Notre Dame Journal of Formal Logic
Volume 32, Number 2, Spring 1991

Finite Axiomatizability and Theories with Trivial Algebraic Closure

DUGALD MACPHERSON*

Abstract It is shown that every quasi-finitely axiomatized complete theory with trivial algebraic closure has the strict order property or is the theory of an indiscernible set, and conjectured that every finitely axiomatized ω-categorical theory with infinite models has the strict order property. It is also shown that complete theories with trivial algebraic closure and (for example) no quantifier-free unstable formulas are rather limited.

1 Introduction The purpose of this note is to present a conjecture about finitely axiomatized ω-categorical theories, and to prove a related theorem as evidence for the conjecture. A complete theory T is said to have the *strict order property* (Shelah [12]; ch. 2, Section 4]) if there is $n \in \omega \setminus \{0\}$, some $\mathcal{M} \models T$, and an $L(M)$-formula $\phi(\bar{x},\bar{y})$ with $l(\bar{x}) = l(\bar{y}) = n$, such that, if we write $\bar{x} \leq \bar{y}$ whenever $\bar{x},\bar{y} \in M^n$ and $\mathcal{M} \models \phi(\bar{x},\bar{y})$, then $\leq$ is a partial ordering on M^n with an infinite totally ordered subset. Clearly any theory with the strict order property is unstable.

Conjecture 1.1 *Every finitely axiomatized ω-categorical theory with infinite models has the strict order property.*

The motivation for this conjecture is threefold. First, if true it would generalize the theorem of Zil'ber [13] that a totally categorical theory cannot be finitely axiomatized. Note here that the result of Zil'ber was generalized (in [5], Section 7.4) to all ω-categorical ω-stable theories. Second, there are two obvious ways to say in a finite number of first-order sentences that a structure is infinite. One is to say that there is a dense linear ordering somewhere in the structure; this yields the strict order property. The other is to say that there is a definable injective function f such that range$(f) \subsetneq$ domain(f); this is not compatible with ω-categoricity.

I thank Jim Schmerl and Frank Wagner for several helpful conversations. I also thank the referee for pointing out that results on normalization shorten the proof of Proposition 1.3.

Received June 1, 1989; revised January 22, 1990

The third reason for the conjecture is that it is supported by a variety of examples. The most obvious example is the countable dense linear ordering; but the countable atomless Boolean algebra, the countable dense local order (see [3]), and the countable universal homogeneous poset (see [2]) all have ω-categorical finitely axiomatized theories with the strict order property. Further examples of interesting ω-categorical structures with finitely axiomatized theories are (in the terminology of [6]) the countable 2-homogeneous trees with finite ramification order. Other structures related to trees, including some of those in Cameron [4] and Adeleke and Neumann [1], also have finitely axiomatized theories. There are also relevant results on ω-categorical posets in [10] and [11].

If $\mathcal{M}$ is a structure and $A \subseteq M$, let $\mathrm{acl}(A)$ denote the *algebraic closure of A in $\mathcal{M}$* in the usual model-theoretic sense (the structure $\mathcal{M}$ will be clear from the context). A theory T is said to have *trivial algebraic closure* if, whenever $\mathcal{M} \models T$ and $A \subseteq M$, we have $\mathrm{acl}(A) = A$. Note that any theory with trivial algebraic closure will have infinite models. Next, a theory is said to be *quasi-finitely axiomatizable* if it is axiomatized by a finite set of sentences together with, for each $k < \omega$, a sentence asserting that there are at least k elements. Our main theorem is the following.

Theorem 1.2 *If T is a complete quasi-finitely axiomatized theory with trivial algebraic closure, then either T has the strict order property or every model of T is an indiscernible set.*

Note that from this it follows that every finitely axiomatized complete theory with trivial algebraic closure has the strict order property; also, that every ω-categorical quasi-finitely axiomatized theory with trivial algebraic closure, apart from the theory of an indiscernible set, is finitely axiomatized.

The assumption of trivial algebraic closure in Theorem 1.2 is very restrictive. In particular, the following proposition suggests that there are no interesting stable theories with trivial algebraic closure.

Proposition 1.3 *Let T be a complete theory with trivial algebraic closure, and suppose that some stable formula defines an infinite coinfinite subset of some model of T. Then there is a parameter-free formula $\phi(x,y)$ which defines a nontrivial equivalence relation on the domain of each model of T.*

Structures will generally be denoted by $\mathcal{M}$ or $\mathcal{N}$, with domains M, N respectively. If P is a property of certain theories, we say a structure $\mathcal{M}$ has P if $\mathrm{Th}(\mathcal{M})$ has P. If $\bar{x}$ is a tuple, then $l(\bar{x})$ denotes the number of entries of $\bar{x}$, and $\mathrm{rng}(\bar{x})$ denotes the set of entries.

2 *Proofs of the results*

The following lemma is easy, and we omit the proof.

Lemma 2.1 *Let T be a complete theory with trivial algebraic closure, and suppose there is $\mathcal{M} \models T$ such that no quantifier-free formula defines an infinite coinfinite subset of M. Then every model of T is an indiscernible set.*

Proof of Theorem 1.2: Let T be a complete quasi-finitely axiomatized theory with trivial algebraic closure. We may assume the language of T is finite relational. For each $k < \omega$ let τ_k be the sentence asserting that there are at least k elements, and let T be axiomatized by $\{\sigma_1, \dots, \sigma_n\} \cup \{\tau_k : k \in \omega\}$.

Claim *We may assume each of $\sigma_1, \dots, \sigma_n$ is universal or of the form $\forall \bar{x} \exists y \phi(\bar{x}, y)$, where $\phi(\bar{x}, y)$ is quantifier-free (and possibly $l(\bar{x}) = 0$).*

Proof of Claim: If σ is an axiom in $\{\sigma_1, \dots, \sigma_n\}$ of the form $\forall \bar{x} \exists y \chi(\bar{x}, y)$ where $\chi(\bar{x}, y)$ is not quantifier-free (and possibly $l(\bar{x}) = 0$), introduce a relation symbol R of arity $l(\bar{x}) + 1$, and replace σ by the axioms $\forall \bar{x} \exists y R\bar{x}y$ and $\forall \bar{x} \forall y (R\bar{x}y \leftrightarrow \chi(\bar{x}, y))$. By repeating this procedure we expand T to a complete quasi-finitely axiomatized theory with trivial algebraic closure T^* over a finite relational language L^*, with axioms of the appropriate form. Clearly both or neither of T, T^* have the strict order property.

Given the claim, we may assume that for some $m \in \{1, \dots, n\}$ each of $\{\sigma_1, \dots, \sigma_n\} \setminus \{\sigma_1, \dots, \sigma_m\}$ is universal, and that if $1 \leq i \leq m$ then σ_i is of the form $\forall \bar{x}_i \exists y \phi_i(\bar{x}_i, y)$ and $\phi_i(\bar{x}_i, y)$ is quantifier-free, and $l(\bar{x}_i) = l_i$ (possibly zero).

Fix a countable $\mathcal{M} \models T$. If $\phi(\bar{x}, y)$ is an L-formula and $\bar{a} \in M^{l(\bar{x})}$, then $\phi(\bar{a}, y)$ will denote $\{y : \mathcal{M} \models \phi(\bar{a}, y)\}$, even if there are models other than $\mathcal{M}$ around. By Lemma 2.1 we may suppose that there is an atomic formula $\psi(\bar{x}, y)$ and $\bar{b} \in M^{l(\bar{x})}$ such that $\psi(\bar{b}, y)$ is an infinite coinfinite subset of M.

We shall construct simultaneously

(a) a chain $\mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \dots \subsetneq \mathcal{M}_i \subsetneq \dots$ of substructures of $\mathcal{M}$,
(b) a sequence $(n(i) : i \in \omega)$ of natural numbers, with $1 \leq n(i) \leq m$ for all $i \in \omega$,
(c) a sequence $(\bar{a}_i : i \in \omega)$ where, for each $i \in \omega$, $\bar{a}_i$ is an $l_{n(i)}$-tuple of elements of M_i.

We shall arrange things so that for each $i \in \omega$,

(d) $\mathcal{M}_i \models \neg (\exists y) \phi_{n(i)}(\bar{a}_i, y)$, and
(e) $\phi_{n(i)}(\bar{a}_i, y) \supsetneq \phi_{n(i+1)}(\bar{a}_{i+1}, y)$.

Part (e) will ensure that T has the strict order property.

