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Matrix Representation of Husserl's Part–Whole–Foundation Theory*

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Abstract This paper pursues two aims, a general one and a more specific one. The general aim is to introduce and illustrate the use of Boolean matrices in representing the logical properties of one- and (mainly) two-place predicates over small finite universes, and hence of providing matrix characterizations of finite models for sets of axioms containing such predicates. This method is treated only to the extent required to pursue the more specific aim, which is to consider axiomatic systems involving the part–whole relation together with a relation of foundation employed by Husserl.

1 Husserl structures We present an axiom system which is a first-order formalization of the theory of part–whole–foundation relations suggested by Husserl ([11], [12]: Third Investigation). Our axiom system employs two primitive predicates '$\leq$' and '$\mathfrak{F}$' which denote the *part* and *foundation* relations. Subsequent to Husserl's own work, the part, but not the foundation relation, was studied independently by Stanislaw Leśniewski ([16]; [17]; [18]; [19]) and his student Alfred Tarski ([28], pp. 24–29; [29], pp. 161–172), and later by Henry Leonard and Nelson Goodman ([14]; [15]; [7]; [8]). Quine has also contributed to this development [24], and the whole topic has been studied extensively by Rolf Eberle [5]. Several part–whole concepts developed within the Leśniewski–Tarski–Leonard–Goodman tradition[1] are involved in the present study.

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We begin with the following definitions:

Definition 1
(Proper Part) $x < y$ means $(x \leq y \ \& \ x \neq y)$.
(Nonatomic whole) x is nonatomic means $\exists y \, y < x$.
(Overlap) $\mathfrak{O}xy$ means $\exists u(u \leq x \ \& \ u \leq y)$.
We say x *overlaps* y iff $\mathfrak{O}xy$.

Definition 2 The Part Axioms

A1 $\leq$ is reflexive.
A2 $\leq$ is antisymmetric.
A3 $\leq$ is transitive.
A4 (Overlap axiom) $\forall y \forall z [\forall x(x \leq y \supset \mathfrak{O}xz) \supset y \leq z]$.

Our part axioms A1–A4 are equivalent to an axiomatization offered by David Bostock ([3], p. 113; viz. axiom P1 and definition D2). The converse of the antisymmetry of the part relation (A2) follows from the reflexivity of the part relation (A1), and so we have as a theorem Bostock's definition D2 that two wholes are identical if and only if they are parts of each other. Together with transitivity, we obtain as a further theorem the *principle of individuation* of our part-whole system (cf. [5], p. 37; 2.3.1 and 2.3.2). Two wholes are identical just in case they share all parts. The system is not atomistic in that no axiom requires that every whole contain a part which itself has no proper part. However, both atomistic and finitistic limitations are intrinsic to our method of matrix representation for computer implementation. All models of the system calculable by the methods studied below will have finite, atomistic universes.

Definition 3 The Foundation Axioms

A5 $\forall x \forall y [x \leq y \supset \mathfrak{F}yx]$
A6 $\mathfrak{F}$ is transitive.
A7 $\mathfrak{F}$ is not symmetric.
A8 $\mathfrak{F}$ is not antisymmetric.

The second primitive predicate '$\mathfrak{F}$' of our system is a reflexive, transitive, nonsymmetric and nonantisymmetric relation. This primitive, the raison d'être of the present system, is a formalization of a relation intrinsic to Locke's concept that "In some of our ideas there are certain relations, habitudes, and connexions, so visibly included in the nature of the ideas themselves, that we cannot conceive them separable from them by any power whatsoever" ([20], II, pp. 221–222). As examples of this *necessary connection* of ideas, Locke provides:

> It is true, solidity cannot exist without extension, neither can scarlet colour exist without extension, but this hinders not, but that they are distinct ideas. Many ideas require others, as necessary to their existence or conception, which yet are very distinct ideas. Motion can neither be, nor be conceived, without space; and yet motion is not space, nor space motion; space can exist without it, and they are very distinct ideas. . . . ([20], I, p. 226)

Locke contrasts a second (contingent) type of "connexion of ideas wholly owing to chance or custom" ([20], I, p. 529). As examples, Locke mentions the connection between nausea and honey for a person who had once made himself sick eating honey, and the connection between goblins, spirits, and darkness for a child who has been frightened by horror stories ([20], I, p. 531).

The foundation relation taken as primitive in the present study is the first type of connection mentioned by Locke but ignored by Hume and more recent empiricist philosophers.[2] This relation has been systematically examined in connection with the part relation only in recent studies[3] of Husserl's thought, which involved a notion of necessary connection amongst parts and wholes ([11], [12]: Investigation III; [21], pp. 126–142):

> A content of the species A is *founded upon* a content of the species B, if an A can by its essence (i.e. legally, in virtue of its specific nature) not exist, unless a B also exists: this leaves open whether the coexistence of a C, a D etc. is needed or not. . . . ([12], p. 475)

The intuitive (extrasystemic) idea of the foundation relation is just this idea that x is *founded upon* y just in case x cannot exist in the absence of y.

The axioms for the primitive '$\mathfrak{F}$' follow directly from this extrasystemic idea. The (implied) reflexivity and transitivity of $\mathfrak{F}$ are clear. $\mathfrak{F}$ is not symmetric (A7) because there are cases of unilateral foundation such as Locke's example above of space and motion. $\mathfrak{F}$ is not antisymmetric (A8) because there are cases of bilateral foundation. For example, any two of the three qualities of tone—pitch, loudness, and timbre—are mutually founding. The foundation of a whole on each of its parts (A5) is motivated by our *principle of individuation* in connection with the intuitive idea of foundation guiding our choice of axioms. If the identity of a whole is determined by all of its parts, then the whole must cease to exist if and only if some of its parts cease to exist.

Definition 4 Let U be a set and let $\leq$ and $\mathfrak{F}$ be binary relations on U. A *Husserl H-structure* is a triple $(U, \leq, \mathfrak{F})$ which satisfies each member of the set $\Sigma_{\leq\mathfrak{F}}$ of eight axioms A1–A8.

We call models of these eight axioms Husserl structures (for short, H-structures) because Husserl's six theorems ([12], pp. 463–465) concerning dependent and independent part relations are provable from these axioms [22]. While we also have intuitive reasons for the assumption of A12 and A13, given below in Section 9, the formulation of these axioms depends on a choice of one of the sum axioms included in this study. The need to decide amongst these several sum axioms was part of the reason for undertaking the work reported on below. A1–A8 thus comprise a common basis of several more complete systems, distinguished by different sum axioms and commensurate formulations of A12 and A13.

We now present two infinite Husserl structures, one in number theory, the other from set theory. These are followed by a finite model, a central example used throughout this paper.

Example 1 Let $U = Z^+$, the set of positive integers. Define $x \leq y$* to be the relation (1) $x = y$ or (2) x and y are squarefree† integers greater than 1 and x di-

*The predicate '$\leq$' always denotes the part relation, never the arithmetic 'less than or equal' relation, which we shall always denote by the predicate '$\leqq$'.

†A positive integer >1 is squarefree iff it is a product of distinct primes.

vides y. Define $\mathfrak{F}xy$ to mean y divides a power of x. One can check that $(U,\leq,\mathfrak{F})$ is an H-structure. To see that $\mathfrak{F}$ is not symmetric, note that $(6,2) \in F$ but $(2,6) \notin \mathfrak{F}$. To see that $\mathfrak{F}$ is not antisymmetric, observe that $(2,4) \in \mathfrak{F}$ and $(4,2) \in \mathfrak{F}$, but $2 \neq 4$. For this model $\mathfrak{O}xy$ has the interpretations (1) $x = y$ or (2) x and y are squarefree and not relatively prime to each other.

Example 2 Interpretations of the primitive predicates of $\leq$-$\mathfrak{F}$ in set theory: Let U be the iterated hierarchy of sets and define $x \leq y$ to be $x \subseteq y$ and $\mathfrak{F}xy$ to be $y \subseteq$ the transitive closure of x. Here z is the transitive closure of x means that (i) z contains all members of x and (ii) for each $y \in z$, $u \in y$ implies $u \in z$. Note that $\mathfrak{O}xy$ means that x and y are not disjoint sets.

Example 3 Let $U = \{1,2,3,4\}$, $\leq = \{(1,1),(2,2),(3,3),(4,4),(2,1),(3,1),(4,1)\}$ and $\mathfrak{F} = \{(1,1),(2,2),(3,3),(4,4),(2,1),(3,1),(4,1),(2,4),(4,2)\}$. Then $(U,\leq,\mathfrak{F})$ is a 4 member H-structure. In fact, $\leq$ and $\mathfrak{F}$ are just the predicates of Example 1 restricted to the set $U = \{1,2,3,4\} \subset Z^+$.

