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Expressiveness and Completeness of an Interval Tense Logic

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Abstract We present the syntax and semantics of an interval-based temporal logic which was defined by Halpern and Shoham. It is proved that this logic has a greater capacity to distinguish frames than any temporal logic based on points and we show that neither this nor any other finite set of operators can be functionally complete on the class of dense orders. In the last part of the paper we give sound and complete sets of axioms for several classes of structures. The methods employed in the paper show that it is rewarding to view intervals as points in a plane, in the style of two-dimensional modal logic.

1 Introduction Recent years of research in temporal logic have shown an increasing tendency to concentrate on intervals or events rather than on points. In contrast with the point-based approach, few results are known for modal logics of intervals (see van Benthem [3], Humberstone [11], and Roper [14]), as most authors confine themselves to classical logic. In [10] Halpern and Shoham present a modal logic of intervals which is investigated here. In the first section we define the syntax and semantics of the system and give some basic facts; in Section 2 we present our results on the expressive power of this and other modal logics of intervals, and in the last part of the paper we treat completeness.

1.1 Syntax HS is a tense logic, the formulas of which are built up using the propositional constants $p, q, r, p_0, p_1, \ldots$, the classical connectives $\neg$ and $\wedge$, and the following modal operators: $\langle B \rangle$, $\langle E \rangle$, $\langle A \rangle$, $\langle \underline{B} \rangle$ $\langle \underline{E} \rangle$, and $\langle \underline{A} \rangle$, which have the following intended readings:

- $\langle B \rangle \varphi$ φ holds at a strict beginning interval of the current one
- $\langle E \rangle \varphi$ φ holds at a strict end interval of the current one
- $\langle A \rangle \varphi$ φ holds at an interval met by the current one, i.e., it begins where the current one ends

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$\langle \underline{\mathrm{B}} \rangle \varphi$ φ holds at an interval which has the current one as a beginning interval
$\langle \underline{\mathrm{E}} \rangle \varphi$ φ holds at an interval which has the current one as an ending interval
$\langle \underline{\mathrm{A}} \rangle \varphi$ φ holds at an interval meeting the current one.

Given a set L of propositional constants, Φ_L is the set of HS-formulas using only constants in L. Furthermore, we shall use the classical connectives $\vee, \rightarrow, \leftrightarrow$ and constants $\top, \bot$ in their standard meaning, and also the following abbreviations:

$\Diamond\varphi$ for "somewhere φ"
$[X]\varphi$ for $\neg\langle X \rangle \neg\varphi$, where $\langle X \rangle$ stands for any operator
$[\![\mathrm{BP}]\!]\varphi$ for $([\mathrm{B}]\bot \wedge \varphi) \vee \langle \mathrm{B} \rangle ([\mathrm{B}]\bot \wedge \varphi)$ ("starting point")
$[\![\mathrm{EP}]\!]\varphi$ for $([\mathrm{E}]\bot \wedge \varphi) \vee \langle \mathrm{E} \rangle ([\mathrm{E}]\bot \wedge \varphi)$ ("ending point").

The *mirror-image* of a formula φ is obtained by simultaneous substitution in φ of every occurrence of: B by E, $\underline{\mathrm{B}}$ by $\underline{\mathrm{E}}$, A by $\underline{\mathrm{A}}$, and vice versa.

For the semantics of this logic one has a choice between structures in which intervals themselves form the ontological basis, and structures in which time points are primary and intervals are defined as (convex) sets of points. In this paper we follow the second approach.

1.2 Semantics A (*temporal*) *frame* is a pair $F = (T, <)$, where T is a set of time-points and $<$ is a strict partial order on T. The *interval set* of a frame F is defined as the set $\mathrm{INT}(F)$ of all closed intervals $[s,t] = \{x \in T \mid s \leq x \leq t\}$ in T. An (L-)*model* is a pair (F, V), where F is a frame and V is a *valuation*, i.e., a map $L \mapsto 2^{\mathrm{INT}(F)}$.

A *truth relation* $\models$ is inductively defined as follows:

$F, V \models p[s,t]$ iff $[s,t] \in V(p)$
$F, V \models \neg\varphi\ [s,t]$ iff $F, V \not\models \varphi[s,t]$
$F, V \models (\varphi \wedge \psi)\ [s,t]$ iff $F, V \models \varphi\ [s,t]$ and $F, V \models \psi\ [s,t]$
$F, V \models \langle \mathrm{B} \rangle \varphi\ [s,t]$ iff there is a u such that $s \leq u < t$ and $F, V \models \varphi\ [s,u]$
$F, V \models \langle \underline{\mathrm{B}} \rangle \varphi\ [s,t]$ iff there is a u such that $t < u$ and $F, V \models \varphi\ [s,u]$
$F, V \models \langle \mathrm{E} \rangle \varphi\ [s,t]$ iff there is a u such that $s < u \leq t$ and $F, V \models \varphi\ [u,t]$
$F, V \models \langle \underline{\mathrm{E}} \rangle \varphi\ [s,t]$ iff there is a u such that $u < s$ and $F, V \models \varphi\ [u,t]$
$F, V \models \langle \mathrm{A} \rangle \varphi\ [s,t]$ iff there is a u such that $t < u$ and $F, V \models \varphi\ [t,u]$
$F, V \models \langle \underline{\mathrm{A}} \rangle \varphi\ [s,t]$ iff there is a u such that $u < s$ and $F, V \models \varphi\ [u,s]$.

In the usual manner we define the concepts of *validity* and *satisfiability* of formulas with respect to models, frames, and classes of frames, and of the *theory* of a model, frame, or class of frames.

1.3 Intervals and points In the truth definition we implicitly defined the following relations:

$[s,u] \subset_{\mathrm{B}} [s,t]$ iff $s \leq u < t$ iff $[s,u]$ is a beginning interval of $[s,t]$
$[u,t] \subset_{\mathrm{E}} [s,t]$ iff $s < u \leq t$ iff $[u,t]$ is an ending interval of $[s,t]$
$[s,t] <_{\mathrm{A}} [t,u]$ iff $t < u$ iff $[s,t]$ meets $[t,u]$.

Note that, with this definition, no interval is a beginning/end interval of itself, and that intervals $[t,t]$ exist that have no beginning or end intervals at all;

stretched intervals and *point intervals* are distinguished by the formula $[\mathrm{B}]\bot$, which holds precisely on the latter ones. The operators $[\![\mathrm{BP}]\!]$ and $[\![\mathrm{EP}]\!]$ are used to describe the relation between intervals and their beginning (respectively, end) points: $F, V \models [\![\mathrm{BP}]\!]\varphi\ [s,t]$ iff $F, V \models \varphi\ [s,s]$, and likewise for $[\![\mathrm{EP}]\!]\varphi$.

The set of point intervals, together with a suitable ordering relation, may be looked upon as a point-structure, which is of course isomorphic to F itself. So implicitly the system HS has a point-based tense logic as a "sublogic".

Given the existence of point intervals, we can and will define the operators $\langle \mathrm{A}\rangle$ and $\langle \underline{\mathrm{A}}\rangle$ using the other ones: $\langle \mathrm{A}\rangle\varphi \equiv [\![\mathrm{EP}]\!]\langle \mathrm{B}\rangle\varphi$ and $\langle \underline{\mathrm{A}}\rangle\varphi \equiv [\![\mathrm{BP}]\!]\langle \underline{\mathrm{E}}\rangle\varphi$.