For the 0^{th} step, choose $\mathcal{M}_0$ to be the substructure of $\mathcal{M}$ with domain $\mathrm{rng}(\bar{b}) \cup \psi(\bar{b}, y) \cup \{z\}$ for some $z \in M \setminus (\mathrm{rng}(\bar{b}) \cup \psi(\bar{b}, y))$. Then because T has trivial algebraic closure and z is algebraic in $\mathcal{M}_0$ over $\bar{b}$, $\mathcal{M}_0 \not\models T$. Clearly $\mathcal{M}_0 \models \tau_k$ for each $k \in \omega$, so as universal axioms of T hold in $\mathcal{M}_0$, $\mathcal{M}_0 \models \neg \sigma_{n(0)}$ for some $1 \leq n(0) \leq m$. Hence there is a (possibly empty) $l_{n(0)}$-tuple $\bar{a}_0$ in M_0 such that $\phi_{n(0)}(\bar{a}_0, y) \subseteq M \setminus M_0$.

After i steps we will have chosen structures $\mathcal{M}_0, \dots, \mathcal{M}_i$, natural numbers $n(0), \dots, n(i)$, and tuples $\bar{a}_0, \dots, \bar{a}_i$ satisfying the obvious restrictions of (a)–(e). At the $(i+1)^{th}$ step, choose $\mathcal{M}_{i+1}$ to contain $M \setminus \phi_{n(i)}(\bar{a}_i, y)$ together with a single point of $\phi_{n(i)}(\bar{a}_i, y)$. Then, as T has trivial algebraic closure, $\mathcal{M}_{i+1} \not\models T$. Hence one of $\sigma_1, \dots, \sigma_m$, say $\sigma_{n(i+1)}$, is violated in $\mathcal{M}_{i+1}$. Thus there is an $l_{n(i+1)}$-tuple $\bar{a}_{i+1}$ in M_{i+1} such that $\mathcal{M}_{i+1} \models \neg (\exists y) \phi_{n(i+1)}(\bar{a}_{i+1}, y)$. So $\phi_{n(i)}(\bar{a}_i, y) \supsetneq \phi_{n(i+1)}(\bar{a}_{i+1}, y)$, and the construction goes through.

Since m is finite, there is an infinite constant subsequence of $(n(i) : i \in \omega)$. To ease notation we shall suppose that $n_i = 1$ for all $i \in \omega$. Put $l := l_1$. Now define a poset on M^l by the rule: if $\bar{b}, \bar{c} \in M^l$, then $\bar{b} \leq \bar{c}$ holds if and only if $\bar{b} = \bar{c}$ or $\phi_1(\bar{b}, y) \subsetneq \phi_1(\bar{c}, y)$. Clearly $\leq$ is a partial ordering on M^l. Since $\bar{a}_{i+1} < \bar{a}_i$ for all $i \in \omega$, the partial ordering has infinite chains, so T has the strict order property.

Remark 1 The above argument does not use the full strength of the assumption that T has trivial algebraic closure. We merely used the fact that for each $i \in \{1, \ldots, m\}$ and each l_i-tuple $\bar{a}$ in M, $\phi_i(\bar{a}, y)$ is infinite or contained in $\mathrm{rng}(\bar{a})$.

2 Theorem 1.2 is a long way from yielding Conjecture 1.1. However, as noted in the proof of Theorem 1.2, it suffices to consider Conjecture 1.1 for model-complete theories in a finite relational language. Unfortunately, the conjecture seems difficult even for theories which have quantifier-elimination in a finite relational language. Note that, if such a theory were a counterexample to the conjecture, then it would have the independence property. For by [9] and [5] (Section 7.4), such a theory would be unstable, and by Shelah ([12], ch. 2, Section 4.7]) an unstable theory without the strict order property must have the independence property. Note too that there are unstable ω-categorical quasi-finitely axiomatized theories without the strict order property. An example is the theory of an infinite dimensional vector space, endowed with a symplectic form, over a finite field [8].

Proof of Proposition 1.3: Let T be a theory satisfying the hypotheses of the theorem, and let $\mathcal{M}$ be an ω-saturated model of T. We again identify formulas with the subsets of M they define. Let $\phi(x, \bar{y})$ be a stable formula defining an infinite coinfinite subset of M. By [12] (ch. 2, Section 2), there are $\bar{c} \in M^{l(\bar{y})}$ and a formula $\psi(x, \bar{z})$ which is a Boolean combination of formulas of the form $\phi(x, \bar{u})$, such that, if

$$\Delta = \{\psi(x, \bar{y}),\ x = y\}, \text{ then } R(\psi(x, \bar{c}), \Delta, \omega) = \mathrm{Mt}(\psi(x, \bar{c}), \Delta, \omega) = 1.$$

From Theorem 9.3 of Harnik and Harrington [7], there is a formula $\chi(x, \bar{d})$ (a positive Boolean combination of conjugates of $\psi(x, \bar{c})$) such that

(i) the symmetric differences of $\psi(x, \bar{c})$ and $\chi(x, \bar{d})$ is finite, and
(ii) any two conjugates of $\chi(x, \bar{d})$ which are distinct as sets have finite intersection.