2 Instantiation and dependence We now define two types of wholes of interest in $\leq$-$\mathfrak{F}$ contexts of the sort Goodman termed "realistic" ([7], p. 142); specifically, contexts in which concrete individuals are considered sums of qualities (cf. [1], pp. 7–10, 76–84). We share a major goal of Goodman ([7]: VII) in establishing a distinction between abstract and concrete individuals. Under the intended interpretation, a whole is *abstract* if it satisfies the predicate '$\mathfrak{D}$' ('is dependent', intended as Husserl's "unselbständig") defined below, and is *concrete* otherwise. The dependence of qualities is a strictly extrasystemic assumption in systems in which the part predicate is the only primitive, but it can be defined systemically if the foundation predicate is available. Husserl's idea was that dependent wholes (which he also called "moments") "only exist (if at all) as parts of more inclusive wholes" ([12], p. 447). The central idea is that being founded on anything other than its own parts makes a whole dependent.

Definition 5 (Dependent whole) $\mathfrak{D}x$ means $\exists y(\sim\mathfrak{O}yx \mathbin{\&} \mathfrak{F}xy)$.

We say x is a dependent whole if $\mathfrak{D}x$; otherwise, x is an independent whole.[4] For example, the pitch, loudness, and timbre of a tone are mutually founding atomic moments of the tone. Thus, the tone is independent, so long as further foundation relations are ignored, while each of the three mutually founding parts satisfies the predicate '$\mathfrak{D}$'.

Goodman has been criticized ([5], p. 40; [7], pp. 149, 155–156) for his principle that an individual "need not have personal integration" ([8], p. 156), a principle which has led to objections to his sum axiom as too strong ([5], pp. 40–41, 62, 90–91, 97–98, 138–139, 192). According to this principle, the sum of "Plato, this sheet of paper, and the Taj Mahal" ([8], p. 155) is as much an individual as is a quality of Plato, or Plato himself. As with dependence for qualities, the integration felt necessary for individuals, whether abstract or concrete, by Goodman's critics can be defined systematically in $\leq$-$\mathfrak{F}$ primitive systems.

To do this, Husserl introduces his *pregnant concept of whole* (die prägnante Begriff des Ganzen): "a range of contents which are all covered by a *single foundation* without the help of further contents . . . *singleness of the foundation* implies that every [part] is *foundationally connected* . . . with every [part]" ([12], p. 475). His definition can be interpreted as the criterion that each two parts be reciprocally founded moments ([26], p. 141).

Definition 6 (Instance[5]) $\mathfrak{I}x$ means $\forall y \forall z[(y \leq x \;\&\; z \leq x) \supset \mathfrak{F}yz]$.[6]

The sum of discrete (nonoverlapping) independent instances cannot be an instance. Insofar as Plato, this piece of paper, and the Taj Mahal are considered to be independent instances, their mereological sum is independent, but not an instance, because it lacks the necessary integration required by Definition 6. By our axioms A1–A8,

Proposition 1 *If x is an instance, then every proper part of x is a dependent instance.*

Proof: Suppose y is a proper part of x. By antisymmetry, $x \not\leq y$. By the contrapositive of the overlap axiom, there exists a z such that $z \leq x$ and $\sim Ozy$. In other words, y and z are nonoverlapping parts of x. Since x is an instance, we have $\mathfrak{F}yz$, and thus y is dependent. It is easy to see from Definition 6 and A3 that y is an instance. (For a detailed discussion of instance and dependence, see [22].[7])

Example 4 We return to the number theory model in Example 1. If $x \in Z^+$ is not squarefree then $y \leq x$ if and only if $y = x$. If x is a squarefree composite integer, let p and q be distinct prime factors of x. Then $p \leq x$ and $q \leq x$, but neither p nor q is founded on each other. Hence $\mathfrak{I}x$ holds if and only if (1) x is not squarefree or (2) x is prime. To interpret $\mathfrak{D}x$, note that if $x \geqq 2$ then $y = x^2$ satisfies $\mathfrak{F}xy$ but not $\mathfrak{O}xy$ (since y is not squarefree). Thus all integers $\geqq 2$ are dependent with 1 being the only independent whole.

3 Boolean matrices and relations The interest of the matrix method developed in this paper is two-fold. Our main application is the computer implementation of modelhood. A second objective is to prove theorems about finite models using matrix calculations (cf. Proposition 27).

We say that an $n \times n$ matrix $A = [a_{ij}]$ is a *boolean matrix* iff $a_{ij} \in \{0,1\}$ for each i,j, $1 \leqq i, j \leqq n$. On occasion we shall be dealing with matrices whose entries are integers other than 0 or 1. To distinguish these from boolean matrices, we shall refer to them as *integral matrices.*

We define the boolean operations $\vee$ and $\cdot$ on the set $\{0,1\}$ to be the standard logical connectives of disjunction and conjunction, respectively, when we interpret 1 as true and 0 as false. As in algebra, we often denote $\cdot$ by adjacency of symbols, e.g. $ab = a \cdot b$. We also define the negation predicate $\bar{x}$ as $\bar{x} = 1 - x$.

The definitions of $\vee$ and $\cdot$ for the boolean values 0 and 1 coincide with the usual arithmetic + and $\cdot$ operations for the integers 0 and 1, with the single exception that in the boolean case, $1 \vee 1 = 1$, not 2. In this same spirit we define the comparison relations $=, <, \leqq, >$, and $\geqq$ between the boolean elements 0 and 1 as in arithmetic, where 0 and 1 are thought of as integers.

We extend these comparisons to matrices as follows. For the rest of this section let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $n \times n$ boolean matrices. We say that a comparison relation $(=,<,\leqq,>,\geqq)$ between A and B holds if and only if the comparison holds between every pair of entries a_{ij} and b_{ij}. For example, $A < B$ means $a_{ij} < b_{ij}$ for every i,j, $1 \leqq i, j \leqq n$. As in arithmetic, we denote the negation of each comparison relation by a slash: $A \not< B$ means it is false that $A < B$. In manipulating "matrix inequalities", care must be exercised, because not all standard properties of the comparison relations for integers hold in the matrix case. For example, for matrices, $A \leqq B$ and $B \leqq A$ together imply that $A = B$, just as in the familiar arithmetic rule. On the other hand, trichotomy fails in the matrix case. For instance, if $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then neither $A \leqq B$ nor $A \geqq B$ is true. In particular, $A \not< B$ does not imply $A \geqq B$.

We now define the following operations on A and B. The *transpose* of A, denoted by A^t, is obtained by interchanging the rows and columns of A, i.e. $A^t = [a_{ij}]^t = [a_{ji}]$. The *matrix join* $A \vee B$ and the *entrywise product* $A \times B$ are defined by $A \vee B = [a_{ij} \vee b_{ij}]$ and $A \times B = [a_{ij} b_{ij}]$. In contrast, the *(relative) matrix product* AB is defined by $AB = [c_{ij}]$ if and only if

$$(1) \qquad c_{ij} = \bigvee_{k=1}^{n} a_{ik} b_{kj}, \qquad 1 \leqq i,j \leqq n.$$

where $\bigvee_{k=1}^{n} x_k = x_1 \vee x_2 \vee \cdots \vee x_n$. We define the *matrix negation* $\bar{A} = [\bar{a}_{ij}]$. Observe that the join, negation, and both products of boolean matrices are again boolean matrices. We will also need the standard *identity* matrix I, with 1's on the main diagonal and 0's elsewhere, as well as the *full* matrix J whose entries are all 1's.

Now fix n, a positive integer, and let $U = \{1, \ldots, n\}$. We associate with the binary relation R on U the $n \times n$ boolean matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, & \text{if } Rij \\ 0, & \text{otherwise.} \end{cases}$$

A is called the *incidence* matrix of R.[8]

It is easy to see how to describe standard properties of R in terms of conditions on A.

Proposition 2

(i) *R is reflexive if and only if $A \geqq I$.*
(ii) *R is symmetric if and only if $A^t = A$.*
(iii) *R is antisymmetric if and only if $A \times A^t \leqq I$.*
(iv) *R is transitive if and only if $A^2 \leqq A$.*

In general we do not have $A^2 = A$ for a transitive relation. For example, suppose $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Since (2,1) is the only pair in the relation, R is clearly transitive, but $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. If R is reflexive, however, we can strengthen the above result to

Proposition 3 *If R is reflexive, then R is transitive if and only if $A^2 = A$.*[9]

We now explore the interplay between two relations on the same universe U. In addition to the relation R, with incidence matrix A, assume that S is another binary relation on U and let B be the incidence matrix of S. The next proposition shows that implications between R and S can be expressed in terms of A and B.

Proposition 4 *$\forall x \forall y\, Rxy \supset Sxy$ if and only if $A \leqq B$.*

4 Matrix representation of Husserl structures For this section assume that $U = \{1, \ldots, n\}$. Let $\leq$ and $\mathfrak{F}$ be binary relations on U, with incidence matrices P and F, respectively. We call the incidence matrix of the overlap relation $\mathfrak{O}$ the *overlap matrix* $O = [o_{ij}]$. It is easy to prove

Proposition 5 $O = P^tP$.

Proposition 6 *O is symmetric.*

Proof: $O^t = (P^tP)^t = P^t(P^t)^t = P^tP = O$.

The next theorem shows how to translate the eight axioms of a finite Husserl structure in terms of the incidence matrices P, F, and O.

Theorem 7
(i) A1 *is equivalent to* $I \leqq P$.
(ii) A2 *is equivalent to* $P \times P^t \leqq I$.
(iii) A3 *is equivalent to* $P^2 \leqq P$.
(iv) A4 *is equivalent to* $P \vee P^t\bar{O} = J$.
(v) A5 *is equivalent to* $P^t \leqq F$.
(vi) A6 *is equivalent to* $F^2 \leqq F$.
(vii) A7 *is equivalent to* $F^t \neq F$.
(viii) A8 *is equivalent to* $F \times F^t \not\leqq I$.

Proof: Every statement except (iv) is immediate from Propositions 2 and 4. To prove (iv), let $A = [a_{ij}] = P^t = [p_{ji}]$, $B = [b_{ij}] = \bar{O} = [\bar{o}_{ij}]$, and $C = [c_{ij}] = AB$. Then the overlap axiom A4 holds

$$\Leftrightarrow \forall y \forall z[\forall x(x \leq y \supset \mathfrak{O}xz) \supset y \leq z] \Leftrightarrow \forall y \forall z[y \leq z \text{ or } \exists x(x \leq y \text{ and } \sim\mathfrak{O}xz)]$$

$$\Leftrightarrow \forall y \forall z[p_{yz} = 1 \text{ or } \exists x(a_{yx} = 1 \text{ and } b_{xz} = 1)] \Leftrightarrow \forall y \forall z\left[p_{yz} = 1 \text{ or } \bigvee_{x=1}^{n} a_{yx}b_{xz} = 1\right]$$

$$\Leftrightarrow \forall y \forall z[p_{yz} \vee c_{yz} = 1] \Leftrightarrow P \vee C = J.$$

Example 5 Let $(U, \leq, \mathfrak{F})$ be the 4-dimensional model of Example 3 above. Then

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Testing axioms A1–A3 and A5–A8 is purely mechanical. By Proposition 5, we have that

$$O = P^tP = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Further calculation shows

$$P^t\bar{O} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Thus, $P \vee P^t\bar{O} = J$, verifying the overlap axiom for this model. Notice how perfect the fit is here, i.e. $\bar{P}$ is exactly $P^t\bar{O}$.

5 Vector representation of instance and dependence So far we have dealt exclusively with binary relations. Assume now that R is a unary relation on $U = \{1,2,\ldots,n\}$. Then the *incidence vector* associated with R is

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ defined by } x_i = \begin{cases} 1, & \text{if } Ri \\ 0, & \text{otherwise.} \end{cases}$$

As in Section 3, a vector X whose entries are 0 or 1 is called a *boolean vector.* Let $A = [a_{ij}]$ be an $n \times n$ boolean matrix. Then the product AX is the boolean vector

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

satisfying

(2) $$b_i = \bigvee_{j=1}^{n} a_{ij}b_j, \qquad l \leqq i \leqq n.$$

We say that A is a *diagonal* matrix if $a_{ij} = 0$ for all $i \neq j$, $1 \leqq i, j \leqq n$, and we define $\operatorname{diag}(X)$ to be the diagonal matrix consisting of the elements $x_1, \ldots, x_n$ along the main diagonal. It is clear that diag is a bijection from the set of all n-dimensional boolean vectors onto the set of all $n \times n$ boolean diagonal matrices. Going backwards, if $A = [a_{ij}]$ is any $n \times n$ matrix, then we will write

(3) $$\operatorname{diag}^{-1}(A) = \begin{bmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{nn} \end{bmatrix}.$$

As in Section 3, we define the relations $=$, $<$, $\leqq$, $>$, and $\geqq$ between boolean vectors by requiring that these relations hold entrywise.

Now let I and D be the incidence vectors associated respectively with $\mathfrak{I}$ and $\mathfrak{D}$. As usual, P, F, and O are the incidence matrices of $\leq$, $\mathfrak{F}$, and $\mathfrak{O}$, respectively. Then

Proposition 8 $\bar{I} = \mathrm{diag}^{-1}[P^t\bar{F}P]$.

Proof: Let $A = [a_{ij}] = \bar{F}$, $B = [b_{ij}] = P^t$, $C = [c_{ij}] = AP$, $D = [d_{ij}] = BC$. We have

$$\mathfrak{I}x \Leftrightarrow \forall y \forall z[(y \le x \text{ and } z \le x) \supset \mathfrak{F}yz] \Leftrightarrow \forall y \forall z(f_{yz} = 1 \text{ or } p_{yx} = 0 \text{ or } p_{zx} = 0)$$

$$\Leftrightarrow \bigvee_{y=1}^{n} p_{yx} \bigvee_{z=1}^{n} a_{yz} p_{zx} = 0 \Leftrightarrow \bigvee_{y=1}^{n} b_{xy} c_{yx} = 0 \Leftrightarrow d_{xx} = 0.$$

Note: Since multiplication of boolean matrices is associative, the triple product in Proposition 8 may be parenthesized $P^t(\bar{F}P)$ or $(P^t\bar{F})P$.

Proposition 9 $D = \mathrm{diag}^{-1}(F\bar{O})$.

Proof: Let $A = [a_{ij}] = \bar{O}$ and $B = [b_{ij}] = FA$. We have

$$\mathfrak{D}x \Leftrightarrow \exists y(\sim \mathfrak{O}yx \;\&\; \mathfrak{F}xy) \Leftrightarrow \exists y(o_{yx} = 0 \text{ and } f_{xy} = 1) \Leftrightarrow b_{xx} = \bigvee_{y=1}^{n} f_{xy} a_{yx} = 1.$$

Example 6 For the finite model of Example 5 above, we compute

$$D = \mathrm{diag}^{-1}(F\bar{O}) = \mathrm{diag}^{-1}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus 1 and 3 are independent wholes, while 2 and 4 are dependent. To determine the instances for this model, a routine calculation gives

$$B = P^t(\bar{F})P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Hence

$$I = \mathrm{diag}^{-1}\bar{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

so every element except 1 is an instance.

6 Sum axioms We begin with the least upper bound notion. We say that u is an upper bound of (all wholes satisfying) a formula ϕ iff every whole satisfying ϕ is part of u. We say that y is the least upper bound (LUB) of ϕ iff (i) y itself is an upper bound of ϕ and (ii) y is a part of any other upper bound of ϕ. The motivation for the other two sum concepts originates from the set theory model of Example 2, where U is a collection of sets and the part relation is the ordinary subset relation. We want to define the 'sum' of two sets x and y to coincide with the set theoretical union of x and y. Of course this definition must be solely in terms of $\leq$ itself (or the defined predicate $\mathcal{O}$). So we search for an abstract condition which characterizes $x \cup y$. Observe that if $z = x \cup y$, then z has the property that any set which intersects z must intersect either x or y, and conversely. This property is the basis of the Goodman–Leonard binary sum. We extend this concept by defining the Goodman-Leonard sum y of all wholes satisfying a formula ϕ to mean anything overlaps y just in case it overlaps some whole which satisfies ϕ. We present an alternative sum formulation originating from the work of Leśniewski and Tarski,[10] which requires that every whole satisfying ϕ is a part of the sum y and no part of y is disjoint from such wholes.

Definition 7 (Summation) Let ϕ be a metalinguistic variable ranging over formulas of the standard first-order language of $\Sigma_{\leq\mathfrak{F}}$.

$$y = \text{LUB}\,\phi \text{ means } \forall z(\phi z \supset z \leq y) \;\&\; \forall u[\forall z(\phi z \supset z \leq u) \supset y \leq u]$$

$$y = \text{Gsum}\,\phi \text{ means } \forall z[\mathcal{O}zy \equiv \exists x(\phi x \;\&\; \mathcal{O}zx)]$$

$$y = \text{Lsum}\,\phi \text{ means } \forall z(\phi z \supset z \leq y) \;\&\; \forall w[w \leq y \supset \exists z(\phi z \;\&\; \mathcal{O}wz)].$$

Remark If $y = \text{LUB}\,\phi$ exists, then it is unique. For suppose that y' is another LUB of S, then we have $y \leq y'$ and $y' \leq y$. By antisymmetry, $y = y'$. Similarly, the other sum concepts are well-defined.

The existence axioms A9–A11 assert that every nonempty formula has a sum.

Definition 8 (Summation Existence Axioms)

A9 $\exists x \phi x \supset \exists y(y = \text{LUB}\,\phi)$.
A10 $\exists x \phi x \supset \exists y(y = \text{Gsum}\,\phi)$.
A11 $\exists x \phi x \supset \exists y(y = \text{Lsum}\,\phi)$.

7 Sum axioms for a finite universe From now on, $U = \{1,2,\ldots,n\}$, where n is a fixed positive integer, and $(U,\leq,\mathfrak{F})$ is a Husserl structure. Let $x_1,\ldots,x_t$ be a sequence of (not necessarily distinct) elements of U. By $\text{LUB}(\{x_1,\ldots,x_t\})$ we mean $\text{LUB}\,\phi$ where ϕ is the formula

$$x = x_1 \vee x = x_2 \vee \cdots \vee x = x_t.$$

Similarly for Gsum and Lsum.[11]

Definition 9 (Binary Summation)[12]

(LUB) $z = x \oplus y$ means $x \leq z \;\&\; y \leq z \;\&\; \forall u[(x \leq u \;\&\; y \leq u) \supset z \leq u]$
(G–L) $z = x + y$ means $\forall u[\mathcal{O}uz \equiv (\mathcal{O}ux \vee \mathcal{O}uy)]$
(L–T) $z = x +_{\text{L}} y$ means $x \leq z \;\&\; y \leq z \;\&\; \forall w[w \leq z \supset (\mathcal{O}wx \vee \mathcal{O}wy)]$.

As expected, all sums (for a finite universe) may be constructed from binary sums.

Lemma 10
(i) *If* $x_1 \oplus x_2 \oplus \cdots \oplus x_t = y$ *then* $y = \mathrm{LUB}(\{x_1, \ldots, x_t\})$.
(ii) *If* $x_1 + \cdots + x_t = y$, *then* $y = \mathrm{Gsum}(\{x_1, \ldots, x_t\})$.
(iii) *If* $x_1 +_{\mathrm{L}} \cdots +_{\mathrm{L}} x_t = y$, *then* $y = \mathrm{Lsum}(\{x_1, \ldots, x_t\})$.

Proof of (iii): (The same induction scheme applies to prove (i) and (ii). The details are straightforward.) By induction on t. The lemma clearly holds if $t = 1$ by reflexivity. Assume for some fixed $t \geqq 1$ that the statement of the lemma holds for each t-tuple. Let $x_1, \ldots, x_{t+1}$ be elements of U, not necessarily distinct, and assume that the sum $x_1 +_{\mathrm{L}} \cdots +_{\mathrm{L}} x_{t+1}$ exists, say $x_1 +_{\mathrm{L}} \cdots +_{\mathrm{L}} x_{t+1} = z$. Then $z = y +_{\mathrm{L}} x_{t+1}$, where $y = x_1 +_{\mathrm{L}} \cdots +_{\mathrm{L}} x_t$. By the induction hypothesis, $y = \mathrm{Lsum}(\{x_1, \ldots, x_t\})$, which implies

(4) $$x_k \leq y \quad \text{for } 1 \leqq k \leqq t$$

and

(5) $$\forall w[w \leq y \supset (\mathcal{O}wx_1 \vee \cdots \vee \mathcal{O}wx_t)].$$

The equation $z = y +_{\mathrm{L}} x_{t+1}$ implies

(6) $$y \leq z, \quad x_{t+1} \leq z$$

and

(7) $$\forall w[w \leq z \supset (\mathcal{O}wy \vee \mathcal{O}wx_{t+1})].$$

Using (4), (6), and transitivity, we have that $x_k \leq z$ for $1 \leqq k \leqq t + 1$. It remains to show that

(8) $$\forall w[w \leq z \supset (\mathcal{O}wx_1 \vee \cdots \vee \mathcal{O}wx_{t+1})].$$

Let $w \in U$ and assume $w \leq z$. We will prove $\mathcal{O}wx_1 \vee \cdots \vee \mathcal{O}wx_{t+1}$. For consider any $v \in U$ which satisfies $v \leq w$. By transitivity, $v \leq z$. Hence by (7), $\mathcal{O}vy$ or $\mathcal{O}vx_{t+1}$. If $\mathcal{O}vx_{t+1}$, then we have $\mathcal{O}wx_{t+1}$, since $v \leq w$, and we are done. Suppose, on the other hand, that $\mathcal{O}vx_{t+1}$ fails to hold for every v such that $v \leq w$. Then $\mathcal{O}vy$ holds whenever $v \leq w$. Applying A4 gives $w \leq y$. By (5) we have $\mathcal{O}wx_1 \vee \cdots \vee \mathcal{O}wx_t$. Thus the lemma is true for $t + 1$.

As an easy consequence of Lemma 10, we have

Theorem 11
(i) A9 *holds if and only if* $\forall x \forall y \exists z(x \oplus y = z)$
(ii) A10 *holds if and only if* $\forall x \forall y \exists z(x + y = z)$.
(iii) A11 *holds if and only if* $\forall x \forall y \exists z(x +_{\mathrm{L}} y = z)$.

Corollary 12 *If* A9 *holds then* $(U, \oplus)$ *is an abelian semigroup.*

Proof: Theorem 11(i) guarantees that any elements x and y have an LUB sum, which by Lemma 10(i) equals $\mathrm{LUB}(\{x,y\})$. Since the LUB is unique, it follows that $\oplus$ is a binary operation. Moreover, $\oplus$ is commutative since $x \oplus y = \mathrm{LUB}(\{x,y\}) = \mathrm{LUB}(\{y,x\}) = y \oplus x$. To see that $\oplus$ is associative, we use the fact that both $(x \oplus y) \oplus z$ and $x \oplus (y \oplus z)$ are equal to $\mathrm{LUB}(\{x,y,z\})$.

We now wish to show that the Goodman–Leonard sum is as strong as the LUB sum. This result follows from four easy lemmas.

Lemma 13 $\forall x \forall y [x \leq y \supset \mathcal{O}xy]$.

Proof: Immediate from reflexivity.

Lemma 14 $\forall x \forall y [x \leq y \supset \forall u(\mathcal{O}ux \supset \mathcal{O}uy)]$.

Proof: Assume $x \leq y$. Let $u \in U$ and suppose $\mathcal{O}ux$. By the definition of overlap, there exists a w such that $w \leq u$ and $w \leq x$. By transitivity, $w \leq x$ and $x \leq y$ implies $w \leq y$. From $w \leq u$ and $w \leq y$ we conclude $\mathcal{O}uy$. Thus, $\forall u(\mathcal{O}ux \supset \mathcal{O}uy)$, as was to be shown.

Lemma 15 *If* $y = \text{Gsum}\, \phi$ and ϕx, then $x \leq y$.

Proof: Assume $y = \text{Gsum}\, \phi$ and ϕx. Let $u \in U$. By Lemma 13, $u \leq x$ implies $\mathcal{O}ux$, which in turn implies $\mathcal{O}uy$ by the definition of Gsum. We have thus shown $\forall u[u \leq x \supset \mathcal{O}uy]$. By the Overlap axiom, $x \leq y$.

As a special case of Lemma 15, we state

Lemma 16 $\forall x \forall y \forall z[(x + y = z) \supset (x \leq z \;\&\; y \leq z)]$.

Theorem 17 *If* $x + y = z$, *then* $x \oplus y = z$.

Proof: Assume $x + y = z$. Immediately from Lemma 16 we have $x \leq z$ and $y \leq z$. In order to show that $x \oplus y = z$, it remains to establish that $\forall v[(x \leq v \;\&\; y \leq v) \supset z \leq v]$. So let $v \in U$ and assume that $x \leq v$ and $y \leq v$. We must show that $z \leq v$. We claim that $\forall u[u \leq z \supset \mathcal{O}uv]$. To see this, let $u \in U$ and assume $u \leq z$. Then $\mathcal{O}uz$ by Lemma 13. Hence $\mathcal{O}ux$ or $\mathcal{O}uy$ by the definition of $+$. Since $x \leq v$, $\mathcal{O}ux$ implies $\mathcal{O}uv$ by Lemma 14. Similarly, $\mathcal{O}uy$ implies $\mathcal{O}uv$. In either case, we have $\mathcal{O}uv$, which proves the claim. Applying axiom A4 to the claim yields $z \leq v$, as required.

Corollary 18 A10 *implies* A9.

Corollary 19 *If* A10 *holds, then* $(U,+)$ *is an abelian semigroup.*

In comparing the G–L and L–T binary sums, we have the following result.

Theorem 20 $x +_{\mathrm{L}} y = z$ *if and only if* $x + y = z$.

Proof: ("if") Assume $x + y = z$. By Lemma 16, $x \leq z$ and $y \leq z$. It follows from Lemma 13 and the equivalence $\mathcal{O}wz \equiv (\mathcal{O}wx \vee \mathcal{O}wy)$ that $w \leq z \supset (\mathcal{O}wx \vee \mathcal{O}wy)$. This proves $x +_{\mathrm{L}} y = z$.

("only if") Assume $x +_{\mathrm{L}} y = z$. Then (i) $x \leq z$ and $y \leq z$ and (ii) $\forall t[t \leq z \supset (\mathcal{O}tx \vee \mathcal{O}ty)]$. Let $w \in U$. We must show that $\mathcal{O}wz \equiv (\mathcal{O}wx \vee \mathcal{O}wy)$. To do this, we prove that each side of this equivalence implies the other side.

To see: $\mathcal{O}wz$ implies $(\mathcal{O}wx \vee \mathcal{O}wy)$. Assume $\mathcal{O}wz$. Then there exists a t such that $t \leq w$ and $t \leq z$. By (ii), $\mathcal{O}tx$ or $\mathcal{O}ty$. Now $\mathcal{O}tx$ implies $\mathcal{O}wx$ since $t \leq w$. Similarly, $\mathcal{O}ty$ implies $\mathcal{O}wy$. Hence, $\mathcal{O}wx \vee \mathcal{O}wy$.

To see: $(\mathcal{O}wx \vee \mathcal{O}wy)$ implies $\mathcal{O}wz$. Assume $\mathcal{O}wx \vee \mathcal{O}wy$. Then $\mathcal{O}wx$ implies $\mathcal{O}wz$ since $x \leq z$ from (i), while $\mathcal{O}wy$ also implies $\mathcal{O}wz$ since $y \leq z$. Therefore, $\mathcal{O}wz$.

Corollary 21 A10 *is equivalent to* A11.

8 Matrix representation of sum axioms Let $P = [p_{ij}]$ be the incidence matrix of $\leq$. Our goal is to transform A9 and A10 into matrix conditions on P. We begin with a necessary condition.

Proposition 22 *If* A9 *holds, then one column of P contains all* 1's.

Proof: Take $S = U$ in Proposition 8. Then there exists a y such that for all $x \in U$, $x \leq y$. Thus, the y^{th} column of P will contain all 1's.

By Corollary 18, the necessary condition above holds for the G–L sum axiom.

Proposition 23 *If* A10 *holds, then one column of P contains all* 1's.

In order to obtain a necessary and sufficient condition for the LUB axiom in terms of the binary operator $\oplus$, we need the following intermediate result, whose derivation is left to the reader.

Proposition 24 $x \oplus y = z$ *if and only if* $\forall u[(x \leq u \,\&\, y \leq u) \equiv z \leq u]$.

We now return to matrices. We need a matrix function which multiplies the i^{th} row of a matrix by 2^{i-1}.

Definition 10 The *binary power function* $\mathbf{B}$ is defined as follows: if $A = [a_{ij}]$ is an $n \times n$ boolean matrix, then $\mathbf{B}(A)$ is the $n \times n$ integral matrix $B = [b_{ij}]$ where $b_{ij} = a_{ij}2^{i-1}$.

Proposition 25 $A\mathbf{B}(A^t)$ *is symmetric.*

Proof: Put $B = [b_{ij}] = \mathbf{B}(A^t) = [a_{ji}2^{i-1}]$. Then $\sum_{k=1}^n a_{ik}b_{kj} = \sum_{k=1}^n a_{ik}a_{jk}2^{k-1} = \sum_{k=1}^n a_{jk}b_{ki}$.

Definition 11 We say the matrix $A = [a_{ij}]$ is *diagonally closed* if the set of all entries of A is a subset of the entries along the main diagonal, i.e. if $\{a_{ij}: 1 \leqq i, j \leqq n\} \subseteq \{a_{mm}: 1 \leqq m \leqq n\}$.

We use Theorem 11(i) to obtain a necessary and sufficient condition for the LUB axiom in terms of the matrix P.

Theorem 26 *The* LUB *axiom holds if and only if* $P\mathbf{B}(P^t)$ *is diagonally closed.*

Proof: Put $A = [a_{ij}] = P^t$, $B = [b_{ij}] = \mathbf{B}(A)$, and $C = [c_{ij}] = PB$. By Theorem 11(i), $(U,\leq)$ satisfies the LUB axiom

$\Leftrightarrow \forall i \forall j \exists m (i \oplus j = m)$

$\Leftrightarrow \forall i \forall j \exists m \forall k [(i \leq k \;\&\; j \leq k) \equiv m \leq k]$ [by Prop. 24]

$\Leftrightarrow \forall i \forall j \exists m \forall k [(p_{ik} p_{jk} = 1) \equiv (p_{mk} = 1)]$

$\Leftrightarrow \forall i \forall j \exists m \forall k (p_{ik} p_{jk} = p_{mk}^2)$ [since $p_{mk} = p_{mk}^2$]

$\Leftrightarrow \forall i \forall j \exists m \sum_{k=1}^{n} p_{ik} a_{kj} 2^{k-1} = \sum_{k=1}^{n} p_{mk} a_{km} 2^{k-1}$

[by the uniqueness of binary representation of integers]

$\Leftrightarrow \forall i \forall j \exists m \sum_{k=1}^{n} p_{ik} b_{kj} = \sum_{k=1}^{n} p_{mk} b_{km}$

$\Leftrightarrow \forall i \forall j \exists m \; c_{ij} = c_{mm}$

$\Leftrightarrow C$ is diagonally closed.

As an application of Theorem 26, we have

Proposition 27 *For $U = \{1,2,\ldots,n\}$, define $\leq = \{(x,x) : x \in U\} \cup \{(x,1) : x \in U\}$. Then $(U,\leq)$ satisfies the* LUB *axiom.*

Proof: We calculate

$$\mathbf{PB}(P^t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & & 0 \\ 1 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 0 & & 0 \\ 0 & 0 & 4 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & 2^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 1 & & 1 \\ 1 & 1 & 5 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & & 1 & 2^{n-1}+1 \end{bmatrix},$$

the matrix whose entries off the main diagonal are all 1's, and whose diagonal entries $a_{ii} = 1 + 2^{i-1}$. Clearly $\mathbf{PB}(P^t)$ is diagonally closed. Thus $(U,\leq)$ satisfies A9 by Theorem 26.

Remark In applying Theorem 26, we need only check, according to Proposition 25, that the entries below (or above) the main diagonal of $\mathbf{PB}(P^t)$ lie on the diagonal.

We derive a result for the G–L sum axiom analogous to Theorem 26 for the LUB axiom.

Theorem 28 A10 *holds if and only if $\bar{O}\mathbf{B}(\bar{O})$ is diagonally closed.*

Proof: Let $A = [a_{ij}] = \bar{O}$, $B = [b_{ij}] = \mathbf{B}(A)$, and $C = [c_{ij}] = AB$. By Theorem 11(ii), $(U,\leq)$ satisfies the sum axiom A10

$$\Leftrightarrow \forall i \forall j \exists m (i + j = m)$$

$$\Leftrightarrow \forall i \forall j \exists m \forall k [(\mathrm{O}ki \vee \mathrm{O}kj) \equiv \mathrm{O}km]$$

$$\Leftrightarrow \forall i \forall j \exists m \forall k (a_{ki} a_{kj} = a_{km}^2)$$

$$\Leftrightarrow \forall i \forall j \exists m \sum_{k=1}^{n} a_{ik} a_{kj} 2^{k-1} = \sum_{k=1}^{n} a_{mk} a_{km} 2^{k-1}$$

$[A = \bar{O}$ is symmetric, since O is$]$

$$\Leftrightarrow \forall i \forall j \exists m \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} a_{mk} b_{km}$$

$$\Leftrightarrow \forall i \forall j \exists m \; c_{ij} = c_{mm}$$

$\Leftrightarrow C$ is diagonally closed.

Example 7 *Consider the $\leq$ predicate defined by the part matrix*

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

One can check that $\leq$ satisfies axioms A1–A4. To see that $\leq$ satisfies A9, we compute

$$P\mathbf{B}(P^t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 1 \\ 1 & 3 & 7 & 3 & 1 \\ 1 & 3 & 3 & 11 & 1 \\ 1 & 1 & 1 & 1 & 17 \end{bmatrix},$$

which is diagonally closed. To test axiom A10, we find

$$\bar{O}\mathbf{B}(\bar{O}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 8 & 0 & 8 \\ 0 & 16 & 16 & 16 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 16 & 16 & 0 \\ 0 & 16 & 24 & 16 & 8 \\ 0 & 16 & 16 & 20 & 4 \\ 0 & 0 & 8 & 4 & 14 \end{bmatrix},$$

which is not diagonally closed since 4 and 8 do not occur on the main diagonal. This gives us an example where A9 holds, but not A10.

Example 9 Let $n = 7$ and consider the $\leq$ predicate determined by the part predicate

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Then } \bar{O}\mathbf{B}(\bar{O}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 64 & 0 & 0 & 64 & 64 & 0 \\ 0 & 0 & 32 & 0 & 32 & 0 & 32 \\ 0 & 0 & 0 & 16 & 0 & 16 & 16 \\ 0 & 64 & 32 & 0 & 104 & 64 & 32 \\ 0 & 64 & 0 & 16 & 64 & 84 & 16 \\ 0 & 0 & 32 & 16 & 32 & 16 & 50 \end{bmatrix},$$

which is diagonally closed. Thus $\leq$ satisfies A10. One can verify that $(U,\leq)$ is actually the model from boolean algebra of cardinality $2^3 - 1 = 7$ obtained by taking $U = \{a,b,c\}$, $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{a,b\}$, $E = \{a,c\}$, and $F = \{b,c\}$, where $\leq$ is the set theoretical subset predicate.

Remarks 1. Since $\bar{O}$ is symmetric, we have that $\bar{O}\mathbf{B}(\bar{O}) = \bar{O}\mathbf{B}(\bar{O}^t)$. By Proposition 25, $\bar{O}\mathbf{B}(\bar{O})$ is symmetric. Thus, in applying Theorem 28, we need only check that the entries below (or above) the main diagonal of $\bar{O}\mathbf{B}(\bar{O})$ lie on the diagonal.

2. Let $A = [a_{ij}] = \bar{O}\mathbf{B}(\bar{O})$ and define the boolean matrix $S = [s_{ij}]$ by

$$s_{ij} = \begin{cases} 1, & \text{if } a_{ij} \in \{a_{k,k} : 1 \leqq k \leqq n\} \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Then Theorem 28 says that A10 holds if and only if $S = J$. One can verify that $s_{ij} = 1$ if and only if the Goodman sum $i + j$ exists in U.

9 Additional axioms involving the foundation predicate

Definition 12 (Additional $\leq$-$\mathfrak{F}$ axioms)

A12 (Independence Axiom)

$$\forall y\{\forall z[\mathcal{O}zy \equiv \exists w(w \leq y \;\&\; {\sim}\mathfrak{D}w \;\&\; \mathcal{O}zw)] \supset {\sim}\mathfrak{D}y\}$$

A13 (Transitive Closure over Arbitrary L-Summation)

$$\forall x \forall y\{[y = \text{Lsum}\,\phi \;\&\; \forall u(\phi u \supset \mathfrak{F}xu)] \supset \mathfrak{F}xy\}$$

A14 ($\mathfrak{F}$-Restricted Sum Axiom)

$$\forall w\{\exists x(\phi x \,\&\, \mathfrak{F}wx) \supset \exists y \forall z[\mathfrak{O}zy \equiv \exists x(\phi x \,\&\, \mathfrak{F}wx \,\&\, \mathfrak{O}zx)]\}.$$

The independence axiom A12 ensures that the sum of independent wholes is independent, just as in ([7], VII) sums of two or more concrete wholes are concrete ([7], p. 249). A12 is adopted in order to exclude the counterintuitive possibility that one might be able to create a dependent (i.e., abstract) object simply by heaping many independent (i.e., concrete) objects together.[13]

The transitive closure axiom A13 asserts that any whole x is founded on the sum of wholes on which it is founded. Since all models of the system computable by the present methods are atomistic, we provide the following extrasystemic support for the adoption of A13: Let $S = S(x)$ be the sum of all y on which x is founded. If S fails to exist, then in the spirit of our principle of individuation, some atomic part of S fails to exist. Since every atomic part of S is an atomic part of some y on which x is founded, that y fails to exist, and hence x fails to exist. While this extrasystemic argument becomes less persuasive if the adjective "atomic" is eliminated throughout, the assumption of A13 constrains all models (atomistic or not) to conform to the intuition expressed in it.

Finally, the restriction of summation by the foundation relation (A14) (considered as one of four possible sum axioms in this study) is of interest in relation to the feeling ([5], pp. 40–41, 62, 90–91, 97–98, 138–139, 192) that unrestricted sum axioms are too strong. A14 ensures that the sums on which a whole is founded by A13 exist, and is a weaker sum axiom compatible with the intuitions involved in the incorporation of the relation of foundation into the system.

10 The independence axiom The intent of the independence axiom is that the sum of independent wholes is independent.

Definition 13 y is a *sum* of *independent wholes* means

$$\forall z[\mathfrak{O}zy \equiv \exists w(w \leq y \,\&\, \sim\mathfrak{D}w \,\&\, \mathfrak{O}zw)].$$

We wish to translate the independence axiom into matrix language using the incidence vector D associated with $\mathfrak{D}$, translated in Proposition 8. To do this, we introduce some new notation. Let '+' denote the 'exclusive or' boolean operation. (Number theorists can substitute '(mod 2) addition'.) We extend + to matrices in the usual way by defining $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$.

Our main result is

Theorem 29 *The independence axiom holds if and only if*

$$D \leqq (O + O\,\mathrm{diag}(\bar{D})P)^t \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Proof: Let $A = [a_{ij}] = \mathrm{diag}(\bar{D})$, $B = [b_{ij}] = OAP$, $C = [c_{ij}] = (O + B)^t$ and $X = C[1 \cdots 1]^t$. Then the independence axiom holds

$$\Leftrightarrow \forall y\{\forall z[\mathfrak{O}zy \equiv \exists w(w \leq y \ \& \ {\sim}\mathfrak{D}w \ \& \ Ozw)] \supset {\sim}\mathfrak{D}y\}$$

$$\Leftrightarrow \forall y\left\{\forall z\left[(o_{zy} = 1) \equiv \left(\bigvee_{w=1}^{n} o_{zw}a_{ww}p_{wy} = 1\right)\right] \supset (d_y = 0)\right\}$$

$$\Leftrightarrow \forall y\{\forall z(o_{zy} = b_{zy}) \supset (d_y = 0)\}$$

$$\Leftrightarrow \forall y\{\forall z(o_{zy} + b_{zy} = 0) \supset (d_y = 0)\}$$

$$\Leftrightarrow \forall y\left\{\left(\bigvee_{z=1}^{n} c_{zy} = 0\right) \supset (d_y = 0)\right\}$$

$$\Leftrightarrow \forall y\{x_y = 0 \supset d_y = 0\} \Leftrightarrow D \leqq X.$$

Example 9 We return to the finite model of Example 5. Using the vector D calculated in Example 6, we find

$$(O + O\,\mathrm{diag}(\bar{D})P)^t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = D.$$

Hence the independence axiom holds for this model.

11 Transitive closure of foundation over summation The transitive closure axiom asserts that for every x and y, if y is the sum of wholes on which x is founded, then x is founded on y.

Definition 14 y is an *Lsum of wholes* on which x *is founded* means

(10) $$\forall w[w \leq y \supset \exists z(\mathfrak{O}wz \ \& \ \mathfrak{F}xz)].$$

It is possible to omit the metalinquistic variable 'ϕ' in the formulation of A13 as follows:

Theorem 30 A13 *holds if and only if*

(11) $$\forall x\forall y\{\forall w[w \leq y \supset \exists z(\mathfrak{O}wz \ \& \ \mathfrak{F}xz)] \supset \mathfrak{F}xy\}.$$

Proof: Fix x and y.

("if") Assume (11). Let ϕ be any nonempty formula of $\Sigma_{\leq\mathfrak{F}}$ and suppose $y = \mathrm{Lsum}\phi$ and $\forall u(\phi u \supset \mathfrak{F}xu)$. By the definition of Lsum, $\forall w[w \leq y \supset \exists z(\phi z \ \& \ \mathfrak{O}wz)]$, which implies $\forall w[w \leq y \supset \exists z(\mathfrak{O}wz \ \& \ \mathfrak{F}xz)]$. By (11), $\mathfrak{F}xy$ follows.

("only if") Assume A13 and suppose the hypothesis of (11) holds, that is, assume (10). Define the formula ϕ by ϕz means $z \leq y$ and $\mathfrak{F}xz$. We first show that $y = \mathrm{Lsum}\phi$. It is immediate that $z \leq y$ for every z satisfying ϕ. It remains to prove

(12) $$\forall w[w \leq y \supset \exists u(\phi u \ \& \ \mathfrak{O}wu)].$$

So pick $w \in U$ and assume $w \leq y$. By (10), there exists a z satisfying $\mathfrak{O}wz$ and $\mathfrak{F}xz$. Using the definition of overlap, there exists a u such that $u \leq w$ and $u \leq z$. Now $u \leq z$ implies $\mathfrak{F}zu$. Using the transitivity axioms for $\mathfrak{F}$ and $\leq$, $\mathfrak{F}xz$ and $\mathfrak{F}zu$ imply $\mathfrak{F}xu$, while $u \leq w$ and $w \leq y$ imply $u \leq y$. Thus u satisfies ϕ. Moreover, $u \leq w$ implies $\mathfrak{O}wu$. This proves (12). It follows that $y = \text{Lsum}\phi$. The above argument also shows that since $y \leq y$, there exists some u satisfying ϕ, i.e. ϕ is nonempty. Finally, the definition of ϕ gives that $\forall z(\phi z \supset \mathfrak{F}xz)$. Thus, the premise of A13 is satisfied, from which we conclude $\mathfrak{F}xy$.

Theorem 31 *The transitive closure of* $\mathfrak{F}$ *over L-Summation (or G-Summation) holds if and only if* $F \vee (\overline{FO})P = J$.

Proof: Let $A = [a_{ij}] = FO$, $B = [b_{ij}] = \bar{A}$, and $C = [c_{ij}] = BP$. Then the transitive closure of $\mathfrak{F}$ over L-Summation holds

$$\Leftrightarrow \forall x \forall y \{\forall w[w \leq y \supset \exists z(\mathfrak{O}wz \;\&\; \mathfrak{F}xz)] \supset \mathfrak{F}xy\}$$

$$\Leftrightarrow \forall x \forall y \left[\forall w\left(w \leq y \supset \bigvee_{z=1}^{n} f_{xz} o_{zw} = 1\right) \supset \mathfrak{F}xy\right] \qquad [\text{since } o_{zw} = o_{wz}]$$

$$\Leftrightarrow \forall x \forall y [\forall w(p_{wy} = 1 \supset a_{xw} = 1) \supset (f_{xy} = 1)]$$

$$\Leftrightarrow \forall x \forall y [f_{xy} = 1 \text{ or } \exists w(b_{xw} = 1 \text{ and } p_{wy} = 1)]$$

$$\Leftrightarrow \forall x \forall y \left[f_{xy} = 1 \text{ or } \bigvee_{w=1}^{n} b_{xw} p_{wy} = 1\right]$$

$$\Leftrightarrow \forall x \forall y [f_{xy} = 1 \text{ or } c_{xy} = 1] \Leftrightarrow F \vee C = J.$$

Example 10 For the finite model of Example 5, we compute

$$FO = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \text{ and } F \vee (\overline{FO})P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = J.$$

Thus A13 holds for this model.

Since in a finite universe $\text{Gsum}\,\phi = \text{Lsum}\,\phi$,

Proposition 32 A13 *is equivalent to*

A13′ $\forall x \forall y \{[y = \text{Gsum}\,\phi \;\&\; \forall u(\phi u \supset \mathfrak{F}xu)] \supset \mathfrak{F}xy\}$.[14]

12 **$\mathfrak{F}$-Restricted sum axiom** In restricting the Goodman-Leonard sum axiom by the relation $\mathfrak{F}$, our intent is that for any whole w, the sum of all wholes on which w is founded exists. As with the regular sum axioms, we would like to express A14 in terms of the binary G-L sum $+$. This requires the transitive closure axiom of $\mathfrak{F}$ over $+$ (or equivalently $+_{\mathrm{L}}$), in order to make the induction work properly.

Proposition 33 *Assume* A13 *holds. Then* A14 *holds if and only if*

$$\forall w \forall x \forall y [(\mathfrak{F}wx \;\&\; \mathfrak{F}wy) \supset \exists z(x + y = z)].$$

We translate this binary condition in terms of matrices.

Theorem 34 *Assume* A13 *holds. Define the* $n \times n$ *boolean matrix* $S = [s_{ij}]$ *by*

$$s_{ij} = 1 \quad \textit{iff} \quad \exists k(i + j = k), \quad 1 \leqq i, j \leqq n. \tag{13}$$

Then A14 *holds if and only if* $F^tF \leqq S$.

Proof: Let $A = [a_{ij}] = F^t$ and $B = [b_{ij}] = AF$. Then A14 holds

$$\Leftrightarrow \forall w \forall x \forall y[(\mathfrak{F}wx \;\&\; \mathfrak{F}wy) \supset \exists z(x + y = z)]$$

$$\Leftrightarrow \forall x \forall y \forall w[(a_{xw}f_{wy} = 1) \supset (s_{xy} = 1)]$$

$$\Leftrightarrow \forall x \forall y[\forall w(a_{xw}f_{wy} = 0) \text{ or } s_{xy} = 1]$$

$$\Leftrightarrow \forall x \forall y\left[\bigvee_{w=1}^{n} a_{xw}f_{wy} = 0 \text{ or } s_{xy} = 1\right]$$

$$\Leftrightarrow \forall x \forall y[b_{xy} \leqq s_{xy}] \Leftrightarrow B \leqq S.$$

Remark S is the matrix in the second remark following Theorem 28.

Example 11 Let $n = 4$ and define $\leq$ and $\mathfrak{F}$ by the incidence matrices

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ with } O = P^tP = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is routine to check that $(U, \leq, \mathfrak{F})$ is a Husserl structure. Using (9) to compute the matrix $S = [s_{ij}]$ of Theorem 34, we have

$$A = [a_{ij}] = \bar{O}B(\bar{O}) = \begin{pmatrix} 8 & 8 & 8 & 0 \\ 8 & 12 & 8 & 4 \\ 8 & 8 & 10 & 2 \\ 0 & 4 & 2 & 7 \end{pmatrix}, \text{ and so } S = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $S \neq J$, the G–L Sum axiom fails (as does the LUB axiom). In order to apply Theorem 34, we first verify that $F \vee \overline{FOP} = J$, so A13 holds by Theorem 31. Since $F^tF = S$, we have an example where G–L Summation fails to exist in general, but the $\mathfrak{F}$-restricted sum axiom holds.

13 Summary We summarize the matrix criteria for the axioms A1 through A14. P = incidence matrix of $\leq$, F = incidence matrix of $\mathfrak{F}$, $\mathcal{O}$ = overlap matrix = P^tP, and $D = \text{diag}^{-1}(F\bar{O})$.

Axiom		Definition	Matrix Condition
A1		$\leq$ is reflexive	$I \leqq P$
A2		$\leq$ is antisymmetric	$P \times P^t \leqq I$
A3		$\leq$ is transitive	$P^2 \leqq P$
A4	Overlap	$\forall y \forall z\{\forall x[x \leq y \supset \mathfrak{O}xz] \supset y \leq z\}$	$P \vee P^t\bar{O} = J$
A5		$\forall x \forall y(x \leq y \supset \mathfrak{F}yx)$	$P^t \leqq F$
A6		$\mathfrak{F}$ is transitive	$F^2 \leqq F$
A7		$\mathfrak{F}$ is not symmetric	$F^t \neq F$
A8		$\mathfrak{F}$ is not antisymmetric	$F \times F^t \nleqq I$

A9 Least Upper Bound — $P\mathfrak{B}(P^t)$ is diag. closed

$\forall S \neq \varnothing \exists y\{\forall z(z \in S \supset z \leq y) \;\&\; \forall u[\forall z(z \in S \supset z \leq u) \supset y \leq u]\}$

A10 Goodman–Leonard Sum Existence — $\bar{O}\mathfrak{B}(\bar{O})$ is diag. closed

$\forall S \neq \varnothing \exists y[\mathfrak{O}zy \equiv \exists x(x \in S \;\&\; \mathfrak{O}zx)]$

A11 Leśniewski–Tarski Sum Existence — Same as for A10

$\forall S \neq \varnothing \exists y\{\forall z(z \in S \supset z \leq y) \;\&\; \forall w[w \leq y \supset \exists z(z \in S \;\&\; \mathfrak{O}wz)]\}$

A12 Independence — $D \leqq (O + O\,\mathrm{diag}(\bar{D})P)^t \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$\forall y\{\forall z[\mathfrak{O}zy \equiv \exists w(w \leq y \;\&\; {\sim}\mathfrak{D}w \;\&\; \mathfrak{O}zw)] \supset {\sim}\mathfrak{D}y\}$

A13 Transitive Closure over L-Summation — $F \vee (\overline{FO})P = J$

$\forall x \forall y\{\forall w[w \leq y \supset \exists z(\mathfrak{O}wz \;\&\; \mathfrak{F}xz)] \supset \mathfrak{F}xy\}$

A13′ Transitive Closure over G-Summation — Same as for A13

$\forall x \forall y\{\forall z[\mathfrak{O}zy \equiv \exists u(u \leq y \;\&\; \mathfrak{O}zu \;\&\; \mathfrak{F}xu)] \supset \mathfrak{F}xy\}$