1.4 The relative position of two intervals If one considers a fixed interval in a linear temporal structure—let's call it the current interval—then there are essentially twelve possible positions for a distinct interval (or seventeen if one considers the position of a point interval to differ from a stretched one's). Every position of such a distinct interval can be described in HS, as is easily verified by Figure 1.

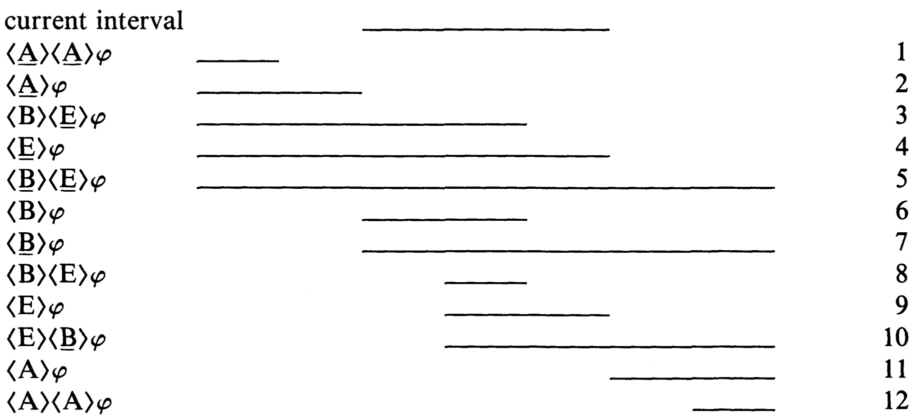


Figure 1.

1.5 Representation of INT(F) as a subset of F^2 An interval $[s,t]$ in a linear structure is only defined if $s \leq t$, and because any interval is completely determined by its beginning and end point, we can easily construct an isomorphism between $\mathrm{INT}(F)$ and $F^{2\mathrm{NW}} = \{(x,y) \in F^2 \mid x \leq y\}$. This means that we can represent the interval structure spatially as the "northwestern halfplane" of $F \times F$. A rewarding consequence is that we can now interpret the operators spatially as well, e.g., $\langle \underline{\mathrm{B}}\rangle\varphi$ as "φ is true at a point right above the current one". As this way of thinking about intervals will be used frequently in the sequel, we will give a new notation for the operators, which reflects their spatial nature more clearly:

Definition 1.5.1

$\Diamond\varphi \equiv \langle \mathrm{B}\rangle\varphi$: φ holds at a point right below the current one
$\Diamond\varphi \equiv \langle \underline{\mathrm{B}}\rangle\varphi$: φ holds at a point right above the current one
$\Diamond\varphi \equiv \langle \mathrm{E}\rangle\varphi$: somewhere left from the current point, φ holds

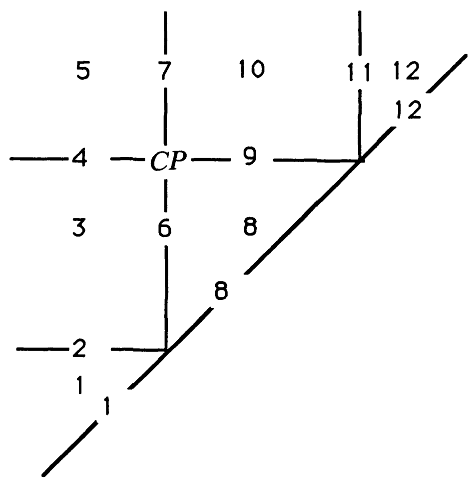
$\Diamond_{\leftarrow}\varphi \equiv \langle \underline{\mathrm{E}} \rangle \varphi$: somewhere right from the current point, φ holds

$\Diamond_{|}\varphi \equiv \Diamond_{\downarrow}\varphi \vee \Diamond_{\uparrow}\varphi$: φ is true at a point with the same latitude and a different longitude

$\Diamond_{-}\varphi \equiv \Diamond_{\rightarrow}\varphi \vee \Diamond_{\leftarrow}\varphi$: φ is true at a point with the same longitude and a different latitude.

Note that to obtain a mirror-image of a formula φ written in the new notation one should simultaneously replace all $\Diamond_{\downarrow}$ by $\Diamond_{\rightarrow}$, $\Diamond_{\uparrow}$ by $\Diamond_{\leftarrow}$, and vice versa, everywhere in φ. The "spatial-semantic counterpart" of this operation is to interchange the dimensions of a structure (or, to reflect the model in a line "orthogonal" to the diagonal).

We can now equate the "twelve possible positions of one interval with respect to another" of Section 1.4 with the equivalence classes of $\mathbb{R}^{2NW}$ shown in Figure 2.



CP denotes the current point. The area numbered n contains all intervals of type n in Figure 1.

Figure 2.

1.6 Some correspondences As is usual in modal or tense logic (cf. van Benthem [4]), there exists a straightforward translation τ mapping any HS-formula φ onto a formula φ^{τ} of first-order predicate logic. This formula φ^{τ} has two free variables x and y and is written in a language with equality ($=$) and dyadic predicate symbols $<, P_0, P_1, \ldots$. The formulas φ and φ^{τ} are locally equivalent on the model level; i.e., $F, V \models \varphi\ [s,t]$ iff $F, V \models \varphi^{\tau}(x,y)\ [s,t]$. In order to obtain equivalence on the frame level, we have to quantify φ^{τ} over all valuations, whence we get a second-order formula. The following examples show that some HS-formulas describe first-order properties of frames, so that in these cases we can do without second-order logic.

Example 1.6.1 – Linearity Call a frame *linear* if it is both left-linear and right-linear, where $F = (T, \leq)$ is *right-linear* if $\forall s,t,u \in T[(s \leq t \wedge s \leq u) \rightarrow (t < u \vee t = u \vee u < t)]$; left-linearity is defined likewise. (Note that this definition does not exclude frames consisting of "parallel" time lines.) Halpern and Shoham remark that a frame is right-linear iff for any two distinct intervals starting at the same point, one is a prefix of the other. This condition is easily shown to be expressed by the following formula: RLIN $\equiv \langle A\rangle p \to [A]\&(p \vee \langle B\rangle p \vee \langle \underline{B}\rangle p)$. Using a similar definition for left-linearity, and setting LIN $\equiv$ RLIN $\wedge$ LLIN, one gets $F \models$ LIN $\Leftrightarrow F$ is linear.

Example 1.6.2 – Linear intervals One can impose less severe constraints upon frames than linearity, e.g., one could permit tree-like structures while rejecting nonlinear intervals in frames. A frame has *linear intervals* if

$$\forall s,t,u,v[(s \leq t \leq v \wedge s \leq u \leq v) \to (t < u \vee t = u \vee u < t)].$$

Now define the following HS-formula:

$$\text{LINITV: } (\langle B\rangle p \wedge \langle B\rangle q) \to \{\langle B\rangle(p \wedge \langle B\rangle q) \vee \langle B\rangle(p \wedge q) \vee \langle B\rangle(\langle B\rangle p \wedge q)\}.$$

Then using a standard correspondence-theoretic argument, we can prove that F has linear intervals iff $F \models$ LINITV.

Example 1.6.3 – Denseness and discreteness Two important classes of orders are the dense and the discrete orders. Recall that a frame is dense if $s < t$ implies that there is a point u between s and t; an ordering is discrete if every point having a successor (predecessor) has an immediate successor (predecessor). In [10] it is shown that there are formulas DISC and DENSE that are valid exactly over the class of discrete, respectively dense, orders. First, define the formulas:

$$\text{length0} \equiv [B]\bot$$
$$\text{length1} \equiv \langle B\rangle\top \wedge [B][B]\bot,$$

the last one being true for an interval iff it has exactly one beginning interval: the beginning-point interval. So $[s,t] \models$ length1 iff $s \neq t$ and there are no points between s and t. Defining DISC $\equiv$ length0 $\vee$ length1 $\vee$ ($\langle B\rangle$length1 $\wedge$ $\langle E\rangle$length1) and DENSE $\equiv$ $\neg$length1, one can easily prove the following:

Claim 1.6.3.1 *F is dense* $\Leftrightarrow$ F $\models$ DENSE.

Claim 1.6.3.2 *F is discrete* $\Leftrightarrow$ F $\models$ DISC.

Note that one could also define these properties in a similar way as in point-logic, e.g., density by $\langle \underline{B}\rangle p \to \langle \underline{B}\rangle\langle \underline{B}\rangle p$. The formulas DENSE and DISC, however, do not require any propositional constants other than $\top$ and $\bot$, which turns out to be an advantage later on.

2 Expressiveness

In this section we state and prove some results concerning the expressive power of HS and other interval tense logics. First we show that HS can distinguish more frames than any point-based logic, and in the second part we show that no interval tense logic with finitely many operators is as strong as first-order logic with dyadic predicates. For a logical system L, call two frames *L-equivalent* if they have the same L-theory. Let UMT be the universal monadic second-order logic; i.e., second-order quantification is only allowed over monadic predicates. The following two lemmas express that on the class of linear frames HS-equivalence is a strictly finer sieve than UMT-equivalence:

Lemma 2.1 *Any two linear frames with the same* HS-*theory are* UMT-*equivalent.*

Proof: By a result of J. Stavi we know that the set of temporal connectives SUSS $= \{S(p,q), U(p,q), S'(p,q), U'(p,q)\}$ is functionally complete for the class of linear orders, where these operators are defined as follows (read $F, V \models p\ [s]$ for Ps):

$F, V \models U(p,q)\ [t]$ if $\exists y > t\ (Py \wedge \forall u\ (t < u < y \rightarrow Qu))$,
$F, V \models U'(p,q)\ [t]$ if

$$
\begin{array}{ll}
 & \text{(a)}\ (\exists v > t)\ \forall u (t < u < v \rightarrow Qu) \\
\wedge & \text{(b)}\ \forall v > t\ (\forall u (t < u < v \rightarrow Qu) \rightarrow Qv \wedge \exists w > v\ \forall u (v < u < w \rightarrow Qu)) \\
\wedge & \text{(c)}\ \exists y > t\ (\neg Qy \wedge Py \wedge \\
 & \qquad \forall v\ ((t < v < y \wedge \exists u (t < u < v \wedge \neg Qu)) \rightarrow Pv)).
\end{array}
$$

$S(p,q)$ and $S'(p,q)$ are defined likewise with respect to the past. These operators can easily be defined in HS, e.g., $U'(p,q)$ by:

$$
\begin{array}{ll}
 & [B]\bot \\
\wedge & \text{(a)}\ \langle \underline{B} \rangle qq \\
\wedge & \text{(b)}\ [\underline{B}]\ (qq \rightarrow ([\![EP]\!] q \wedge \langle A \rangle qq)) \\
\wedge & \text{(c)}\ \langle \underline{B} \rangle ([\![EP]\!] (\neg q \wedge p) \wedge [B] (\neg qq \rightarrow [\![EP]\!] p))
\end{array}
$$

where qq is a formula meaning 'q holds at every point inside the interval'; take, e.g., $qq \equiv [B][E]([B]\bot \rightarrow q)$.

Now suppose F and F' are two linear frames which are not UMT-equivalent; then clearly F and F' do not satisfy the same first-order formulas in the language with only monadic predicates, except for a dyadic $<$. By the functional completeness of SUSS this implies that F and F' are not SUSS-equivalent. But then they cannot be HS-equivalent either.

Lemma 2.2 *There are two* UMT-*equivalent linear frames not having the same* HS-*theory.*

Proof: We use the following results of Büchi and Siefkes [5], p. 91: Every ordinal α has a unique representation $\alpha = \omega^{\omega} \cdot \nu + \omega^{q-1} \cdot k_{q-1} + \ldots + \omega^0 \cdot k_0$, where $\omega^{\omega} \cdot \nu$ is the ω-*head* and $\omega^{q-1} \cdot k_{q-1} + \ldots + \omega^0 \cdot k_0$ is the ω-*tail* of α. Büchi and Siefkes prove that two countable ordinals α and β are UMT-equivalent iff either $\alpha = \beta < \omega^{\omega}$ or $\omega^{\omega} \leq \alpha, \beta$ and α, β have the same ω-tail. So the ordinals ω^{ω} and $\omega^{\omega} + \omega^{\omega}$ are UMT-equivalent. It remains to be proved that there is an HS-formula valid in one of the frames and not in the other. Call a frame *isochoppable* if it can be decomposed into a head and a tail that are isomorphic. Then clearly $\omega^{\omega} + \omega^{\omega}$ is isochoppable and ω^{ω} is not.

Now a frame G is iso-choppable iff the following holds: There is a bijection f from a head P of G to the corresponding tail Q that is order-preserving, i.e., $s > t$ implies $f(s) > f(t)$.

Consider the following HS-formula φ:

	$\Box(p \to [B]\bot) \wedge \Box(q \to [B]\bot)$	(G is the disjunct union of the "pointsets" $V(p)$ and $V(q)$)
$\wedge$	$\Box([B]\bot \to (p \leftrightarrow \neg q))$	
$\wedge$	$\Diamond p \wedge \Diamond q$	$(V(p, V(q) \neq \varnothing)$
$\wedge$	$\Box(p \to [\underline{E}]\,[\![BP]\!]\,\neg q)$	$(V(q)$ "<" $V(p))$
$\wedge$	$\Box(f \to ([\![BP]\!]p \wedge [\![EP]\!]q))$	$(V(q) \subseteq P \times Q)$
$\vee$	$\Box(p \to \langle \underline{B}\rangle(f \wedge [B]\neg f \wedge [\underline{B}]\neg f))$	(f is a function)
$\wedge$	$\Box(q \to \langle \underline{E}\rangle f)$	(f is surjective)
$\wedge$	$\Box(f \to [E]\neg f)$	(f is injective)
$\wedge$	$\Box(f \to [B][E]\neg f)$.	(f is order-preserving)

Then clearly F is iso-choppable iff φ is satisfiable in F. So $\omega^\omega \models \neg\varphi$, $\omega^\omega + \omega^\omega \not\models \varphi$, whence these frames have a different HS-theory.

As every point-logical formula φ has a UMT-equivalent φ° on the frame level (i.e., for every frame F, $F \models \varphi$ iff $F \models \varphi^\circ$), we obtain the following theorem, which expresses that HS has a greater capacity of distinguishing linear frames than any point-based logic.

Theorem 2.3

(1) *For any point-logic P, two* HS-*equivalent linear frames have the same P-theory.*

(2) *The ordinals defined in the previous proof have the same P-theory for any point-logic P, but they are not* HS-*equivalent.*

2.4 Functional completeness for interval tense logic In 1.6 we saw that every HS-formula φ can be translated into a first-order equivalent φ^τ. One might ask whether the converse holds as well, i.e., given a first-order formula ψ in the appropriate language and with two free variables, is there an HS-equivalent ψ' of ψ? The answer to this question is negative, since the following operator CHOP(φ, ψ) is undefinable in HS:

$F, V \models$ CHOP$(\varphi, \psi)\,[s, t]$ iff
there is u such that for $s < u < t$,
$F, V \models \varphi\ [s, u]$ and $F, V \models \psi\ [u, t]$.

Now of course one may add the CHOP-operator to HS and pose the same question for the new system (possibly restricting the class of structures in which the equivalence should hold). In general, it is an interesting question whether there exists any finite set FC of tense logical operators that is *functionally complete* over a certain class of structures K.

For point logic, Kamp was the first one to answer the analogous question in the affirmative for the class of (Dedekind-)compelte linear orders (see Kuhn [13]). Gabbay [9] proved that a class of orders K with arbitrary (monadic) predicates admits such a finite basis for the temporal connectives if and only if, for some natural number k, it satisfies the *k-variable property*, which says that every first-order formula is equivalent over K to a first-order formula with at most k bound variables (possibly reused). This "rather syntactic property" can be interpreted in a more model-theoretic sense, as Immerman and Kozen [12] showed, by using a variant of Ehrenfeucht/Fraïsse games.

Here I will prove that the same method can be used to obtain the following result for interval tense logic:

There is *no* finite functionally complete set of operators over the class of dense linear orders.

Definition 2.5 L is the set of first-order formulas in the language with $=$ and dyadic predicates $<, P_1, P_2, \ldots$.

L_k is the set of formulas with at most k (possibly reused) variables $x_1, x_2, \ldots, x_k$.

$L_{k,n}$ is the set of L_k-formulas of quantifier-depth at most n.

$L(x_1,x_2)$ (respectively $L_k(x_1,x_2)$, respectively $L_{k,n}(x_1,x_2)$), is the set of formulas in L (respectively L_k, respectively $L_{k,n}$) having two free variables x_1 and x_2.

Definition 2.6 A *table* for an interval-tense logical connective $\nabla(p_1, \ldots, p_p)$, is an $L(x_1,x_2)$-formula $\psi(x_1,x_2,<,=,P_1,\ldots,P_p)$ such that for any frame F, valuation V, and interval $[s,t]$ in $\mathrm{INT}(F)$:

$$F, V \models \nabla(p_1, \ldots, p_p)[s,t] \text{ iff } F, V \models \psi(x_1,x_2,<,=,P_1,\ldots,P_p)[s,t].$$

(Example: The table for $\mathrm{CHOP}(p,q)$ is the following formula:

$$\exists x_3 (x_1 < x_3 < x_2 \wedge Px_1x_3 \wedge Qx_3x_2).)$$

Lemma 2.7 *If FC is a set of tense logical formulas, using a finite number of operators* $\{\nabla_1, \ldots, \nabla_c\}$, *then there is a k such that each FC-formula has an* $L_k(x_1,x_2)$*-equivalent* φ°.

Proof: Suppose ∇_i is a $p(i)$-place operator, with a table ψ_i using $v(i)$ variables. Define $k = \max\{v(i) | 1 \leq i < c\}$.

By (FC-)formula induction we prove that any φ in IL has an L_k-variant $\varphi^\circ(x_1,x_2,\ldots,x_k)$ with x_1 and x_2 free in φ°.

- For atomic formulas the proof is easy: $p^\circ = Px_1x_2$.
- If $\varphi \equiv \neg\psi$ one simply takes $\varphi^\circ = \neg\psi^\circ$ and $(\psi \wedge \chi)^\circ = \psi^\circ \wedge \chi^\circ$.
- So, suppose $\varphi \equiv \nabla(\varphi_1, \ldots, \varphi_p)$. Let $\psi(x_1,x_2,<,=,P_1,\ldots,P_p)$ be the table of $\nabla(p_1,\ldots,p_p)$. By the induction hypothesis we have an $L_k(x_1,x_2)$-equivalent φ_i° for every φ_i.

  We know that we get a classical equivalent for φ by substituting the formulas φ_i° for P_i in ψ. To obtain an L_k-equivalent, we have to attune the free variables in the formulas $\varphi_i^\circ(x_1,\ldots,x_k)$ to the variables of P_i in ψ.

  To this end, consider an occurrence l of P_i in ψ. Suppose P_i occurs at l with the variables x_j and x_m ($j,m \in \{1,\ldots,k\}$ by definition of k). Permute the variables $x_1,\ldots,x_k$ of $\varphi_i^\circ(x_1,\ldots,x_k)$ in such a way that x_j and x_m appear where formerly x_1 and x_2 stood. So we get a formula $\varphi_{i,l}^\circ(x_j, x_m, \ldots)$ using only the variables $x_1,\ldots,x_k$.

  The required formula φ° is then obtained by simultaneously substituting in ψ, for every occurrence l of every predicate P_i in ψ, the formula $\varphi_{i,l}$ for P_i.

By this Lemma we know that, if there were a finite functional complete set of operators over a class K, every $L(x_1,x_2)$-formula would have a K-equivalent in $L_k(x_1,x_2)$. The remaining part of this section is devoted to showing that this cannot be the case for the class of dense linear orders. As was said before, in [12] a game-theoretic treatment of this subject is given. We need the following definitions:

Definition 2.8 A *partial valuation* over a temporal order $F = (T,<)$ is a partial function u: $\{x_1,x_2,\dots\} \mapsto T$.

A *k-configuration* over F_1,F_2 is a pair (u,v) where u and v are partial valuations over F_1,F_2, respectively, such that the *domains* σu of u and σv of v satisfy $\sigma u = \sigma v \subset \{x_1,\dots,x_k\}$.

For $L' \subset L$, u and v are said to be *L′-equivalent* if for all $\varphi \in L'$ with free variables in σu, $F_1,u \models \varphi$ iff $F_2,v \models \varphi$.

In the following we assume familiarity with Ehrenfeucht games and now give only an informal definition of the special version used in [12]:

Definition 2.9 Let F_1,F_2 be two orders and (u,v) a k-configuration over $F_1,F_2,k,n \geq 0$. $G(u,v,k,n)$ is an Ehrenfeucht game of n moves with only $2k$ pebbles, pairwise colored $x_1,\dots,x_k$. (This means that during the game pebbles have to be lifted from the board in order to be replaced on another element—one 'loses information'.) After each move a new k-configuration is generated, so after n moves one has a sequence $(u,v) = (u_0,v_0), (u_1,v_1),\dots,(u_n,v_n)$. Now player II *has a forced win for*, or simply *wins*, the game if he can play so that every k-configuration (u_i,v_i) is a *local isomorphism*, i.e., if the map $u_i(x_j) \mapsto v_i(x_j)$ is an isomorphism of the substructures $\{u_i(x) | x \in \sigma u\}$ of F_1 and $\{v_i(x) | x \in \sigma v\}$ of F_2. Player I has a forced win if player II hasn't.

The restriction to k pebbles is just the game-theoretical counterpart of the syntactic restriction of formulas to k variables:

Lemma 2.10 *Let F and F' be temporal orders and (u,v) a k-configuration over F,F'. Then player II wins $G(u,v,k,n)$ iff u and v are $L_{k,n}$-equivalent.*

Proof: See Lemma 3.4 of Immerman and Kozen [12].

Again using [12], we can now easily prove the following:

Lemma 2.11 *For K a class of orders, and $k \geq 0$: (1) ⇒ (2) ⇒ (3).*
(1) *For every φ in $L(x_1,x_2)$ there is a K-equivalent ψ in $L_k(x_1,x_2)$;*
(2) *For all orders F,F' in K and 2-configurations (u,v) over F,F': if u and v are L_k-equivalent they are L-equivalent;*
(3) *For all temporal orders F,F' and 2-configurations (u,v) over F,F': if player II wins every game $G(u,v,k,n)$, he can win every $G(u,v,m,n)$ $(m \geq k)$.*

Proof: (1) ⇒ (2): See Lemma 2.1 of [12].

(2) ⇒ (3): $L_k = \bigcup_{n\in\omega} L_{k,n}$, so if player II wins every $G(u,v,k,n)$, u and v are L_k-equivalent. By assumption, then, they are L-equivalent, so they are $L_{m,n}$-equivalent for all $m,n \geq 0$.

Lemma 2.12 *There is no k such that every $L(x_1,x_2)$-formula is equivalent to an $L_k(x_1,x_2)$-formula over the dense linear orders.*

Proof: By Lemma 2.11 it is sufficient to find dense linear models F_1 and F_2 and a 2-configuration (u,v) over F_1, F_2 such that player II has a forced win in every game $G(u,v,k,n)$, while player I can win a game $G(u,v,k+1,n)$.

For the definition of F_1, consider a *k-shuffle* of $\mathbb{Q}$, i.e., a partition of $\mathbb{Q}$ into k subsets (henceforth called the *shuffle sets*) $C_1,\ldots,C_k$, each of which is dense in $\mathbb{Q}$. Define

$$V(P) = \{[s,t] \mid s \in C_i, t \in C_j, s \neq t \text{ and } i = j\}.$$

Likewise, for F_2 one divides $\mathbb{Q}$ into $k+1$ shuffle sets $D_1,\ldots,D_{k+1}$ and defines:

$$V(P) = \{[s,t] \mid s \in D_i, t \in D_j, s \neq t \text{ and } i = j\}.$$

Claim *Player II wins $G(u,v,k,n)$ iff (u,v) is a local isomorphism.*

Proof: Use the following terminology: F,F' for F_1,F_2 or F_2,F_1 (in this proof F_1 and F_2 are interchangeable), p_i (p_i') for the position of the pebble with color x_i on the board F (F'), and $S(i)$ ($S'(i)$) for the shuffle set p_i is in. It is not hard to see that

(#) (u,v) is a local isomorphism

$$\text{iff} \quad (\bigwedge (p_i < p_j \Leftrightarrow p_i' < p_j') \wedge \bigwedge (S(i) = S(j) \Leftrightarrow S'(i) = S'(j))).$$

We prove the claim by induction on n.

$^*n = 0$: The claim follows by the definition of a win.
$^*n > 0$:
$\Rightarrow$: Again by definition.
$\Leftarrow$: Suppose σu and σv have k elements (the other case is simpler) and let (u,v) be a local isomorphism.

With every possible move player I must pick up a pebble x_i *from* the board F_i, so only $k-1$ pebbles are left behind. If she places this pebble on an already existing position p_j, the strategy for player II will be clear. So suppose the pebble x_i is moved between p_j and p_l and is in the shuffle set $S(i)$ (the case in which x_i is placed above all other pebbles can be treated likewise).

Player II now picks up pebble x_i' and has to put it somewhere between position $p_{j'}$ and $p_{l'}$; this is quite possible, though he has to be careful in which shuffle set S' to put x_i':

If $S(i) = S(j)$ for some $j \neq i$, x_i' is of course to be put in $S'(j)$.

If $S(i) \neq S(1),\ldots,S(i-1),S(i+1),\ldots,S(k)$, x_i must be placed in a shuffle set S' not appearing in the sequence $S'(1),\ldots,S'(i-1)$, $S'(i+1),\ldots,S'(k)$. Such a set S' exists: F' is partitioned into k or $k+1$ shuffle sets.

In both cases there is always an S'-element between p_j' and p_l', as each shuffle set is *dense* in $\mathbb{Q}$.

By the assumption that (u,v) is a local isomorphism, and the equivalence (#), it will be clear that the new k-configuration (u°, v°) is a local isomorphism as well. By the induction hypothesis, then, player II has a winning strategy in $G(u,v,k,n-1)$.

So, the above sketched strategy for player II yields a forced win for him in $G(u,v,k,n)$.

Now, let $u = v = \emptyset$. By the Claim, player II can win every game $G(u,v,k,n)$.

It remains to be shown that the following strategy in the game $G(u,v,k+1, k+1)$ gives player I a forced win: She should in $k+1$ successive steps pick out positions $p_1, \ldots, p_{k+1}$ belonging to $k+1$ *different* shuffle sets in F_2. This will mean that for all intervals $[p_i, p_j]$ one has $F_2 \models \neg P p_i p_j$. At this moment, however, the set of $k+1$ timepoints chosen by player II in F_1 must contain two different elements q_i and q_j belonging to the same shuffle set. So $F_1 \models P q_i q_j$.

In this way, no (u,v) reached in $k+1$ steps can be a local isomorphism, i.e., player I has a winning strategy.

Theorem 2.13 *There is no finite functionally complete set of interval tense operators over the class* DL *of dense linear orders.*

Proof: Suppose there were such a set FC; then every $L(x_1,x_2)$-formula φ would be DL-equivalent to an FC-formula φ', and so, by Lemma 2.7, to an $L_k(x_1,x_2)$-formula φ°. By Lemma 2.12 this is impossible.

Question 2.14 Theorem 2.13 can be easily generalized to the case of any class containing an order with a dense substructure. For scattered orderings such as $\mathbb{N}$ or $\mathbb{Z}$, however, it seems to be harder to answer the question whether there exists a finite functional complete set of interval tense operators.

3 Completeness Concerning the complexity of the validity problem for certain classes of structures, Halpern and Shoham [10] prove, by constructing formulas encoding the computation of a Turing machine, the first two of the following facts; the third one then follows easily, and the fourth is a consequence of the fact that the whole second-order theory of $\mathbb{Q}$ is recursively axiomatizable:

Fact 3.1
(1) *The validity problem for any class of temporal structures containing a frame having an infinitely ascending sequence of time points is r.e.-hard;*
(2) *If every frame in such a class is complete, the validity problem is* Π^1_1*-hard;*
(3) *The theories of* $\mathbb{N}$, $\mathbb{Z}$, *and* $\mathbb{R}$ *are not recursively axiomatizable;*
(4) *The theory of* $\mathbb{Q}$ *is recursively axiomatizable.*

Here we will give finite sets of axioms for the following (classes of) structures: K_{itv} (all frames with linear intervals), K_{lin} (linear structures), K_{dis} (discrete structures), and $\mathbb{Q}$, and prove their completeness. Only for the class of linear structures will the proof be given in detail. The general line of the proof is as in Burgess [7], while the use of a special derivation rule originates with Gabbay [8]. In a countable number of stages we will construct for a consistent formula φ a frame F in which this φ is satisfiable. Important in this enterprise is the concept of a "lattice", which is to be considered a finite approximation of F. Because the geometrical view of the matter is dominant in this chapter, I will represent the operators as defined in 1.5.1.

Definition 3.2. The *linear interval tense logic* L_{lin} consists of:
(1) (All substitution instances of) the following *axioms* and their mirror-images:

(I0)		all propositional tautologies	
(I1)	**a**	$[\downarrow](p \to q) \to ([\downarrow]p \to [\downarrow]q)$	(distribution)
	b	$[\uparrow](p \to q) \to ([\uparrow]p \to [\uparrow]q)$	(distribution)
(I2)	**a**	$\langle\downarrow\rangle\langle\downarrow\rangle p \to \langle\downarrow\rangle p$	(transitivity)
	b	$\langle\uparrow\rangle\langle\uparrow\rangle p \to \langle\uparrow\rangle p$	(transitivity)
(I3)	**a**	$\langle\downarrow\rangle[\uparrow]p \to p$	($\downarrow$ and $\uparrow$ are . . .
	b	$\langle\uparrow\rangle[\downarrow]p \to p$	converses)
(I4)	**a**	$\langle\downarrow\rangle[\downarrow]\bot \vee [\downarrow]\bot$	(endpoints)
(I5)		$[\downarrow]\bot \to [\leftarrow]\bot$	(id.)
(I6)	**a**	$\langle\uparrow\rangle\langle\leftarrow\rangle p \to \langle\leftarrow\rangle\langle\uparrow\rangle p$	(SW-directedness)
	b	$\langle\downarrow\rangle\langle\leftarrow\rangle p \to \langle\leftarrow\rangle\langle\downarrow\rangle p$	(NW-directnedness)
	c	$\langle\downarrow\rangle\langle\rightarrow\rangle p \to \langle\rightarrow\rangle\langle\downarrow\rangle p$	(NE-directedness)
(I7)	**a**	$(\langle\uparrow\rangle p \wedge \langle\uparrow\rangle q) \to$ $[\langle\uparrow\rangle(p \wedge \langle\uparrow\rangle q) \vee \langle\uparrow\rangle(p \wedge q) \vee \langle\uparrow\rangle(\langle\uparrow\rangle p \wedge q)]$	(linearity)

Note that by looking at the syntax alone, one can easily see that each axiom corresponds to a first-order condition on the *i*-frame.

(2) *The rules of inference*:

(R1) (Modus ponens): infer ψ from φ and $\varphi \to \psi$.
(R2) (Temporal Generalization): infer $[\leftarrow]\varphi$, $[\rightarrow]\varphi$, $[\uparrow]\varphi$ and $[\downarrow]\varphi$ from φ.
(R3) (Redundancy of Distinguishing Propositionals)
 a infer φ from $\mathrm{hor}(h) \to \varphi$, provided h does not occur in φ;
 b infer φ from $\mathrm{ver}(v) \to \varphi$, provided v does not occur in φ, where $\mathrm{hor}(\varphi)$ and $\mathrm{ver}(\psi)$ are defined in 3.11.

The *theorems* of the system are the formulas obtainable from its axioms by its rules. A formula is *consistent* if its negation cannot be derived as a theorem; a set of formulas is consistent if the conjunction of every finite subset is.

Remark 3.3 (1) Reading "F" (future) for "$\langle\uparrow\rangle$" and "P" (past) for "$\langle\downarrow\rangle$", one easily proves (cf. [7]) that the (point-based!) logic axiomatized by Axioms (I0)–(I4) and (I7) is complete for the class of linear point structures with a beginning point. This fact will implicitly be used in the sequel.

(2) The odd-looking rule (R3) is the one mentioned above. The formulas $\mathrm{hor}(h)$ and $\mathrm{ver}(v)$ state that the "environment" of the current interval is "(point)-frame-like". So (R3) more or less expresses the following: If φ is a "theorem in a frame-like environment" it is a theorem.

We want to prove the following:

(A) L_{lin} is *sound* for K_{lin}: Every theorem of L_{lin} is valid over K_{lin}.
(B) L_{lin} is *complete* for K_{lin}: Every formula valid over K_{lin} is a theorem of L_{lin}, or equivalently, every consistent formula is satisfiable in K_{lin}.

As usual, the first of these problems is easier to solve:

Theorem 3.4. (Soundness) *Every theorem of L_{lin} is valid over K_{lin}.*

Proof: To show the soundness of the axioms and the first two derivation rules is routine and is left to the reader. (In checking Axiom (I6) its name will become clear.) For (R3), suppose $\not\models \varphi$, then there are $F, V, s < t$ in F such that $F, V \models \neg\varphi$ $[s,t]$. Now consider a valuation V' which only differs from V on h; set $V'(h) = \{[x,t] \mid x \leq t\}$. Then $F, V' \models \mathrm{hor}(h)$ and, as h does not occur in φ, $F, V' \models \neg\varphi$. So $\not\models \mathrm{hor}(h) \to \varphi$. This means that $\models \mathrm{hor}(h) \to \varphi$ implies $\models \varphi$, so (R3) is sound.

Definition 3.5
(1) If L is a set of propositional constants, then an *L-MCS* (*maximal consistent set*) is a consistent set $\Delta \subset \Phi_L$ satisfying: For all $\varphi \in \Phi_L$, $\varphi \in \Delta$ iff $\neg\varphi \notin \Delta$.
(2) If Δ' is an L'-MCS and $L' \supseteq L$ then $\Delta' | L$ is defined as $\Delta' \cap \Phi_L$.

Definition 3.6 If Γ and Δ are MCS's, we say Γ *is* (potentially) *below* Δ, or Δ *lies below* Σ, and write $\Gamma <_\uparrow \Delta$, if one of the following equivalent conditions is met:

(i) For all φ, if $\Box_\downarrow\varphi \in \Delta$ then $\varphi \in \Gamma$;
(ii) For all φ, if $\Box^\uparrow\varphi \in \Gamma$ then $\varphi \in \Delta$;
(iii) For all φ, if $\varphi \in \Delta$ then $\Diamond^\uparrow\varphi \in \Gamma$;
(iv) For all φ, if $\varphi \in \Gamma$ then $\Diamond_\downarrow\varphi \in \Delta$.

In a similar way we define: $\Sigma >_\rightarrow \Pi$, Π *lies* (potentially) *to the right of* Σ, *or* Σ *is situated to the left of* Π.

Without proof I mention the following facts (of course, their mirror images hold as well):

Lemma 3.7
(0) *Every consistent set* $\Sigma \subseteq \Phi_L$ *is contained in an* L-MCS Σ';
(1) *If* $\Diamond^\uparrow\varphi \in \Gamma$ *then there is an* MCS Δ *above* Γ *such that* $\varphi \in \Delta$;
(2) *If* $\Diamond_\downarrow\varphi \in \Gamma$ *then there is an* MCS Δ *below* Γ *such that* $\varphi \in \Delta$;
(3) *If* $\Gamma <_\uparrow \Delta$ *and* $\Delta <_\uparrow \Sigma$ *then* $\Gamma <_\uparrow \Sigma$;
(4) *If* Δ *and* Σ *are both above* Γ *then* $\Sigma <_\uparrow \Delta$ *or* $\Sigma = \Delta$ *or* $\Delta <_\uparrow \Sigma$;
(5) *If* Γ *is above both* Δ *and* Σ *then* $\Sigma <_\uparrow \Delta$ *or* $\Sigma = \Delta$ *or* $\Delta <_\uparrow \Sigma$;
(6) *If* Γ *is below* Γ' *and to the right of* Δ *then there is a* Δ' *above* Δ *and to the left of* Γ';
(7) *If* Γ *is below* Γ' *and to the left of* Δ *then there is a* Δ' *above* Δ *and to the right of* Γ';
(8) If Γ *is above* Γ' *and to the right of* Δ *then there is a* Δ' *below* Δ *and to the left of* Γ',

viz., (6) $\begin{array}{cc} \Delta' & \Gamma' \\ \Delta & \Gamma \end{array}$ (7) $\begin{array}{cc} \Gamma' & \Delta' \\ \Gamma & \Delta \end{array}$ (8) $\begin{array}{cc} \Delta & \Gamma \\ \Delta' & \Gamma'. \end{array}$

Definition 3.8 Let W be a countably infinite set. An *L-lattice* G (in W) *of size* n is a pair (F, U), where F is a finite (linear) frame $(T, <)$, $T \subseteq W$ has n elements, and U is a *chronicle* on $\mathrm{INT}(F)$, i.e., a map assigning L-MCS's to F-intervals. This chronicle U must be *respecting point-intervals*, i.e., $\Box_\downarrow\bot$ is in $U(t,t)$ for all $t \in T$, and *coherent*, which means that for all $s, t, u \in T$, $U(s,t)$ is below $U(s,u)$ if $s \leq t < u$, $U(s,u)$ lies to the left of $U(t,u)$ if $s < t \leq u$, etc.

Remark 3.9 In the completeness proof the spatial representation of $\mathrm{INT}(F)$ will ordinarily be used. In that case I will give names $C, D, P, S, \ldots$ to the points representing intervals and use the corresponding Greek capitals Γ, Δ, Π, $\Sigma, \ldots$ for their U-images.

Definition 3.10 For frames and lattices, *inclusion* (notation: $\subseteq$) is defined in an obvious way, as is, for two languages $L \subseteq L'$, the *restriction* of an L'-lattice G' to L (notation: $G'|L$). If G is an L-lattice, $L \subset L'$, and G' is an L'-lattice, then G' is said to *extend* G ($G \subseteq G'$) if $G'|L$ does, i.e., if $G \subseteq G'|L$. For the *union* of a *chain* of frames, i.e., a sequence $(F_n)_{n\in\omega}$ of frames with $F_n \subseteq F_{n+1}$ for all $n \in \omega$, the obvious definition holds as well.

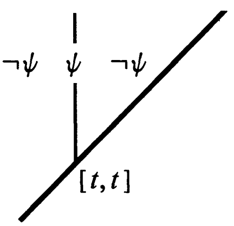
Definition 3.11 A lattice $G = (F, U)$ is *maximally distinguishing* (*md*), if for every $t \in T$ there are formulas φ_t and ψ_t such that

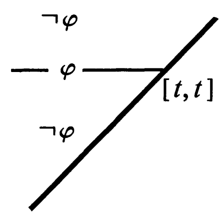
$$\mathrm{ver}(\psi_t) = (\psi_t \wedge \boxbar\psi_t) \wedge \boxbar\boxminus\neg\psi_t \wedge \boxminus(\neg\psi_t \wedge \boxbar\neg\psi_t)$$

and

$$\mathrm{hor}(\varphi_t) = (\varphi_t \wedge \boxminus\varphi_t) \wedge \boxminus\boxbar\neg\varphi_t \wedge \boxbar(\neg\varphi_t \wedge \boxminus\neg\varphi_t)$$

are in $U(t,t)$, viz.





Lemma 3.12 *If G is a maximally distinguishing lattice then for every t in F there are formulas μ and ν such that*

$$(\mu \wedge \boxminus\mu) \wedge \boxbar(\mu \wedge \boxminus\mu) \qquad \textit{is in } U(x,t) \textit{ for all } x \in F$$

and

$$(\nu \wedge \boxbar\nu) \wedge \boxminus(\nu \wedge \boxbar\nu) \qquad \textit{is in } U(t,y) \textit{ for all } y \in F.$$

Proof: Take $\mu = [\![\mathrm{EP}]\!]\varphi_t$ and $\nu = [\![\mathrm{BP}]\!]\psi_t$. The proof is easy, though laborious.

As was said before, a lattice is meant to be only an approximation of a model: it doesn't have to be perfect. One way of describing shortcomings of this approximation is given by:

Definition 3.13 A *possible* (L_0)-*defect* of a lattice is a quadruple (k,s,t,φ), where $1 \leq k \leq 4$, $s,t \in W$ (recall that lattices were defined "within" a set W), and φ is a formula. Such a quadruple (k,s,t,φ) is called an (*actual*) *defect* of a lattice $G = (T, <, \cup)$ if $s,t \in T$, $s \leq t$, and one of the following is the case: $k =$ 1, $\Diamond\varphi \in U(s,t)$, and there is no $u \in T$ such that $s \leq u < t$ and $\varphi \in U(s,u)$; 2, $\Diamond\varphi \in U(s,t)$, and there is no $u \in T$ such that $t < u$ and $\varphi \in U(s,u)$; 3, $\Diamond\varphi \in U(s,t)$, and there is no $u \in T$ such that $s < u \leq t$ and $\varphi \in U(u,t)$; 4, $\Diamond\varphi \in U(s,t)$, and there is no $u \in T$ such that $u < s$ and $\varphi \in U(u,t)$.

A defect (k, s, t, φ) is said to be *of type* k. As W is countable, we may and will assume the existence of an enumeration $d_1, d_2, \ldots$ of all possible defects.

Lemma 3.14 *Suppose* $G = ((T, <), U)$ *is an* md-*lattice of size* n, *and* s *is an actual defect of* G; *then there is a lattice* $G' = ((T', <'), U')$ *of size* $n + 1$, *extending* G, *in which the defect* s *is removed.*

Proof: (Only defects of type 1 and 2 are treated; the proof for their mirror images 3 and 4 is the mirror image of this proof.) The adding of points/intervals runs in two stages:

(A) First one point-with-an-MCS is added.
(B) In the second stage this "pseudo-lattice" is made into a proper lattice again.

(A) Suppose $T = \{u_1, \ldots, u_n\}$, where $t_i < t_j$ if $i < j$. As we have a defect of type 1 or 2, there must be $s, t \in T$ with $s \leq t$, $\diamondsuit\varphi$ (respectively $\diamondsuit\varphi$) $\in U(s, t)$, while there is no $u \in T$ such that $t < u$ (respectively $s \leq u < t$), and $\varphi \in U(s, u)$. Consider the "vertical line of intervals" $\{[s, x] \mid x \in T\}$. Somewhere on this line, above (respectively below) $[s, t]$ we have to add, in a coherent way, an MCS Γ with $\varphi \in \Gamma$.

By Remark 3.3(1) one can use a simple argument, which has a completely standard counterpart in point-logic (cf. [7]), to show that this is possible, somewhere between a $[s, x_k]$ and a $[s, x_{k+1}]$ or above $[s, x_n]$.

So we add a new point $u \in W \backslash T$ to T, set $u_i < u$ if $i \leq k$, $u < u_i$ if $i \geq k + 1$ in the first case and $u_i < u$ for all i in the second, viz., the examples in the figures below. Then we define $U'(s, u)$ to be the Γ we found.

(B) In the second stage of the construction we are dealing with an md-lattice to which one interval-with-an-MCS is added. To make this "pseudo-lattice" into a real lattice again, we have to show that it is possible to extend coherently the definition of U' to the other newly arisen intervals.

What exactly must be done now is best illustrated in Figures 3 and 4 of the spatial representation of the "pseudo-lattice". In these figures + stands for an old interval (i.e., an interval $[s, t]$ with $s, t \in T$), $*$ for the one new interval already accompanied by an MCS, and $\circ$ for the new intervals for which we still need to define U'-images.

$\circ$	$\circ$	$_*[s,u]$	$\circ$	$\circ$	$\circ$
+	+	+	+	+	
+	+	$_+[s,t]$	+		
+	+	+			
+	+				
+					

Figure 3.

+	$_+[s,t]$	+	$\circ$	+	+
+	+	+	$\circ$	+	
$\circ$	$_*[s,u]$	$\circ$	$\circ$		
+	+	+			
+	+				
+					

Figure 4.

We treat only the second (and more difficult) case, in which the new interval lies somewhere between two older ones:

+	+	+	$_\circ P_6$	+	+
$_+D_1$	$_+D_2$	$_+D_3$	$_\circ P_5$	$_+D_4$	
$_\circ P_1$	$_\circ P_2$	$_\circ P_3$	$_\circ P_4$		
$_+C_1$	$_+C_2$	$_+C_3$			

This means that in the first part of the construction we added the interval (represented by) P_2 to the lattice G, together with an MCS Π_2 of which we know that it lies above Γ_2 and below Δ_2.

Now we have to prove the existence of MCS's Π_1, Π_3, Π_4, Π_5, and Π_6, which are coherent with each other and with the old lattice.

By Lemma 3.7.8 there must be a Π_1 situated to the left of Π_2 and below Δ_1. Now the problem is to prove that Π_1 is above Γ_1. In any case, because both Γ_1 and Π_1 are below Δ_1, by Lemma 3.7.5 we know that either Π_1 is below Γ_1 or $\Pi_1 = \Gamma_1$ or Π_1 is above Γ_1. To rule out the first two possibilities, we need the fact that G is maximally distinguishing; by Lemma 3.1 there is a formula γ such that $(\gamma \wedge \boxminus\gamma) \wedge \boxbar(\neg\gamma \wedge \boxminus\neg\gamma)$ belongs to each MCS "on the C-line", whence $\gamma \in \Gamma_1$ and $\boxdot\boxminus\neg\gamma$ is in $\Gamma_2 \Rightarrow \boxdot\boxdot\boxminus\neg\gamma \in \Gamma_2 \Rightarrow \boxdot\boxminus\boxdot\neg\gamma \in \Gamma_2$. So both $\boxminus\neg\gamma$ and $\boxminus\boxdot\neg\gamma$ are in Π_2, as Π_2 is above Γ_2. We get

(1) $\neg\gamma$ is in Π_1, so Π_1 cannot be on the C-line: $\Pi_1 \neq \Gamma_1$
(2) $\boxdot\neg\gamma$ is in Π_1, so Π_1 cannot be below Γ_1.

So we have found a Π_1 with the desired properties.

The existence of the MCS's Π_3, Π_4, Π_5, and Π_6 can be proved similarly. So, adding the pairs (P_i, Π_i) to the chronicle U', we get a lattice $G' \supseteq G$ of size $n + 1$, in which the old defect is removed.

Now as the construction of new lattices out of old ones has to be repeated a possibly infinite number of times, the previous lemma is not enough: we have to obtain a new lattice which itself is maximally distinguishing:

Lemma 3.15 *Let G be an* md *L-lattice of size n and s a defect of G. Then there are a set of propositional constants $L'' \supseteq L$ and an* md *L''-lattice G'' of size $n + 1$, extending G but not having the defect s.*

Proof: Using the same terminology as in Lemma 3.14, we will show that there is a language L'' and an md L''-lattice G'' such that $G' = G'' \mid L''$, where G' is the lattice obtained in Lemma 3.14.

The only way for G' not to be maximally distinguishing is for there to be no formulas φ and ψ such that $\mathrm{hor}(\varphi)$ and $\mathrm{ver}(\psi)$ are in $U(t,t)$, where t is the one point newly added to T.

So our first step in order to turn G' into G'' is to add two new constants h and v to L and put $\mathrm{hor}(h)$ and $\mathrm{ver}(v)$ in $U'(t,t)$, and then extend this set to an MCS $U''(t,t)$. That this is possible is given by the following claim:

Claim 3.15.1 *If $\boxcircle\bot \in \Pi$, Π is consistent, and the propositional constants h and v do not appear in Π, then $\Pi \cup \{\mathrm{hor}(h), \mathrm{ver}(v)\}$ is consistent.*

Proof: Suppose $\Pi \cup \{\mathrm{hor}(h)\}$ is inconsistent; then there is a formula $\pi \in \Pi$ such that $\vdash \mathrm{hor}(h) \rightarrow \neg\pi$. As h does not occur in Π, (R3) gives $\vdash \neg\pi$, contradicting the consistency of Π. In the same way one can prove the consistency of $\Pi \cup \{\mathrm{hor}(h), \mathrm{ver}(v)\} = (\Pi \cup \{\mathrm{hor}(h)\} \cup \{\mathrm{ver}(v)\}$.

After this, one by one every L'-MCS $U'(s,u)$ of G' will be coherently extended to an L''-MCS $U''(s,u)$. As this construction is very similar to the one in the previous lemma, the proof is omitted here.

Theorem 3.16 *L_{lin} is complete for K_{lin}.*

Proof: Suppose φ is consistent. We now construct a linear frame in which φ is satisfiable.

Define L_0 as the set of propositional constants in φ and fix an L_0-MCS Σ with $\varphi \in \Sigma$. