{Lim}_\kappa \bar{Q}$, if it is a $\bar{Q}$-named atomic condition.

The set of conditions in $R\mathrm{Lim}_\kappa \bar{Q}$ is

$\{p$: $p$ a set of $\lambda$ atomic conditions for some $\lambda < \kappa$; and for every $\beta < \alpha$, $p \restriction \beta =^{dsf} \{r \restriction \beta$: $r \in p\} \in P_\beta$, and $p \restriction \beta \Vdash_{P_\beta}$ "the set $\{r \restriction \{\beta\}$: $r \in p\}$ has an upper bound in $\underset{\sim}{Q}_\beta$"$\}$.

We define $p \restriction \beta = \{r \restriction \beta$: $r \in p\}$.

The order is inclusion.

Now we have to show $P_\beta <\circ R\mathrm{Lim}_\kappa \bar{Q}$ (for $\beta < \alpha$). Note that any $\bar{Q}$-named $[j, \beta)$-ordinal (or condition) is a $\bar{Q}$-named $[j, \alpha)$-ordinal (or condition), and see Claim 1.4(1) below.

Remark 1.1A: Note that for the sake of 1.5(3) we allow $\kappa$ to be not a cardinal and then we really use $|\kappa|^+$.

Remark 1.1B: We can obviously define $\bar{Q}$-named sets; but for conditions (and ordinals for them) we want to avoid the vicious circle of using names which are interpreted only after forcing with them below.

**Definition 1.2**

(1) Suppose $\bar{Q}$ is a $\kappa$-RS iteration, $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j, \alpha)$-ordinal above $r$, $\beta \leq \alpha$, $r \in G \in \mathrm{Gen}(\bar{Q})$ (see Definition (3) below). We define $\underset{\sim}{\zeta}[G]$ by:

(i) $\underset{\sim}{\zeta}[G] = i$ if for some $\gamma \leq \beta > \alpha$ and $p \in Dom(\underset{\sim}{\zeta}) \cap G_\gamma$ we have $\underset{\sim}{\zeta}(p) = i$.

(ii) otherwise (i.e., $G \cap \mathrm{D}_{\underset{\sim}{\zeta}} = \phi$ or $r \notin G$) $\underset{\sim}{\zeta}[G]$ is not defined.

For a $\bar{Q}$-named $[j, \alpha)$-condition above $r, \underset{\sim}{q}$, we defined $\underset{\sim}{q}[G]$ similarly.

(2) We denote the set of $G \subseteq \bigcup_{i<\alpha} P_{i+1}$ such that $G \cap P_{i+1}$ is generic over $V$ for each $i < \alpha$ by $\mathrm{Gen}(\bar{Q})$.

(3) For $\underset{\sim}{\zeta}$ a $\bar{Q}$-named $[j, \alpha)$-ordinal (above $r$) and $q \in \bigcup_\alpha P_i$ let $q \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} = \xi$" if for every $G \in \mathrm{Gen}(\bar{Q})$ such that $r \in G$: $q \in G \Rightarrow \underset{\sim}{\zeta}[G] = \xi$.

Remark 1.3: From where is $G$ taken in (2), (3)? e.g., $V$ is a countable model of set theory, $G$ taken from the "true" universe.

Now we point out some properties of $\kappa$-RS iteration.

Claim 1.4: Let $\bar{Q} = \langle P_i \underset{\sim}{Q}_i$: $i < \alpha\rangle$ be a $\kappa$-RS iteration, $P_\alpha = R\mathrm{lim}_\kappa \bar{Q}$.

(1) *If* $\beta < \alpha$ then: $P_\beta \subseteq P_\alpha$; for $p_1, p_2 \in P_\beta$, $P_\beta \models p_1 \leq p_2$ iff $P_\alpha \models p_1 \leq p_2$: and $P_\beta <\circ P_\alpha$. Moreover, if $q \in P_\beta$, $p \in P_\alpha$, then $q, p$ are compatible iff $q$, $p \restriction \beta$ are compatible.

(2) If $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j, \alpha)$-ordinal $G$, $G' \in \mathrm{Gen}(\bar{Q})$ $G \cap P_\xi = G' \cap P_\xi$ and $\underset{\sim}{\zeta}[G] = \xi$ then $\underset{\sim}{\zeta}[G'] = \xi$; hence we write $\underset{\sim}{\zeta}[G \cap P_\xi] = \xi$.

(3) *If* $\underset{\sim}{\beta}, \underset{\sim}{\gamma}$ are $\bar{Q}$-named $[j, l(\bar{Q}))$-ordinals, then $\mathrm{Max}\{\underset{\sim}{\beta}, \underset{\sim}{\gamma}\}$ (defined naturally) is a $\bar{Q}$-named $[j, l(\bar{Q}))$-ordinal.

(4) If $\alpha = \beta_0 + 1$, in Definition 1.1(D), in defining the set of elements of $P_\alpha$ we can restrict ourselves to $\beta = \beta_0$. Also in such a case, $P_\alpha =$

$P_{\beta_0} * \underset{\sim}{Q}_{\beta_0}$ (essentially). More exactly, $\{p \cup \{\underset{\sim}{q}\}: p \in P_{\beta_0}, \underset{\sim}{q}$ a $P_{\beta_0}$-name of a member of $\underset{\sim}{Q}_{\beta_0}\}$ is a dense subset of $P_\alpha$, and the order $p_1 \cup \{\underset{\sim}{q}_1\} \le p_2 \cup \{\underset{\sim}{q}_2\}$ iff $p_1 \le p_2$, $p_2 \Vdash \underset{\sim}{q}_1 \le \underset{\sim}{q}_2$ is equivalent to that of $\tilde{P}_\alpha$; i.e., we get the same Boolean algebra.

(5) The following set is dense in $P_\alpha$: $\{p \in P_\alpha$; for every $\beta < \alpha$, if $r_1, r_2 \in p$, then $\Vdash_{P_\beta}$ "if $r_1 \restriction \{\beta\} \neq \emptyset$, $r_2 \restriction \{\beta\} \neq \emptyset$ then they are equal"$\}$.

(6) $|P_\alpha \le (\Sigma_{i<\alpha} 2^{P_i})^\kappa$, for limit $\alpha$.

(7) If $\Vdash_{P_i}$"$|\underset{\sim}{Q}_i| \le \lambda$", $\alpha$ a cardinal, then $|P_{i+1}| \le 2^{|P_i|} + \lambda$ (assuming, e.g., that the set of elements of $G$ is $\lambda$).

*Proof:* By induction on $\alpha$.

**Lemma 1.5** *The Iteration Lemma*

(1) *Suppose $F$ is a function, then for every ordinal $\alpha$ there is one and only one $\kappa$-RS-iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i: i < \alpha^\dagger\rangle$, such that:*

(a) *for every $i$, $\underset{\sim}{Q}_i = F(\underset{\sim}{Q} \restriction i)$,*

(b) $\alpha^\dagger \le \alpha$,

(c) *either $\alpha^\dagger = \alpha$ or $F(\bar{Q})$ is not an $(R\mathrm{Lim}_\kappa \bar{Q})$-name of a forcing notion.*

(2) *Suppose $\bar{Q}$ is a $\kappa$-RS-iteration, $\alpha = l(\bar{Q})$, $\beta < \alpha$, $G_\beta \subseteq P_\beta$ is generic over $V$. Then in $V[G_\beta]$, $\bar{Q}/G_\beta = \langle P_i/G_\beta, \tilde{Q}_i: \beta \le i < \kappa\rangle$ is a $\kappa$-RS-iteration and $R\mathrm{Lim}_\kappa$ $\tilde{Q} = P_\beta$ $*(R\mathrm{Lim}$ $\bar{Q}/G_\beta)$ (essentially).*

(3) *The Associative Law: If $\alpha_\xi$ $(\xi \le \xi(0))$ is increasing and continuous, $\alpha_0 = 0$; $\bar{Q} = \langle P_i, \tilde{Q}_i: i < \alpha_{\xi(0)}\rangle$ is a $\kappa$-RS-iteration, $P_{\xi(0)} = R\mathrm{Lim}_\kappa \bar{Q}$; then so are $\langle P_{\alpha(\xi)}, P_{\alpha(\xi+1)}/P_{\alpha(\xi)}: \xi < \xi(0)\rangle$ and $\langle P_i/P_{\alpha(\xi)}, \underset{\sim}{Q}_i: \alpha(\xi) \le i < \alpha(\xi+1)\rangle$; and vice versa.*

Remark 1.5A: In (3) we can use $\alpha_\xi$'s which are names.

*Proof:* (1) Easy.

(2) Pedantically, we should formalize the assertion as follows:

**(*)** There is a function $F$ (= a definable class) such that for every $\kappa$-RS-iteration $\bar{Q}$ and $l(\bar{Q}) = \alpha$, and $\beta < \alpha$, $F_0(\bar{Q}, \beta)$ is a $P_\beta$-name of $\underset{\sim}{\bar{Q}}^\dagger$ such that:

(a) $\Vdash_{P_\beta}$ "$\underset{\sim}{\bar{Q}}^\dagger$ is a $\kappa$-RS-iteration of length $\alpha - \beta$".

(b) $P_\beta * (R\mathrm{Lim}_\kappa \underset{\sim}{\bar{Q}}^\dagger)$ is equivalent to $P_\alpha = R\mathrm{Lim}_\kappa$ $\bar{Q}$, by $F_1(\bar{Q}, \beta)$ (i.e., $F_1(\bar{q}, \beta)$ is an isomorphism between the corresponding completions to Boolean algebras)

(c) if $\beta \le \gamma \le \alpha$ $\Vdash_{P_\beta}$ "$F_0(\bar{Q} \restriction \gamma, \beta) = F(\bar{Q}, \beta) \restriction (\gamma - \beta)$" and $F_1(\bar{Q}, \beta)$ extends $F_1(\bar{Q} \restriction \gamma, \beta)$ and $F_1(\bar{Q} \restriction \gamma, \beta)$ transfer the $P_\gamma$-name $\underset{\sim}{Q}_\gamma$ to a $P_\beta$-name of a $(R\mathrm{Lim}_\kappa(\underset{\sim}{\bar{Q}}^\dagger \restriction (\gamma - \beta))$-name of $\underset{\sim}{Q}^\dagger_{\gamma-\beta}$ (where $\underset{\sim}{Q}^\dagger_{\gamma-\beta} = \langle \underset{\sim}{Q}^\dagger_{\beta+i}: i < \gamma - \beta\rangle$).

The proof is the induction on $\alpha$, and there are no special problems.

(3) Again, pedantically the formulation is

**(**)** For $\bar{Q}$ is an RCS-iteration, $l(\bar{Q}) = \alpha_{\xi(0)}$, $\bar{\alpha} = \langle \alpha_\xi: \xi \le \xi(0)\rangle$ increasing continuous, $F_3(\bar{Q}, \bar{\alpha})$ is a $\kappa$-RS-iteration $\bar{Q}^\dagger$ of length $\alpha_{\xi(0)}$ such that

(a) $F_4(\bar{Q}, \bar{\alpha})$ is an equivalence of the forcing notions $R\mathrm{Lim}_\kappa \bar{Q}$. $R\mathrm{Lim}_\kappa \bar{Q}^\dagger$.

(b) $F_3(\bar{Q} \restriction \alpha_\xi, \alpha \restriction (\zeta + 1)) = F_3(\bar{Q}), \bar{\alpha}) \restriction \zeta$
(c) $\underset{\sim}{Q}_\xi^\dagger$ is the image by $F_4(\bar{Q} \restriction \alpha_\xi, \bar{\alpha} \restriction (\xi + 1))$ of the $P_{\alpha_\xi} = R\mathrm{Lim}_\kappa(\bar{Q} \restriction \alpha_\xi)$-name $F_0(\bar{Q} \restriction \alpha_{\xi+1}, \alpha_\xi)$.

The proof again poses no special problems.

Claim 1.6: Suppose we add in Definition 1.1(B) also:

(5) if $\alpha$ is inaccessible, and for some $\beta < \alpha$ for every $\gamma$ satisfying $\beta \leq \gamma < \alpha$, $\Vdash_{P_\beta}$ "$|P_\gamma/P_\beta| < \alpha$" then $(\exists\beta < \alpha)$ $[\mathrm{Dom}\ \underset{\sim}{\zeta} \subseteq P_\beta]$.

Then nothing changes in the above (only we have to prove everything by simultaneous induction on $\alpha$), and if $\lambda$ is an inaccessible cardinal $> \alpha$ and $|P_i| < \lambda$ for every $i < \lambda$ and $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \lambda \rangle$ is a $\kappa$-RS iteration, then

(1) every $\bar{Q}$-named ordinal is in fact a $(\bar{Q} \restriction i)$-named ordinal for some $i < \alpha$,
(2) like (1) for $\bar{Q}$-named conditions.
(3) $P_\kappa = \bigcup_{i<\kappa} P_i$.
(4) if $\kappa$ is a Mahlo cardinal then $P_\lambda$ satisfies the $\lambda$-c.c. (in a strong way).

## 2 *The κ-finitary revised support*

We deal with forcing notions $Q$ satisfying:

**Definition 2.1** Let $\gamma$ be an ordinal, $S \subseteq \{2\} \cup \{\lambda : \lambda \text{ a regular cardinal}\}$. Now $Q$ satisfies $(S, \gamma) - Pr_1$ if

(i) $Q = (|Q|, \leq, \leq_0)$
(ii) as a forcing $Q = (|Q|, \leq)$
(iii) $\leq_0$ is a partial order
(iv) $[p \leq_0 q \Rightarrow p \leq q]$
(v) *for every* cardinal $\kappa \in S$ and $Q$-name $\tau$, such that $\Vdash_Q$ "$\underset{\sim}{\tau} \in \kappa$" and $p \in Q$ *for some* $q \in Q$, $l \in \kappa$, $p \leq_0 q$ and $q \Vdash_Q$ "if $\kappa = 2$, $\tau = l$ and if $\kappa \geq \aleph_0$, $\underset{\sim}{\tau} \leq l$"
(vi) for each $q \in Q$ in the following game player I has a winning strategy: for $i < \gamma$ player I chooses $p_{2i} \in Q$ such that $q \leq_0 p_{2i}$: $\wedge \bigwedge_{j<2i} p_j \leq_0 p_{2i}$ and then player II chooses $p_{2i+1} \in Q$, $p_{2i} \leq_0 p_{2i+1}$.
Player I loses if he has sometimes no legal move which can occur in limit stages only.
Let $(S, \gamma) - Pr_1^-$ means $(\{\kappa\}, \gamma) - Pr_1$ for every $\kappa \in S$.

Fact 2.2:

(1) If $\kappa < \gamma_1, \gamma_2 < \kappa^+$ then $(S, \gamma_1) - Pr_1$ is equivalent to $(S, \gamma_2) - Pr_1$.
(2) If $\kappa + 1 \leq \gamma < \kappa^+$ and $\Box_\kappa$ (i.e., there is a sequence $\langle C_\delta : \delta < \kappa^+ \rangle$, $C_\delta \subseteq \delta$ closed unbounded) $[\delta_1 \in C_\delta, \delta_1 = \sup \delta_1 \cap C_\delta \rightarrow C_{\delta_1} = C_\delta \cap \delta_1]$ and $Q$ satisfies $(S, \gamma) - Pr_1$ then $Q$ satisfies $(S, \kappa^+) - Pr_1$.
(3) If $Q$ satisfies $(S, \gamma) - Pr_1$, $\lambda \leq \gamma$, and $\lambda \in S$ then in $V^Q$ $\lambda$ is still a regular cardinal and when $\lambda = 2$, $Q$ does not add bounded subsets to $\gamma$.

(4) If $Q$ satisfies $(S,\gamma) - Pr_1$, $\lambda \in S$, $\lambda$ regular, and for every regular $\mu$, $\gamma \leq \mu < \lambda \Rightarrow \Vdash_Q$ "$\mu$ is not regular" (e.g., $[\gamma,\lambda)$ contains no regular cardinal) then $\lambda$ is regular in $V^Q$.

*Proof:* Straightforward.

**Definition 2.3** $(S,<\kappa) - Pr_1$, will mean $(S,\gamma) - Pr_1$ for every $\gamma < \kappa$.
Fact 2.4: The following three conditions on forcing notion $Q$, a set $S \subseteq \{2\} \cup \{\lambda: \lambda$ a regular cardinal$\}$ and regular ordinal $\kappa$ are equivalent:

(a) there is $Q' = (Q', \leq, \leq_0)$ such that $(Q',\leq)$, $(Q,\leq)$ are equivalent and $Q'$ satisfies $(S,\kappa) - Pr_1$.

(b) for each $p \in Q$, in the following game (which last $\kappa$ moves) player II has a winning strategy:

*in the* $i$th move player I chooses $\lambda_i \in S$ and a $Q$-name $\tau_i$ of an ordinal $< \lambda_i$ then player II chooses an ordinal $\alpha_i < \lambda_i$.

In the end player II wins if for every $\alpha < \kappa$ there is $p_\alpha \in Q$, $p \leq p_\alpha$ such that for every $i < \alpha$ $p_\alpha \Vdash$ "either $\lambda_1 = 2_i$, $\tau_i = \alpha_i$ or $\lambda_i \geq \aleph_0$ $\tau_i < \alpha_i$".

(c) like (a) but moreover $(Q,\leq_0)$ is $\kappa$-complete.

*Proof:* (c) $\Rightarrow$ (a): trivial.

*Proof:* (a) $\Rightarrow$ (b): Choose $q \in Q'$ which is above $p$. We describe a winning strategy for player II: he plays on the side a play (for $q$) of the game from 2.1 (vi) where he uses a winning strategy (whose existence in guaranteed by (a)). In step $i$ of the play (for 4.2(b)) he already has the initial segment $\langle p_j: j < 2i\rangle$ of the play for 2.1(vi). If player II plays $\lambda_i, \underset{\sim}{\tau}_i$ in the actual game, he plays $p_{2i} \in Q'$ in the simulated play by the winning strategy of player I there and then he chooses $p_{2i+1}$, $p_{2i} \leq_0 p_{2i+1} \in Q'$, which forced the required $\alpha_i$ (exists by 2.1(v)) and then plays $\alpha_i$ in the actual play.

*Proof:* (b) $\Rightarrow$ (c): Find winning strategy for player II in the game from 2.9(b). We define $Q'$: $Q' = \{(p, \langle\lambda_i, \underset{\sim}{\tau}_i, \alpha_i: i < \xi\rangle): p \in Q$, and $\langle\lambda_i, \underset{\sim}{\tau}_i, \alpha_i: i < \alpha\rangle$ is an initial segment of a play of the game from 2.4(b) for $p$ in which II uses his winning strategy.

The order $\leq_0$ is:

$$(p, \langle\lambda_i, \underset{\sim}{\tau}_i, \alpha_i: i < \xi\rangle) \leq_0 (p', \langle\lambda_i', \underset{\sim}{\tau}_i', \alpha_i': i < \xi'\rangle)$$

iff (both are in $Q'$) and

$$Q \models p = p',\ \xi \leq \xi', \text{ and for } i < \xi$$
$$\lambda_i = \lambda_i',\ \underset{\sim}{\tau}_i = \underset{\sim}{\tau}_i',\ \alpha_i = \alpha_i'$$

and the order $\leq$ on $Q'$ is

$$(p, \langle\lambda_i, \underset{\sim}{\tau}_i, \alpha_i: i < \xi\rangle) \leq (p', \langle\lambda_i', \underset{\sim}{\tau}_i', \alpha_i': i < \xi'\rangle)$$

iff (both are in $Q'$ and) $Q \models p \leq p'$. Moreover, $p' \Vdash_Q$ "$\lambda_i = 2$, $\underset{\sim}{\tau}_i = \alpha_i$ or $\lambda_i \geq \aleph_0$, $\underset{\sim}{\tau}_i < \alpha_i$" for $i < \xi$.

The checking is easy.

**Definition 2.5**

(1) Let $\mathrm{Gen}(\bar{Q}) = \left\{G\colon G \subseteq \bigcup_{i<\alpha} P_i \text{ is directed, } G \cap P_i \text{ generic over } V \text{ for } i < \alpha\right\}$. Let $\mathrm{Gen}^r(\bar{Q}) = \left\{G\colon \text{for some (set) forcing notion } P^*, \bigwedge_{i<\alpha} P_i <\circ P^* \text{ and } G^* \subseteq P^* \text{ generic over } V \text{ and } G = G^* \cap \bigcup_{i<\alpha} P_i\right\}$.

(2) If $\bar{Q} = \langle P_i\colon i < \alpha\rangle$ or $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i\colon i < \alpha\rangle$ $P_i$ is $<\circ$-increasing we define a $\bar{Q}$-name $\tau$ almost as we define $\left(\bigcup_{i<\alpha} P_i\right)$-names, but we do not use maximal antichains of $\bigcup_{i<\alpha} P_i$, $G \subseteq \bigcup_{i<\alpha} P_i$:

(*) $\tau$ is a function, $\mathrm{Dom}(\tau) \subseteq \bigcup_{i<\alpha} P_i$ and every directed $G \in \mathrm{Gen}^r(\bar{Q})$, $\tau[G]$ is defined iff $\mathrm{Dom}(\tau) \cap G \neq \emptyset$ and then $\tau[G] \in V[G]$ [where "every $G$..." is taken? e.g., $V$ is countable, $G$ any set from the true universe] and $\tau$ is definable with parameters from $V$ (so $\tau$ is really a first-order formula with the variable $G$ and parameters from $V$).

(3) For $p \in \bar{Q}$ $\left(\text{i.e., } p \in \bigcup_{i<\alpha} P_i\right)$, $\bar{Q}$-names $\tau_0, \ldots, \tau_{n-1}$, and (first-order) formula $\psi$ let $p \Vdash_{\bar{Q}} \psi(\tau_0, \ldots \tau_{n-1})$ means that for every directed $G \in \mathrm{Gen}^r(\bar{G})$, with $p \in G$, $V[G] \models \psi(\tau_0[G], \ldots, \tau_{n-1}[G])$.

(4) A $\bar{Q}$-named $[j, \beta)$-ordinal $\underset{\sim}{\zeta}$ is a $\bar{Q}$-name $\underset{\sim}{\zeta}$ such that if $\underset{\sim}{\zeta}[G] = \xi$ then $j \leq \xi < \beta$ and $(\exists p \in G \cap P_{\xi\cap\alpha})\ p \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} = \xi$" (where $\alpha = l(\bar{Q})$). If we omit "$[j, \beta)$" we mean $[0, l(\bar{Q}))$.

Remark 2.5A: We can restrict in the definition of $\mathrm{Gen}^r(\bar{Q})$ to $P^*$ in some class $K$, and get a $K$-variant of our notions.

Fact 2.6:

(1) For $\bar{Q}$ as above and $\bar{Q}$-named $[j, \beta)$-ordinal $\underset{\sim}{\zeta}$ and $p \in \bigcup_{i<\alpha} P_i$ there are $\xi$, $q$ and $q_1$ such that $p \leq q$, $q \Vdash_{\bar{Q}}$ "$q_1 \in \underset{\sim}{G}$", $q_1 \in P_\xi$, $\xi < \alpha$, and $q_1 \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} = \xi$" or $q \Vdash_{\bar{Q}}$, "$\underset{\sim}{\zeta}$ is not defined".

(2) For $\bar{Q}$ as above, and $\underset{\sim}{\zeta}, \underset{\sim}{\xi}$ $\bar{Q}$-named $[j, \beta)$-ordinals, also $\mathrm{Min}\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}$, $\max\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}$ (naturally defined) are $\bar{Q}$-named $[j, \beta)$-ordinals.

(3) For $\bar{Q}$ as above and $\bar{Q}$-named ordinals $\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n$ and $p \in \bigcup_{i<\alpha} P_i$ there are $\zeta < \alpha$ and $q_0 \in P_\zeta$, $p \leq q$, $q \Vdash_{\bar{Q}}$ "$\zeta = \mathrm{Max}\{\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n\}$". Similarly for Min.

**Definition 2.7** We define and prove by induction on $\alpha$ the following simultaneously:

(A) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i\colon i < \alpha\rangle$ is a $\kappa$-$Sp_2$-iteration.

(B) A $\bar{Q}$-named atomic condition $\underset{\sim}{q}$ (or $[j, \beta)$-condition, $\beta \leq \alpha$) and we define $\underset{\sim}{q} \restriction \xi$, $\underset{\sim}{q} \restriction \{\xi\}$ for a $\bar{Q}$-named atomic condition $\underset{\sim}{q}$ and ordinal $\xi < \alpha$ (or $\bar{Q}$-named ordinal $\underset{\sim}{\xi}$).

(C) If $\underset{\sim}{q}$ is a $\bar{Q}$-named $[j, \beta)$-atomic condition, $\xi < \alpha$, then $\underset{\sim}{q} \restriction \xi$ is a $(\bar{Q} \restriction \xi)$-named $[j, \mathrm{Min}[\beta, \xi])$-condition and $\underset{\sim}{q} \restriction \{\xi\}$ is a $P_\xi$-name of a member of $\underset{\sim}{Q}_\xi$ or undefined (and then it is assigned the value $\emptyset$, the minimal member of $\underset{\sim}{Q}_\xi$ similarly for $\underset{\sim}{\xi}$).

(D) The $\kappa - Sp_2$-limit of $\bar{Q}$, $Sp_2$-$\mathrm{Lim}_\kappa \bar{Q}$, and $p \restriction \xi$ for $p \in Sp_2$-$\mathrm{Lim}_\kappa \bar{Q}$, $\xi$ an ordinal $\le \alpha$ (or $\bar{Q}$-named ordinal).

(E) $P_\beta <\circ Sp_2 \mathrm{Lim}_\kappa \bar{Q}$ (if $\bar{Q} = \langle P_i, Q_i: i < \alpha \rangle$ is a $\kappa$-$Sp_2$-iteration, $\beta < \alpha$, $P_i$, $Q$ satisfying (i)–(iv) of Definition 1.2). In fact $P_\beta \subseteq Sp_2$-$\mathrm{Lim}_\kappa \bar{Q}$ (as models with two partial orders, even compatibility is preserved) and $q \in P_\beta$, $p \in Sp_2 \mathrm{Lim}_\kappa \bar{Q}$ are compatible iff $q$, $p \restriction \beta$ are in $P_\beta$.

*Proof:*

(A) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i: i < \alpha \rangle$ is a $\kappa$-$Sp_2$-iteration if $\bar{Q} \restriction \beta$ is a $\kappa$-$Sp_2$-iteration for $\beta < \alpha$, and if $\alpha = \beta + 1$ then $P_\beta = Sp_2 \mathrm{Lim}_\alpha(\bar{Q} \restriction \beta)$ and $\underset{\sim}{Q}_\beta$ is a $P_\beta$-name of a forcing notion as in Definition 2.1(i)–(iv).

(B) We say $\underset{\sim}{q}$ is a $\bar{Q}$-named atomic $[j, \beta)$-condition when: $\underset{\sim}{q}$ is a $\bar{Q}$-name, and for some $\underset{\sim}{\zeta} = \underset{\sim}{\zeta}_q$ a $\bar{Q}$-named $[j, \beta)$-ordinal $\Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta}$ has a value iff $\underset{\sim}{q}$ has, and if they have then $\underset{\sim}{\zeta} < \mathrm{Min}(\beta, l(\bar{Q}))$, $\underset{\sim}{q} \in \underset{\sim}{Q}_{\underset{\sim}{\zeta}}$". Now $\underset{\sim}{q} \restriction \xi$ will have a value iff $\underset{\sim}{\zeta}_q$ has a value $< \xi$ and then its value is the value of $\underset{\sim}{q}$. Lastly, $\underset{\sim}{q} \restriction \{\xi\}$ will have a value iff $\underset{\sim}{\zeta}_q$ has value $\xi$ and then its value is the value of $\underset{\sim}{q}$ (similarly for $\underset{\sim}{\xi}$).

(C) Left to the reader.

(D) We are defining $Sp_2 \mathrm{Lim}_\kappa \bar{Q}$. It is a triple $P_\alpha = (|P_\alpha|, \le, \le_0)$ where

(a) $|P_\alpha| = \{\{\underset{\sim}{q}_i: i < i(*)\}; i(*) < \kappa$, each $\underset{\sim}{q}_i$ is a $\bar{Q}$-named atomic condition, and for every $\xi < \alpha$, $\Vdash_{P_\xi}$ "$\{\underset{\sim}{q}_i^e \restriction [\xi\}: i < i^*\}$ has an $\le_0$-upper bound in $\underset{\sim}{Q}_\xi$"$\}$.

(b) $P_\alpha \models p_1 \le_0 p_2$ iff for every $\zeta < \alpha \Vdash_{P_\zeta} \{q_i^l \restriction \{\zeta\}: i < i^l(*)\}$ are equal for $l = 1, 2$ or for some $i < i^2(*)$ for every $j_1 < i^1(*) \models q^1_{\gamma_1}\ \underset{\sim}{Q}_\zeta \models q_{j_1} \le_0 q_i^2$ where $p_l = \{q_i^l: i < i^l(*)\}$

(c) $P_\alpha \models p^1 \le p^2$ iff:

(i) for every $\zeta < \alpha$ $(p^2 \restriction \zeta) \Vdash_{P_\zeta}$ "$p^1 \restriction \{\zeta\}, p^2 \restriction \{\zeta\}$ are equal as subsets of $\underset{\sim}{Q}_\zeta$ (remember (F)) or for some $i < i^2(*)$ for every $j < i^1(*) \Vdash_{P_\zeta}$" $\underset{\sim}{Q}_\zeta \models \underset{\sim}{q}_j^1 \le q_i^2$" where $p^l = \{q_i^l: i < i^l(*)\}$

(ii) for some $n < \omega$ and $\bar{Q}$-named ordinals $\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n$ for each $\zeta < l(\bar{Q})$: $p_2 \restriction \Vdash_{\bar{Q}}$ "if $\zeta \notin \{\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n\}$ then for some $r \in p_2$, $\underset{\sim}{\zeta}_r[\underset{\sim}{G}] = \zeta$ and for every $s \in p_1$ $[\underset{\sim}{\zeta}_r = \zeta \Rightarrow s \le_0 r]$". We then say: $p_1 \le p_2$ over $\{\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n\}$.

Remark: We could use names for $\underset{\sim}{n}$ too, but as it is finite this is not necessary.

Now for $\xi \le \alpha$, and $p \in Sp_2 \mathrm{Lim}_\kappa \bar{Q}$, let us define

$$p \restriction \xi = \{r \restriction \xi: r \in p\}$$
$$p \restriction \{\xi\} = \{r \restriction \{\xi\}: r \in p\}.$$

*Proof of* (*E*): Let us check Definition 2.1 for $P_\alpha =_{df} Sp_2 \mathrm{Lim}_\kappa \bar{Q}$:

$\le^{P_\alpha}$ *is a partial order:* Suppose $p_0 \le p_1 \le p_2$. Let $n^l, \underset{\sim}{\xi}_0^l, \ldots, \underset{\sim}{\xi}_n^l$ appear in the definition of $p_l \le p_{l+1}$. Let $n = n^0 + n^1$, and

$$\underset{\sim}{\zeta}_\ell = \begin{cases} \underset{\sim}{\xi}_l^0 \text{ if } l < n^0 \\ \underset{\sim}{\xi}_{l-n^0}^1 \text{ if } l \ge n^0. \end{cases}$$

Now $\Vdash_{\bar{Q}} p_l \restriction \{\underset{\sim}{\zeta}_\ell\} \le p_{l+1} \restriction \{\underset{\sim}{\zeta}_\ell\}$, "hence $\Vdash_{\bar{Q}} p_0 \restriction \{\underset{\sim}{\zeta}_\ell\} \le p_2 \restriction \{\underset{\sim}{\zeta}_\ell\}$".

Also $\Vdash_{\bar{Q}}$ "if $\zeta \notin \{\underset{\sim}{\zeta}_0, \ldots, \underset{\sim}{\zeta}_{n+1}\}$ then $p_0 \upharpoonright \{\zeta\} \leq_0 p_1 \upharpoonright \{\zeta\} \leq_0 p_2 \upharpoonright \{\zeta\}$". So we finish.

$\leq_0$ *is a partial order:* As in I.

$p \leq_0 q \Rightarrow p \leq q$: By the definition; easy.

So in Definition 2.1, (i), (ii), (iii), and (iv) hold. We leave the checking of the rest to the reader.

Remark 2.8: This is a combination of [4], X with the recent Gitik ([2]) (which uses Easton support, each $Q$ is $(\{2\}, \kappa_i)$-complete, where for the important $i$'s $\kappa_i = i$: as his aim was mainly cardinals which remain inaccessible).

**Lemma 2.9** *Suppose $\gamma$ is an ordinal and $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i\colon i < \alpha\rangle$ is a $\kappa$-$Sp_2$-iteration.*
(1) *if $p \leq q$ in $P_\alpha = Sp_2\ \mathrm{Lim}_\kappa \bar{Q}$ then for some $n$ ordinals $\xi_1 < \ldots, < \xi_n$, $r \in P_\alpha$, $q \leq r$, and $p \leq r$ above $\{\xi_1, \ldots, \xi_n\}$.*
(2) *If $\gamma$ is successor cardinal (or not a cardinal) then the parallel of 1.4, 1.5, 1.6 holds.*
(3) *If $\kappa$ is inaccessible but $\Vdash_{P_i}$ "$\kappa$ is a regular cardinal" for each $i < \alpha$ then the parallel of 1.4, 1.5, 1.6 holds.*

*Proof:* Left to the reader.

**Lemma 2.10** *Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i\colon i < \alpha\rangle$ is a $\kappa$-$Sp_2$-iteration, $\kappa > \aleph_0$ a regular cardinal, $S \subseteq \{2\} \cup \{\mu\colon \aleph_0 \leq \mu \leq \kappa,\ \mu$ regular$\}$ and each $Q_i$ (in $V^{P_i}$), has $(S, <\kappa) - Pr_1$, then:*
(1) *$P_\alpha = Sp_2$-$\mathrm{Lim}_\kappa\ \bar{Q}$ has $(S, <\kappa) - Pr_1$, and if each $Q_\alpha$ has $(S, \kappa) - Pr_1$ then $P_\alpha$ has it.*
(2) *If $\kappa \in S$ and $cf(\alpha) = \kappa$ then $\bigcup_{i<\alpha} P_i$ is dense in $P_\alpha$.*
(3) *If $\kappa \in S$, $\alpha$ strongly inaccessible, $\alpha > |P_i| + \kappa$ for $i < \alpha$ then $P_\alpha$ satisfies the $\alpha$-chain condition (in a strong sense).*
(4) *If each $Q_i$ has a power of $\leq \chi$, then $P_\alpha$ has a dense subset of power $\leq (|\alpha| + \chi)^{<\chi}$.*
(5) *If $|Q_i| \leq \chi$, $\chi^{<\chi} = \chi$, $l(\bar{Q}) = \chi^+$ then $\bar{Q}$ satisfies the $\chi^+$-c.c.*
(6) *If $S = \{\kappa\}$, (1) works even for $(S, \kappa) - Pr$ which is defined as the game definition of semiproperness; i.e., using Fact 2.4(b) with winning means:*

$$\bigwedge_{\alpha} (\exists p_\alpha) p_\alpha \Vdash \operatorname*{Sup}_{i<\alpha} \tau_i \leq \operatorname*{Sup}_{i<\alpha} \alpha_i$$

*Proof:*

(1) Let us check Definition 2.1. Now (i)–(iv) hold by 2.7.

For (v) let $\mu \in S$, $\Vdash_{P_\alpha}$ "$\tau < \mu$", $p \in P_\alpha$. For simplicity $\mu \neq 2$. We define by induction on $n$ $p_n$, $p = p^0$, $p^n \leq_0 p^{n+1}$. For each $n$ let $\{\xi_i^n\colon i < \gamma_n < \kappa\}$ be the domain of $p^n$ (i.e., $\{\underset{\sim}{\zeta}_r\colon r \in p^n\}$) and define by induction on $i < \gamma_n$ $p_i^n$, $p_0^n = p_n$. $p_i^n$ is $\leq_0$-increasing (in $i$).

If $p_i^n$ is defined let (writing a little inaccurately) $G \subseteq P_{\xi_i^n+1}$ be generic over $V$. In $V[G]$ if there are $\alpha_i^n < \mu$, $r \in P_\alpha$, $r \upharpoonright (\xi_i^n + 1) \in G$, $p_i^n \leq_0 r$, such that $r \Vdash_{P_\alpha/G}$ "$\tau \leq \alpha_i^n$", let $r_i^n[G]$ be like that; otherwise, let $r_i^n = p_i^n$. So $r_i^n$, $\alpha_i^n$ are $P_{\xi_i^n+1}$-names. Now in $V[G \cap P_{\xi_i^n}]$, $\underset{\sim}{Q}_{\xi_i^n}$ is a forcing notion, $\underset{\sim}{\alpha}_i^n$ a name of an ordinal $< \mu$; hence there are $\beta_i^n < \mu$, $q_i^n$, $p_i^n \restriction \{\xi_i^n\} \leq_0 q_i^n \in Q_{\xi_i^n}$, $V[G \cap P_{\xi_i^n}] \models$ "$q_i^n \Vdash_{Q_{\xi_i^n}}$ "$\alpha_i^n \leq \beta_i^n$"". So $\beta_i^n$ is a $P_{\xi_i^n}$-name, $q_i^n$ a $\bar{Q}$-named atomic condition. Now define $p_{i+1}^n$ as $p_{i+1}^n = p_i^n \cup r_i^n \restriction [\xi_i^n + 1, \alpha) \cup \{q_i^n\}$.

We have an obvious flaw—why is there a limit for $p_i^n(i < \delta)$? (or $p^n(n < \omega)$). For this, use (v) of Definition 2.1, i.e., increase $p_{i+1}^n$ albeit according to the winning strategy. Now $p_{n+1}$ will be ${}_0\geq p_{\gamma_n}^n$ according to the strategy too.

So there is $p^*$, $p^n \leq_0 p^*$ for each $n$. Dom $p^* = \bigcup_{n<\omega}$ Dom $p_n$. We claim that for some $\alpha < \mu$, $p^* \Vdash_{P_\alpha}$ "$\tau \leq \alpha$". If not, let $q \in P_\alpha$, $q \geq p^*$, and $\beta < \mu$ be such that $q \Vdash_{P_\alpha}$ "$\tau = \beta$". So by 2.9(3) w.l.o.g. $q \gtrsim p^*$ above some $\{\xi_0, \ldots, \xi_{n-1}\}$, $\xi_0 < \ldots < \xi_{n-1}$. Choose such number $n$, and ordinals $\xi_l (l < n)$ with minimal $\xi_{n-1}$ (or $n = 0$ is best of all). If $n > 0$, w.l.o.g. for some $m < \omega$ $q \restriction \xi_{n-1} \Vdash_{P_{\xi_{n-1}}}$ " $\xi_n \in$ Dom $p^m$" and we get contr. to the choice of $p^{m+1}$

(vi) is left to the reader.

(2), (3) are left to the reader.

(4), (5) Like [4], Ch. III x.x, use only names which are hereditarily $< \kappa$.

**Definition 2.10** We define $Sp_3$ iteration $\bar{Q}$ and $Sp_3$ $\mathrm{Lim}_\kappa$ $\bar{Q}$ like $\kappa$-$SP_2$ with only one change: instead $p \in P_i$ being of cardinality $< \kappa$, we require:

(*) for every $p \in P_\alpha$, $\lambda \leq l(\bar{Q})$ which is strongly inaccessible, and $(\forall i < \kappa)[|P_i| < \lambda] \Vdash_{\bar{Q} \restriction \lambda}$ "the domain of $p \restriction \lambda$ is bounded below $\lambda$". Hence, for each $\lambda \bigcup_{i<\lambda} P_i$ is dense in $P_\lambda$.

Claim 2.11: The parallel of Definition 2.10 holds.

*3* We can get from the lemma of preservation of forcing with $(S, \gamma) - Pr_1$ by $\kappa$-$Sp_2$ iteration (and on the $\lambda$-c.c. for then) Martin-like axioms. We list below some variations.

**Notation 3.1:** Reasonable choices for $S$ are

(1) $S_\kappa^0 = UR\ Car_{\leq\kappa} = \{\mu: \mu$ a regular cardinal, $\aleph_0 < \mu \leq \kappa\}$

(2) $S_\kappa^1 = RCar_{\leq\kappa} = \{\mu: \mu$ a regular cardinal, $\aleph_0 \leq \mu \leq \kappa\}$

(3) $S_\kappa^2 = \{2\} \cup Car_{\leq\kappa}$

(4) If we write "$< \kappa$" instead $\leq \kappa$ (and $S_{<\kappa}^l$ instead $S_\kappa^l$) the meaning should be clear.

Fact 3.2: Suppose the forcing notion $P$ satisfies $(S, \gamma) - Pr_1$

(1) If $2 \in S$ then $P$ does not add any bounded subset of $\gamma$.

(2) If $\mu$ is regular, and $\lambda_i (i < \mu)$ are regular, and $\{\mu\} \cup \{\lambda_i: i < \mu\} \subseteq S$, $D$ is a uniform ultrafilter on $\mu$, $\theta = cf\left(\prod_{i<\mu} \lambda_i/D\right)$ ($\lambda_i$-as an ordered set) then $P$ satisfies $(S \cup \{\theta\}, \gamma') - Pr_1$ whenever $\mu\gamma' \leq \mu$. (We can do this for all such $\theta$s simultaneously.)

(3) If $\lambda \in S$ is regular, $\mu < \gamma$ then for every $f: \mu \to \lambda$ from $V^P$ for some $g: \mu \to \lambda$ from $V$ for every $\alpha < \mu$, $f(\alpha) < g(\alpha)$.

*Claim 3.3: Suppose $MA_{<\kappa}$ holds* (i.e., *for every $P$ satisfying the $\aleph_1$-c.c. and* dense $D_i \subseteq P$ (for $i < \alpha < \kappa$) there is a directed $G \subseteq Q$, $\bigwedge_{i<\kappa} G \cap D_i \neq \emptyset\Big)$. Then the following forcing notions have expansions (by $\leq_0$) having the $(U\,RCar, \kappa) - Pr_1^0$.

(1) Silver forcing: $\{(w, A)\colon w \subseteq \omega$ finite, $A \subseteq \omega$ infinite$\}$
$(w_1, A_1) \leq (w_2, A_2)$ iff $w_1 \subseteq w_2 \subseteq w_1 \cup A_1$, $A_2 \subseteq A_1$.
(2) The forcing from [5], Section 2 (changed suitably).

*Proof:* (1) Let $P'$ be the set of $(w, A, B)$ satisfying: $w \subseteq \omega$ finite, $B \subseteq \omega$ infinite, $B \subseteq A \subseteq \omega$, with the order

$$(w_1, A_1 B_1) \leq (w_2, A_2, B_2) \text{ iff } (w_1, A_1) \leq (w_2, A_2)$$
$$\text{and } B_2 \subseteq^* B_1 \text{ (i.e., } B_2 - B_1 \text{ finite)}$$

$$(w_1, A_1, B_1) \leq_0 (w_2, A_2, B_2) \text{ if } w_1 = w_2$$
$$A_1 = A_2$$
$$B_2 \subseteq * B_1.$$

Let us check Definition 2.1: (i)–(iv) easy.
Note that $\{(w, A, A)\colon (w, A, A) \in P\}$ is dense in $P$.
(iv) Let $\mu > \aleph_0$ be a regular cardinal, $\tau$ a $P'$-name, $\Vdash_P$ "$\tau < \mu$. Let $p = (w, A, B)$ be given. Choose by induction on $i < \omega, n_i$, $A_i$ such that

(a) $A_0 = B(\subseteq A)$
(b) $n_i = \text{Min } A_i$
(c) $A_{i+1} \subseteq A_i - \{n_i\}$
(d) for every $u \subseteq \{0, 1, 2, \ldots, n_i\}$ for some $\alpha_{i,u} < \mu$, $(u, A_{i+1}, A_{i+1})$ $\Vdash_{P'}$ "$\tau = \alpha_{i+1}$" or for no $B \subseteq \omega$ and $\alpha < \mu (u, B, B) \Vdash$ "$\tau = \alpha_{i,u}$".
There is no problem to do this, now $q =_{df} (w, A, \{n_i\colon i < \omega\})$ satisfies:
(e) $p \leq q \in P'$ and even $p \leq_0 q$.
(f) $q \Vdash_{P'}$ " "$\tau \in \{\alpha_{i,u}\colon i < \omega,\ u \subseteq \{0, 1, 2, \ldots, n_i\}\}$.

So $q$ is as required.
(v): Suppose $p_i (i < \gamma)$ is $\leq_0$-increasing so $p_i = (w, A, B_i)$ $B_i \subseteq A$, $B_i$ is $*$-decreasing. It is well known that for $\gamma < \kappa$, $MA_{<\kappa}$ implies the existence of an infinite $B \subseteq \omega$, $(\forall i < \gamma)$ $B \subseteq * B_i$.

Claim 3.4: The following forcing notions have the $(U\,RCar, \kappa) - Pr_1$:
(1) $\aleph_1$-c.c.
(2) $\kappa$-complete
(3) $\{f\colon f$ a function from $A$ to $\{0, 1\}$, $A \subseteq \omega$, $A = \phi \bmod D\}$ where $D$ is a filter on $\omega$, containing the co-finite sets, such that if $A_i \in D$ for $i < i^* < \kappa$ then for some $B \in D$ $\bigwedge_{i<i^*} B \subseteq * A_i$

Discussion 3.5: Let $\kappa < \lambda$, $\lambda$ regular. Each of the following gives rise naturally to a generalized $MA$, stronger as $\lambda$ is demanded to be a larger cardinal (so if $\lambda$ is supercompact we get parallels to PFA).

*Case I:* We use $\bar{Q}$ of length $\lambda$, a $\kappa$-$SP_2$ iteration, $\Vdash_{P_i}$ "$|Q_i| < \lambda$", each $Q_i$ having $(S^l_\kappa, \kappa) - Pr_1^-$.

Now $P_\lambda = \kappa - SP_2 \operatorname{Lim}_\kappa \bar{Q}$ have the $(S^l_\kappa, \kappa) - Pr_1$ by 2.10, so all regular $\mu \le \kappa$ remain regular and usually every $\lambda' \in (\kappa, \lambda)$ is collapsed. But $\lambda$ is not collapsed if it is strongly inaccessible (by 2.10(3)) and also if $(\forall \chi < \lambda)(\chi^{<\kappa} < \lambda)$ (by 2.10(5)). If $2 \in S^Q_\kappa$, no bounded subset of $\kappa$ is added.

*Case II:* Like Case I with $(\kappa + 1) - Sp_2$ iteration $Sp_2 \operatorname{Lim}_{\kappa+1}$ and every $\lambda' \in (\kappa, \lambda)$ is collapsed. Here we need $\lambda$ to be strongly inaccessible.

*Case III:* $\bar{Q}$ is $Sp_3$-iteration, has length $\kappa$, $|Q_i| < \kappa$ for $i < \kappa$, $\kappa$ is strongly inaccessible, and $Q_i$ have $(S, \gamma_i) - Pr_1^-$.

By 2.11 $P_\kappa = Sp_3 \operatorname{Lim} \bar{Q}$ has the $\kappa$-c.c. (and $|P_i| < \kappa$ of course). Let $S = \{\mu < \kappa;\ \mu$ regular and for some $i$, $\Vdash_{P_i}$ "$\mu$ is regular and $\mu \in S_j$, $\mu \le \gamma_j$, for $j > i\}$ then $\Vdash_{P_\alpha}$.

Fact 3.6: Suppose $\lambda$ is strongly inaccessible, limit of measurables, $\lambda > \kappa$, $\kappa$ regular. Then *for some* $\lambda$-*cc forcing* $P$ *not adding bounded subsets of* $\kappa$, $|P| = \lambda$, and $\Vdash_P$ "$2^\kappa = \lambda = \kappa^+$, and for every $A \subseteq \kappa$ there is a countable subset of $\lambda$ not in $L(A)$.

*Proof:* We use $\kappa$-$SP_2$-iteration $\langle P_i, \underset{\sim}{Q}_i: i < \lambda \rangle$, $|P_i| < \lambda$. For $i$ even: let $\kappa_i$ be the first measurable $> |P_i|$, (but necessarily $< \lambda$) and $\tau$. Then $Q_i$ is Prikry forcing on $\kappa_i$ and $Q_{i+1}$ is Levi collapse of $\kappa_i^+$ to $\kappa$.

## *4*

**Lemma 4.1** *Suppose*

(i) *$R$ is an $\aleph_1$-complete forcing notion.*

(ii) *For $r \in R$, $\bar{Q}^r = \langle P^r_i : i \le \alpha^r_\alpha \rangle$, $P^r_i$ is $<\!\circ$-increasing in $i$ and if $i \le \alpha^r$ has cofinality $\omega_1$, then every countable subset of $V^{P^r_\kappa}$ belongs to $V^{P^r_i}$ for some $i < \alpha$.*

(iii) *If $r^1 \le r^2$ then $\bar{Q}^{r^1} \le \bar{Q}^{r^2}$.*

(iv) *If $r \in R$ and $\underset{\sim}{Q}$ is a $P^r_{\alpha_r}$-name of a forcing notion, then for some $r^1 \ge r$*

$$P^{r_1}_{\alpha_\mu + 1} = P^r_{\alpha_\mu} * \underset{\sim}{Q} \text{ or } \Vdash_{P^{r_1}_{\alpha_M}} \underset{\sim}{Q} \text{ does not satisfy the c.c.c.}$$

(v) *If $r^\zeta (\zeta < \delta)$ is increasing, $\delta \le \omega_1$, then for some $r$*

$$\bigwedge_{\zeta < \delta} r^\zeta \le r \text{ and } \alpha_r = \bigcup_{\zeta < \delta} \alpha_{r^\zeta}.$$

Let $P[\underset{\sim}{G}_R]$ be $\bigcup \{P^r_i : r \in \underset{\sim}{G}_R,\ i \le \alpha_r\}$, so it is an $R$-name of a forcing notion. *Then* $\Vdash_R$ $[\Vdash_{P[\underset{\sim}{G}_R]}$ "for any $\aleph_1$ dense subsets of Sacks forcing, there is a directed subset of Sacks forcing not disjoint to any of them"].

Remark: $Q_{Sacks} = \{\tau : \tau \subseteq^{\omega >} 2$ is closed under initial segments nonempty and $(\forall \eta \in \tau)(\exists \upsilon)(\eta < \upsilon \wedge \upsilon {}^\frown \langle 0 \rangle \in T \wedge \upsilon {}^\frown \langle 1 \rangle \in T)$ and $\tau_i \le \tau_2$ if $\tau_2 \subseteq \tau_1$.

*Proof:* Let $\underset{\sim}{D}_i$ be $R^* P[\underset{\sim}{G}_R]$-name of dense subset of $Q^{R^* P[G_R]}_{Sacks}$ for $i < \omega_1$ ($Q^V_{Sacks}$ is Sacks forcing in the universe $V$).

For a subset $E$ of Sacks forcing let $var(E)$ be $\{(n, T) : T \in E,\ n < \omega\}$ ordered by $(n_1 T_1) \le (n_2, T_2)$ iff $n_1 \le n_2$, $T_2 \subseteq T_1$, and $T_1 \cap^{n_1 \ge} 2 = T_2 \cap^{n_1 \ge} 2$. We now define by induction on $\zeta \le \omega_1$, $r(\zeta)$, and $D_\zeta$ such that:

(a) $r(\zeta) \in R$ is increasing, $\alpha_{r(\zeta)}$-increasing continuous.
(b) $D_\zeta$ is a $P^{r(\zeta+1)}_{\alpha_r(\zeta+1)}$-name of a countable subset of $Q_{Sacks}$.
(c) If $T \in D_\zeta$, $\eta \in T$ then $T_{[\eta]} =_{df} \{v: \eta \frown v \in T\}$ belongs to $D_\zeta$.
(d) If $T_1, T_2 \in D_\zeta$ then $\{\langle\ \rangle, \langle 0\rangle \frown \eta: \eta \in T_1\}$, $\{\langle\ \rangle, \langle 1\rangle \frown \eta: \eta \in T_2\}$ and their union belongs to $D_\zeta$.
(e) Let $\xi < \zeta$, then for $T_1 \in D_\xi$ there is $T_2 \in D_\zeta$, $T_1 \geq T_2$ and for $T_2 \in D_\zeta$ there is $T_1 \in D_\xi$, $T_1 \geq T_2$.
(f) If $T \in D_{\zeta+1}$ then for some $n$ for every $\eta \in {}^n2 \cap T$, $T_{(\eta)} =_{df} \{v \in T: v \leq \eta \text{ or } \eta \leq v\}$ belongs to $\underset{\sim}{D}_\zeta$.
(g) Suppose $\zeta$ is limit, then $P^{r(\zeta+1)}_{\alpha_r(\zeta)+1} = P^{r(\zeta)}_{\alpha_r(\zeta)} * T_\zeta$, $T_\zeta$ is $\left[\operatorname{var} \bigcup_{\xi<\zeta} p_\xi\right]$ if $\zeta < \omega_1$ and $T_\zeta$ is $\left[\operatorname{var} \bigcup_{\xi<\zeta} D_\xi\right]^\omega$ if $\zeta = \omega_1$ (the $\omega$-th power, with finite support).

Next the generic subset of $T_\zeta$ gives a sequence of length $\omega$ of Sacks conditions closing the set of those conditions by $(c) + (d)$ we get $D_\zeta$. We have to prove that $T_\zeta$ satisfies the $\aleph_1$-c.c. in $V^{R^* P_{\alpha_{r(\zeta)}}}$: When $\zeta < \omega_1$ this is trivial (as $T_\zeta$ is countable). Let $\zeta = \omega_1$. It suffices to prove that $\left[\operatorname{var} \bigcup_{\xi<\zeta} D_\xi\right]^n$ satisfies the $\aleph_1$-c.c.where $n < \omega$. So let $\underset{\sim}{I}$ be a $R * \underset{\sim}{P}^{r(\zeta)}_{\alpha_{r(\zeta)}}$ name of a dense subset of $\left[\operatorname{var} \bigcup_{\xi<\zeta} D_\xi\right]^n$. We can find a $\xi < \zeta$, cf $\xi = \aleph_0$ such that $I_\xi = \{x: x \in V^{R * \underset{\sim}{P}^{r(\zeta)}_{\alpha_{r(\xi)}}}$ and every $p \in R * \underset{\sim}{P}^{r(\zeta)}_{\alpha_{r(\zeta)}} / R * \underset{\sim}{P}^{r(\zeta)}_{\alpha_{r(\xi)}}$ force $x$ to be in $I\}$ is predense in $\left[\operatorname{var} \bigcup_{\gamma<\xi} D_\gamma\right]^n$ (exists by (e)). Check the rest.

Remark: This argument works for many other forcing notions like Laver.

## 5

**Definition 5.1** Let $S$ be a subset of $\{2\} \cup \{\lambda: \lambda \text{ is regular cardinal}\}$, $D$ a filter on a cardinal $\lambda$ (or any other set). For any ordinal $\gamma$, we define a game $Gm^*(S, \gamma, D)$. It lasts $\gamma$ moves. In the $i$-th move player I choose a cardinal $\lambda \in S$ and function $F_i$ from $\lambda$ to $\lambda_i$ and then player II chooses $\alpha_i < \lambda_i$.

Player II wins a play if for every $i < \gamma$,

$$d(\langle \lambda_j, F_j, \alpha_j: j < i\rangle) =_{df} \{\zeta < \lambda: \text{for every } j < i\ [\lambda_i = 2 \Rightarrow F_j(\zeta) = \alpha_i]\ [\lambda_i > 2 \Rightarrow F_j(\zeta) < \alpha_i\} \neq \emptyset \ mod\ D.$$

Remark 5.1A:
(1) See [4], Chapter X on this.
(2) If not said, otherwise we assume that $\lambda - \{\zeta\} \in D$ for $\zeta < \lambda$.
(3) If $D$ is an ultrafilter on $\lambda$, $(|\gamma| + \kappa^+)$-complete for each $\kappa \in S$ then player II has a winning strategy.

**Definition 5.2** For $\mathbf{F}$ a winning strategy for player II in $Gm^*(S, \gamma, D)$, $D$ a filter on $\lambda$ (we write $\lambda = \lambda(D)$), we define $\underset{\sim}{Q} = Q_{\mathbf{F},\lambda} = Q_{\mathbf{F},S,\gamma,D}$, $Q = (|Q|, \leq, \leq_0)$.

*Part A:* Let $(T, H) \in Q$ *iff*

(i) $T$ is a nonempty set of finite sequence of ordinals $< \lambda$.

(ii) $\eta \in T \Rightarrow \eta \restriction \ell \in T$, and for some $n$ and $\eta$: $T \cap {}^{n\geq}\lambda = \{\eta \restriction \ell: \ell \leq n\}$, $|T \cap {}^{n+1}\lambda| \geq 2$; we denote $\eta = stam(T) = stam(T, H)$ (it is unique).

(iii) $H$ is a function, $T - \{stam(T) \restriction \ell: \ell < lg(stam(T))\} \subseteq \mathrm{dom}\, H \subseteq {}^{\omega>}\lambda$.

(iv) for each $\eta \in \mathrm{Dom}\, H$, $H(\eta)$ is a proper initial segment of a play of the game $Gm^*(S, \gamma, D)$ in which player II use his strategy $\mathbf{F}$ so $H(\eta) = \langle \lambda_i^{H(\eta)}, F_i^{H(\eta)}, \alpha_i^{H(\eta)}: i < i^{H(\eta)} \rangle$, and $i^{H(\eta)} < \gamma$.

(v) for $\eta \in T$, $d(H(\eta)) = \{\zeta < \lambda: \eta \frown \langle \zeta \rangle \in T\}$.

*Part B:* $(T_1, H_1) \leq (T_2, H_2)$ (where both belong to $Q$) *iff* $T_2 \subseteq T_1$ and for each $\eta \in T_2$, if $stam(T_2) \leq \eta$ then $H_1(\eta)$ is an initial segment of $H_1(\eta)$.

*Part C:* $(T_1, H_1) \leq_0 (T_2, H_2)$ (where both belong to $Q$) *if* $(T_1, H_1) \leq (T_2, H_2)$ and $stam(T_1) = stam(T_2)$.

Remark 5.2A: (1) So if $(T, H) \in Q_{\mathbf{F},\lambda}$ and $\mathbf{F}$, $S(\gamma, D)$ are as above, $\eta \in T$, $\eta \geq stam(T)$ *then* $d(H(\eta)) \neq \emptyset$ *mod D*.

(2) We could restrict $H$ to $T$ in (iii).

Notation 5.2B: For $p = (T, H) \in Q_{\mathbf{F},\lambda}$ and $\eta \in T$ let $p^{[\eta]} = (T^{[\eta]}, H)$, $T^{[\eta]} = \{\nu \in T: \nu \leq \eta \text{ or } \eta \leq \nu\}$. Clearly $p \leq p^{[\eta]} \in Q_{\mathbf{F},\lambda}$.

**Lemma 5.3** *If $Q = Q_{\mathbf{F},S,\gamma,D}$, $D$ a uniform filter on $\lambda(D)$ then $\Vdash_Q cf\, \lambda(D) = \aleph_0$.*

*Proof:* Let $\underset{\sim}{\eta}_Q = \bigcup \{stam(p): p \in \underset{\sim}{G}_Q\}$.

Clearly if $(T_\ell, H_\ell) \in \underset{\sim}{G}_Q$ for $\ell = 1, 2$ then for some $(T, H) \in \underset{\sim}{G}_Q$, $(T_\ell, H_\ell) \leq (T, H)$; hence $stam(T_\ell) \leq stam(T)$, hence $stam(T_1, H_1) \cup stam(T_1, H_2)$ is in ${}^{\omega>}\lambda$. Hence $\underset{\sim}{\eta}_Q$ is a sequence of ordinals of length $\leq \omega$. It has length $\omega$, as for every $p = (T, H) \in Q$, and $n$, there is $\eta \in T \cap {}^n\lambda$, hence $p \leq p^{[\eta]} \in Q$ (see 5.2B), and $p^{[\eta]} \Vdash$ "$lg(\underset{\sim}{\eta}_Q) \geq n$" because $\eta \leq stam(p^{[\eta]})$ and for every $q \in Q$, $q \Vdash_Q$ "$stam(q) \leq \underset{\sim}{\eta}_Q$". Obviously, $\Vdash_Q$ "$Rang(\underset{\sim}{\eta}_Q) \subseteq \lambda$". Why $\Vdash_Q$ sup $Rang(\underset{\sim}{\eta}_Q) = \lambda$? Because for every $(T, H) \in Q$ and $\alpha < \lambda$, letting $\eta =_{df} stam(T)$, clearly $d(H(\eta)) \neq \emptyset$ *mod D* (see Definition 5.12) but $D$ is uniform, hence there is $\beta \in d(H(\eta))$, $\beta > \alpha$, so $\eta \frown \langle \beta \rangle \in T$, and $(T, H) \leq (T, H)^{[\eta \frown \langle \beta \rangle]} \in Q$, $(T, H)^{[\eta \frown \langle \beta \rangle]} \Vdash_Q$ "$\eta \frown \langle \beta \rangle \leq \underset{\sim}{\eta}_Q$" hence $(T, H)^{[\eta \frown \langle \beta \rangle]} \Vdash$ "sup $Rang(\underset{\sim}{\eta}_Q) \geq \beta$", as $\alpha < \beta$ we finish.

**Lemma 5.4** *If $S$, $\gamma$, $D$ are as in Definition 5.1, $\aleph_0 \notin S$, $\mathbf{F}$ a winning strategy of player II in $Gm^*(S, \gamma, D)$, $cf\, \gamma > \aleph_0$, then $Q$ satisfies $(S, cf\, \gamma) - Pr_1$ (see Definition 2.1).*

*Proof:* In Definition 2.1, parts (i), (ii), (iii), (iv), (vi) are clear. So let us check (v). Let $\kappa \in S$, $\underset{\sim}{\tau}$ be a $Q$-name, $\Vdash_Q$ "$\underset{\sim}{\tau} \in \kappa$" and $p = (T, H) \in Q$. We define by induction on $n$, $p_n = (T_n, H_n)$ such that:

(i) $p_0 = p$, $p_n \leq_0 p_{n+1}$, $T_n \cap {}^{n>}\lambda = T_{n+1} \cap {}^{n>}\lambda$

(ii) if $\eta \in T_n \cap {}^n\lambda$, and there are $q, \alpha$ satisfying

(*) "$p_n^{[\eta]} \leq_0 q \in Q$, $\alpha < \kappa$, $q \Vdash$ "if $\kappa = 2$, $\underset{\sim}{\tau} = \alpha$, if $\kappa \geq \aleph_0$, $\underset{\sim}{\tau} < \alpha$"

*then* $p_{n+1}^{[\eta]}$, $\alpha_\eta$ satisfying this.

(iii) if $\eta \in T_{n+1} \cap {}^n\lambda$ and there are $q, \beta$ satisfying

(*) $p_{n+1}^{[\eta]} \leq_0 q \in Q$, and for every $r$, $\beta < \kappa$,

$[q \leq_0 r \in Q \rightarrow \neg(\exists r_1)(r \leq r_1 \in Q \wedge r_1 \Vdash \text{ if } \kappa = 2, \underset{\sim}{\tau} = \beta, \text{ if } \kappa \geq \aleph_0, \underset{\sim}{\tau} < \beta'']$

*then* $p_{n+1}^{[\eta]}$ satisfies (*).

Let $p_\omega$ be the limit of $\langle p_n : n < \omega \rangle$, i.e., $p_\omega = (T_\omega, H_\omega)$, $T_\omega = \bigcap_{n<\omega} T_n$, $H_\omega(\eta)$ is the limit of the sequences $H_n(\eta)$ (for $\eta \in T_\omega - \{stam(T) \restriction \ell : \ell\}$). It is well defined as $cf(\gamma) > \aleph_0$.

Now for each $\eta \in T_\omega$, $H_\omega(\eta)$ is a proper initial segment of a play of the game $Gm^*(S, \gamma, D)$, and it lasts $i^{H_\omega(\eta)}$ moves. Player I could choose in his $i^{H_\omega(\eta)}$-th move the cardinal $\kappa$ and the function $f_\eta : \lambda \rightarrow \kappa$,

$$f_\eta(\zeta) = \begin{cases} \alpha_{\eta^\frown\langle\zeta\rangle} & \text{if defined (which is } < \kappa) \\ 0 & \text{otherwise.} \end{cases}$$

So, for some $\beta_\eta$, $H_\omega(\eta) {}^\frown \langle \alpha, f_\eta, \beta_\eta \rangle$ is also a proper initial segment of a play of $Gm^*(S, \gamma, D)$ in which player II use the strategy **F**. So there is $p_{\omega+1} = (T_{\omega+1}, H_{\omega+1}) \in Q$, $p_\omega \leq_0 p_{\omega+1}$, and for each $\eta \in T_{\omega+1} - \{\nu : \nu < stam(T)\}$, $H_{\omega+1}(\eta) = H_\omega(\eta) {}^\frown \langle \kappa, f_\eta, \beta_\eta \rangle$.

We can easily show

Fact 5.4A: If $p = (T, H) \in Q$, $\kappa \in S$, $f: T \rightarrow \kappa$, *then* for some $p_1 = (T_1, H_1) \in Q$, $p \leq p_1$, and for every $\eta \in T_1$, $[\kappa = 2 \wedge f \restriction Suc_{T_1}(\eta)$ is constant$]$ or $[\kappa \geq \aleph_1 \wedge f \restriction Suc_{T_1}(\eta)$ is bounded below $\kappa]$.

[*Proof:* Define by induction $r^n$, $p \leq_0 r^n \leq_0 r^{n+1} \in Q$, $r^{n+1}$ satisfies the conclusion of 5.4A for $\eta$ of length $n$, now any $r^\omega \in Q$, $(\forall n) r^n \leq_0 r^\omega$ is as required].

Fact 5.4B: If $p = (T, H) \in Q$, $A \subseteq T$ *then* there is $p_1 = (T_1, H_1) \in Q$, $p \leq_0 p_1$ and for every $\eta \in T_1$, and $k < \omega$:

$$(\exists \nu \in A)[\nu \in T_1 \wedge \eta \leq \nu \wedge lg(f) = k] \rightarrow$$
$$(\forall q)[q \in Q \wedge p_1^{[\eta]} \leq_0 q \rightarrow (\exists \nu \in A)(\nu \in q \wedge \eta \leq \nu A lg(\nu) = k)]$$

[*Proof:* Define by induction on $n$ $r^n$, $p \leq_0 r^n \leq_0 r^{n+1} \in Q$, $r^{n+1}$ satisfies the conclusion of 5.4B for $\eta$ of length $\leq n$ and $k \leq n$. Now any $r^\omega \in Q$, $(\forall n) r^n \leq_0 r^\omega$ is as required.]

Let $A = \{\eta \in T_{\omega+1} : \alpha_\eta$ well defined$\}$, and let $q$, $p_{\omega+1} \leq q \in Q$ be as in 5.4B. Now for every $\eta \in T^q$ there is $r \in Q$, $q^{[\eta]} \leq r$, and $r$ force a value for $\underset{\sim}{\tau}$. So $stam(r) \in A$ (as $p_\omega \leq q$, see the definition of the $p_\eta$'s), and $p_\omega^{[stam\, r]}$ force a value to $\underset{\sim}{\tau}$; hence, $q^{[stam\, r]}$ does, and let $k_\eta$ be $lg(stam\ r)$ for such $r$ with minimal $lg(stam(r))$. So by 5.4B,

(*) For every $\eta \in T^q$, and $r$, $q^{[\eta]} \leq_0 r \in Q$, for some $\nu \in q^{[\eta]}$, $\eta \leq \nu$, $lg(\nu) = k_\eta$, and $\nu \in A$.

Now for each $q_1$, $q \leq_0 q_1 \in Q$, $\eta \in T^{q_1}$ we can, by $k_\eta$ applications of 5.4A, get an ordinal $\alpha < \kappa$ and $q_2$, $q_1^{[\eta]} \leq_0 q_2$, and

(*) $(\forall q_3 \in Q)[q_2 \leq_0 q_3 \rightarrow (\exists \nu \in A)(\nu \in T^{q_3} \wedge lg(\nu) = k_\eta \wedge \alpha_\nu \leq \alpha)]$ (or if $\kappa = 2$, $\alpha_\nu = \alpha$).

But this shows that $\beta_\eta$ is defined for every $\eta \in T^q$. Finishing alternatively by repeated application of 5.4A we can define by induction on $n$, $q(n) \in Q$, $q(0) = q$, $q(n) \leq_0 q(n+1)$ and $\beta^n_\eta$ for $\eta \in T^{q(n)}$ such that:

(a) $\beta^0_\eta = \beta_\eta$

(b) when $\kappa \geq \aleph_0$: $\eta \frown \langle \zeta \rangle \in T_{n+1} \Rightarrow \beta^{n+1}_\eta \geq \beta^n_{\eta \frown \langle \zeta \rangle}$

(c) when $\kappa = 2$: $\eta \frown \langle \zeta \rangle \in T_{n+1} \Rightarrow \beta^{n+1}_\eta = \beta^n_{\eta \frown \langle \zeta \rangle}$.

Let $q_\omega \in Q$ be such that $q_n \leq_0 q_\omega$ for $n < \omega$.

Now if $\kappa > \aleph_0$ (is regular), we claim

$$q_\omega \Vdash_Q \underset{\sim}{\tau} \leq \bigcup_{n<\omega} \beta^n_{\langle \rangle}$$

Clearly $p \leq_0 q_\omega \in Q$, $\bigcup_{n<\omega} \beta^n_{\langle \rangle} < \kappa$ so this suffices. Why does this hold? If not, then for some $q'$, $q_\omega \leq q' \in Q$, $q' \Vdash_Q$ "$\underset{\sim}{\tau} \geq \bigcup_n \beta^n_{\langle \rangle}$. Let $\eta = stam(q')$, so $\eta \in T^q$, and $\alpha_\eta \omega$ is well defined, and as $p^{[\eta]}_\omega \leq_0 (q')^{[\eta]}$, $\alpha_\eta > \bigcup_n \beta^n_{\langle \rangle}$. But as $\eta \in \bigcap_{n<\omega} T^{q(n)}$, $\beta^{lg(n)}_{\langle \rangle} \geq \beta_\eta$, and we get a contradiction.

If $\kappa = 2$, we note just that if $\eta \in T^{q(1)}$, $\beta_\eta = \beta^0_\eta = \beta^1_\eta$.

**Lemma 5.5** *Suppose $\bar{Q} = \langle P_i, Q_i \colon i < \lambda \rangle$ is a $\kappa$-$Sp_2$-iteration, $|P_i| < \lambda$ for $i < \lambda$, each $Q_i$ has $(S, <\kappa) - Pr_1$* and $(S, \sigma) - Pr_1$ *$\sigma \leq \kappa$ regular, $S \subseteq \{2\} \cup \{\theta$: $\theta$ regular uncountable $\leq \kappa\}$ and in $V$, $D$ is a normal ultrafilter on $\lambda$ (so $\lambda$ is a measurable cardinal). Then $\Vdash_{P_\lambda}$ "player II wins $Gm^*(S, \kappa, D)$".*

Remark: Also for $\kappa$-$Sp_3$.

*Proof:* Let $A = \{\mu < \lambda \colon (\forall i < \mu) |P_i| < \mu$, $\mu$ strongly inacessible $> \kappa\}$.

Let $G_\lambda \subseteq P_\lambda$ be generic over $V$, $G_\alpha = G \cap P_\alpha$.

W.l.o.g. player I choose $P_\lambda$-names of functions and cardinals in $S$. Now we work in $V$ and describe player II's strategy there. For each $\mu \in A$ the forcing notion $P_\lambda / P_\mu$ has $(S, \sigma) - Pr_2$; hence, player II has a winning strategy $F(P_\lambda / G_\mu) \in V[G_\kappa]$, so $\underset{\sim}{F}(P_\lambda / \underset{\sim}{G}_\mu)$ is a $P_\kappa$-name, $\langle \underset{\sim}{F}(P_\lambda / \underset{\sim}{G}_\mu) \colon \mu \rangle$ a $P_\lambda$-name. Let us describe a winning strategy for player II.

So in the $i$th move player I chooses $\underset{\sim}{\theta}_i \in S$ and $\underset{\sim}{f}_i \colon \lambda \to \underset{\sim}{\theta}_i$. Player II chooses in his $i$-th move not only $\underset{\sim}{\alpha}_i < \underset{\sim}{\theta}_i$ but also $A_i, \underset{\sim}{f}_i, \gamma_i, \langle \langle \underset{\sim}{p}^\mu_j \colon j \leq i \rangle \colon \mu \in A_i \rangle$ such that $\gamma_i$ is an ordinal $< \lambda$,

(1) $j < i \Rightarrow \gamma_j < \gamma_i$.

(2) $A_i \in D$, $A_i \in V$, $A_i \subseteq \bigcap_{j<i} A_j$ and $A_\delta = \bigcap_{j<\delta} A_j$

(3) $\Vdash$ "$\underset{\sim}{f}_i \colon \lambda \to \underset{\sim}{\theta}_i$, $\underset{\sim}{\theta}_i \in S$".

(4) for $\mu \in A_i$,

$$\langle \underset{\sim}{p}^\mu_j \colon j \leq 2i + 2 \rangle$$

is a $P_\kappa$-name of an initial segment of a play as in (vi) of 2.1, for the forcing $P_\lambda / G_\kappa$, $p^\mu_{2j+1} \Vdash_{P_\lambda / G_\mu}$ "$\underset{\sim}{f}_i(\mu) = \underset{\sim}{\alpha}_i$ if $\theta_i = 2$, $f_i(\mu) < \alpha^\mu_i$ if $\theta_i \geq \aleph_0$", $\underset{\sim}{\alpha}^\mu_i$ a $P_{\alpha_i}$-name.

In the $i$-th stage clearly $A^0_i =_{df} \bigcap_{j<i} A_j \cap A$ is in $D$, and let $\gamma^0_i = \sup_{j<i} \gamma_j$, so $\gamma^0_i < \lambda$ and choose $\gamma^1_\mu \in (\gamma^0_\mu, \lambda)$ such that $\underset{\sim}{\theta}_i$ is a $P_{\gamma^1_\mu}$-name. For every $\mu \in A$, $\mu > \gamma'$, we can define $P_\mu$-names $\underset{\sim}{p}^\mu_{2i}$, $\underset{\sim}{p}^\mu_{2i+1}$, $\alpha^\mu_i$ such that:

(a) $\Vdash_{P_\mu}$ "$\langle p_j^\mu : j < 2i+2\rangle$ is an initial segment of a play as in (v) of 2.1 for $P_\lambda/P_\mu$ in which player II uses his winning strategy $F(P_\lambda/\underset{\sim}{G}_\mu)$.

(b) $p_{2i+1}^\mu \Vdash_{P_\lambda/P_\mu}$ "$\underset{\sim}{f}_i(\mu) = \underset{\sim}{\alpha}_i^\mu$ if $\underset{\sim}{\theta}_i = 2$, $\underset{\sim}{f}_i(\mu) < \alpha_i^\mu$ if $\underset{\sim}{\theta}_i \geq \aleph_0$".

Now $\underset{\sim}{\alpha}_i^\mu$ is a $P_\mu$-name of an original $< \kappa \leq \mu$, it is $P_{\beta[\mu]}$-name for some $\beta[\mu] < \mu$ (as $P_\mu$ satisfies the $\mu$-c.c. see 2.x). By the normality of the ultrafilter $D$, on some $A_i^1 \subseteq A_i^0$, $\beta[\mu] = \beta_i$ for every $\mu \in A_i^1$. Let $\gamma_i = \gamma_n^1 + \beta_i$.

Easily for each $i < \sigma$, $\Vdash_{P_\lambda}$ "$\{\mu \in A_i : p_{2i+1}^\mu \in \underset{\sim}{G}_\lambda\} \neq \emptyset$ *mod* $D$", so we finish.

Now we can solve the second Abraham problem.

Conclusion 5.6: Suppose $\lambda$ is strongly inaccessible $\{\mu < \lambda : \mu$ measurable$\}$ is stationary, $\kappa < \lambda$, $S \subseteq \{2\} \cup \{\theta : \theta \leq \kappa$ regular uncountable$\}$. Then for some forcing notion $P$: $|P| = \lambda$, $P$ satisfies $\lambda$-c.c. and $(S, <\kappa) - Pr_1$ (and $(S, \kappa) - Pr_1$, if we want), and $\Vdash_P$ "$\lambda = |\kappa|^+$" (so $\Vdash_{P_\lambda} 2^{|\kappa|} = \lambda$) in $V^P$: and: for every $A \subseteq \lambda$, for some $\delta < \lambda$, there is a countable set $\alpha \subseteq \delta$, which is not in $V[A \cap \delta]$, we can also get suitable axiom (see 3.5).

Remark 5.6A: We can also prove (by the same forcing) the consistency of $D_\lambda + \{\delta < \lambda : cf\,\delta = \aleph_0\}$ is precipitous: if in addition there is a normal ultrafilter on $\lambda$ concentrates on measurables.

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