A14 $\mathfrak{F}$-Restricted Summation — $\begin{cases} F^tF \leqq S \text{ where} \\ \text{S is defined in (9)} \end{cases}$

$\forall w\{\exists x(x \in S \;\&\; \mathfrak{F}wx) \supset \exists y \forall z[\mathfrak{O}zy \equiv \exists x(x \in S \;\&\; \mathfrak{F}wx \;\&\; \mathfrak{O}zx)]\}$

14 Computer implementation A computer program named *Husserl* was written on the IBM PC AT to implement the above matrix criteria for $n \times n$ incidence matrices P and F. The program displays those axioms among A1–A14 which P and F satisfy. We write *Husserl* in the language Pascal in order to employ a modular program structure.[15] The entries of the boolean matrices are declared in the program to be of type *integer*, even though they take on only the values 0 or 1. In several places, such as computing the matrix join $A \vee B$, it would have been convenient to use *boolean* values instead. (Perhaps a language such as C, in which a variable can be treated simultaneously as *integer* and *boolean*, is ideally suited to this implementation.) *Husserl* uses several simple procedures which compute the transpose and negation of a single matrix, and the join, product, entrywise product, and sum of two matrices. Boolean functions compute the truth value of the matrix comparisons '=' and '$\leqq$', and also determine whether an integral matrix is diagonally closed. Once these matrix subroutines were developed, writing procedures to test the various axioms A1–A14 was completely straightforward. We also developed a recursive algorithm for determining all part matrices P (up to isomorphism) which satisfy the first four axioms. This method is effectively computable for small dimensions, i.e. $n \leqq 15$. Source code is available from the first author.

NOTES

1. Leśniewski used both proper part ([16]; [17]; [19], p. 25) and part (ingredient) ([18]; [19], p. 39; [13]) as primitive. Tarski used the part relation as primitive in [28], as does Bostock in [3]. The Leonard-Goodman system uses discreteness (the negation of our overlap predicate '$\mathcal{O}$') as primitive, and defines part in terms of it, but in [7], Goodman used the overlap predicate as primitive, and in [24], Quine and Goodman used the part predicate as primitive.

2. Hume considered the impressions comprising the parts of a substance to be *together* in the same sense that the impressions of honey and nausea are *together* for Locke's unfortunate glutton. Hume's analysis of, and denial of, necessity in the causality connection between perceptions ([10], pp. 155–172) reduces it to a special case of resemblance and contiguity ([10], p. 170). So for Hume, the principles of association ([10], pp. 10–11) are conceived as a relation which, like resemblance and contiguity, is symmetrical and nontransitive. A part-whole system with a second, symmetrical and nontransitive primitive dyadic predicate 'W' has been studied by Nelson Goodman ([7]: Ch. VII), whose mention of Hume ([7], p. 137) is an important historical reference. For a discussion of the historical context within which we intentionally depart from Goodman's reliance on Hume's empiricism, see [9]. For Descartes' use of the concept of foundation (which is a common primitive of rationalist philosophy), see Rule XII ([4], p. 42).

3. Peter Simons used a modal language in the first published attempt ([26], pp. 113–159) to axiomatize Husserl's version of the Brentano school part-whole-foundation theory. Kit Fine has undertaken the task [6] of a first-order axiomatization of Husserl's notion of foundation and Null has published ([22], pp. 463–483) a set of axioms formulated in first-order predicate logic. The eight axioms chosen for translation into matrix theory in this paper are a variation of Null's earlier axioms.

4. See the extended treatment of dependence in Chapter 8 of [27], which appeared after the work in this essay was done.

5. This reading is in partial conformity with Husserl's theory of essence [23], since all (abstract and concrete) pregnant wholes of the *first* type are *instances* of essences for Husserl. The conformity is partial because we have left Husserl's *second* type of pregnant whole ([12], p. 475) out of account.

6. In [22] a weaker definition of the predicate '$I(x)$' is employed, viz. $\forall y \forall z[(y \leq x$ and $z \leq x) \supset (\mathfrak{F}yz \vee \mathfrak{F}zy)]$. Either definition serves to establish a degree of "personal integration" sufficient to satisfy the criticisms of Goodman's concept of individual (cf. [5], pp. 185–186). However, neither permits the use of Goodman's time qualia (or the position-time qualia which he rejected ([7], p. 197)) as atoms in $\leq$-$\mathfrak{F}$ realist systems, because they are independent by Definition 5.

7. The $\leq$-$\mathfrak{F}$ dyadic predicate '$\mathfrak{D}$' is analogous to Goodman's dyadic predicate 'W' in [7]. His *complex* ([7], pp. 210, 227) is a whole, each two discrete parts of which have the relation W, while a $\leq$-$\mathfrak{F}$ instance is a whole, each two discrete parts of which

have the relation $\mathfrak{D}$. Similarly, it is possible to define a predicate 'S' ('is a *substance*': Sx iff ($\mathfrak{I}x$ & $\sim\mathfrak{D}x$)) in place of Goodman's defined predicate 'c' ('is a concretum'). (See [22] for further defined predicates of our system (B-N).) A Goodman *concretum* is a complex which has the togetherness relation W with no individual ([7], pp. 211, 230). A B-N independent instance is founded only on its own parts, and thus on no instance which is discrete from it. Hence, a B-N independent instance is dependent on (i.e. bears the relation $\mathfrak{D}$ to) nothing, just as a Goodman concretum is together with (i.e. bears the relation W to) nothing.

8. Ernst Schröder [25] anticipates a great deal of the usage of boolean matrices, viz. the ideas of incidence matrix, the full matrix J as well as the diagonal unit I, matrix join, relative product, entrywise product, and negation, along with the matrix characterizations of reflexivity, symmetry, and transitivity in Prop. 2. Schröder even predicts and hopes for a mechanical means of calculation ([25], p. 42). See [2] for a modern discussion of lattices and boolean algebras.

9. This gives a method for obtaining the incidence matrix for the transitive closure of a relation. For example, suppose we have a Hasse diagram of a partial order $\mathcal{P}$. We can easily compute from the diagram the incidence matrix A of the dyadic predicate 'below' where 'below(x,y)' means 'x is an immediate predecessor of y in the diagram'. Then the incidence matrix of $\mathcal{P}$ is $(A \vee I)^2$.

10. In 1916 ([16], p. 12; [18], pp. 25, 49) Leśniewski defined the sum of all m's to mean (1) every m is an ingredient (part) of p, and (2) if x is an ingredient of p, then some ingredient of x is an ingredient of some m. Leśniewski claimed to have also used this sum concept in 1918 ([19], p. 38) and in 1920 ([19], p. 39). In 1927 Tarski ([28], p. 25) defined an individual X to be a sum of all elements of a class a of individuals if every element of a is a part of X and if no part of X is disjoint from all elements of a. All nit picking aside, it seems that these all mean the same thing, viz. just what we call their sum definition. And Leśniewski said it first. But in view of Tarski's use of the disjoint predicate (which means not overlapping), we shall call their axiom the Leśniewski–Tarski sum concept.

11. Thus we could translate the LUB axiom for finite models into set theoretic language. In all the axioms we could replace 'ϕx' by '$x \in S$', where S is a nonempty subset of U. Set theoretical versions of all the axioms are given in the summary.

12. One can easily check that these operations are well-defined. See the remark following Definition 7.

13. The adoption of A12 is necessary to give a correct proof of Theorem 67 in [22].

14. For transitive closure over G–L sums we define y is a Gsum of wholes on which x is founded to mean $\forall z[\mathcal{O}zy \equiv \exists u(u \leq y \;\&\; \mathcal{O}zu \;\&\; \mathfrak{F}xu)]$. Analogous to Theorem 30 we can prove

Theorem A13′ *holds if and only if*

$$\forall x \forall y\{\forall z[\mathcal{O}zy \equiv \exists u(u \leq y \;\&\; \mathcal{O}zu \;\&\; \mathfrak{F}xu)] \supset \mathfrak{F}xy\}. \tag{14}$$

For the reader wondering why we chose L–T summation instead of G–L summation for the transitive closure axiom (A13), it turns out that it is easier to express condition (11) than condition (14) in terms of matrices. In fact, we were unable to translate (14) into a concise matrix form. In contrast, the G–L sum existence axiom (A10) translated easily, but not A11.

15. All of the axioms A1–A14 can be formulated directly, without the use of matrices, in the computer language Prolog. We have specifically done this for the LUB axiom, by defining Prolog predicates for upper bound, least upper bound, and finally axiom A9 itself. The Prolog procedure is very slow for $n \geq 10$ since it must run through all 2^n possible subsets of U.

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