(Here, $\chi(x, \bar{a})$ is a *conjugate* of $\chi(x, \bar{b})$ if $\mathrm{tp}(\bar{a}/\varnothing) = \mathrm{tp}(\bar{b}/\varnothing)$.) By compactness there is a natural number k and a formula $\rho(\bar{y}) \in \mathrm{tp}(\bar{d})$ such that if

$$\mathcal{M} \models \rho(\bar{c}_1) \wedge \rho(\bar{c}_2)$$

then $\chi(x, \bar{c}_1) = \chi(x, \bar{c}_2)$ or $|\chi(x, \bar{c}_1) \cap \chi(x, \bar{c}_2)| < k$. Thus the set $\chi(x, \bar{d})$ is definable from any k of its members. By the triviality of algebraic closure, if $\mathcal{M} \models \rho(\bar{c}_1) \wedge \rho(\bar{c}_2)$ and $\chi(x, \bar{c}_1) \neq \chi(x, \bar{c}_2)$, then $\chi(x, \bar{c}_1) \cap \chi(x, \bar{c}_2) = \varnothing$. It follows that there is a nontrivial 0-definable equivalence relation E on M, defined by the formula

$$\forall x \forall y (Exy \leftrightarrow \forall \bar{z}(\rho(\bar{z}) \rightarrow (\chi(x, \bar{z}) \leftrightarrow \chi(y, \bar{z})))).$$

Remark By Lemma 2.1, the conclusion of Proposition 1.3 holds if we know that all quantifier-free formulas are stable and T is not the theory of an indiscernible set. It might be possible to prove stronger results, either by considering the way different equivalence classes can be related, or by weakening the hypothesis on algebraic closure to: $\mathrm{acl}(A) = \bigcup(\mathrm{acl}(a) : a \in A)$ for all $A \subseteq M \models T$.

REFERENCES

[1] Adeleke, S. and P. M. Neumann, "On infinite Jordan groups," preprint.

[2] Albert, M. H. and S. N. Burris, "Finite axiomatisations for existentially closed posets and semilattices," *Order*, vol. 3 (1986), pp. 169–178.

[3] Cameron, P. J., "Orbits of permutation groups on unordered sets II," *Journal of the London Mathematical Society* (2), vol. 23 (1981), pp. 249–264.

[4] Cameron, P. J., "Some treelike objects," *Quarterly Journal of Mathematics at Oxford* (2), vol. 38 (1987), pp. 155–183.

[5] Cherlin, G., L. Harrington, and A. H. Lachlan, "$\aleph_0$-categorical, $\aleph_0$-stable structures," *Annals of Pure and Applied Logic*, vol. 28 (1985), pp. 103–135.

[6] Droste, M. "Structure of partially ordered sets with transitive automorphism groups," *Memoirs of American Mathematical Society*, vol. 57, no. 334 (1985).

[7] Harnik, V. and L. Harrington, "Fundamentals of forking," *Annals of Pure and Applied Logic*, vol. 26 (1984), pp. 245–286.

[8] Kantor, W. M., M. W. Liebeck, and H. D. Macpherson, "$\aleph_0$-categorical structures smoothly approximated by finite substructures," *Proceedings of the London Mathematical Society* (3), vol. 59 (1989), pp. 439–463.

[9] Lachlan, A. H., "On countable stable structures which are homogeneous for a finite relational language," *Israel Journal of Mathematics*, vol. 49 (1984), pp. 69–152.

[10] Schmerl, J. H., "Decidability and $\aleph_0$-categoricity of theories of partially ordered sets," *The Journal of Symbolic Logic*, vol. 45 (1980), pp. 585–611.

[11] Schmerl, J. H., "Decidability and finite axiomatizability of $\aleph_0$-categorical partially ordered sets," *The Journal of Symbolic Logic*, vol. 46 (1981), pp. 101–120.

[12] Shelah, S., *Classification theory and the number of non-isomorphic models*, North-Holland, Amsterdam, 1978.

[13] Zil'ber, B. I., "Totally categorical theories: Structural properties and the non-finite axiomatizability," *Proceedings: Model Theory of Algebra and Arithmetic* (Karpacz, Poland, 1979), ed. L. Pacholski et al., Springer-Verlag, Berlin, 1980, pp. 381–410.

School of Mathematical Sciences
Queen Mary and Westfield College
Mile End Road
London E1 4NS, England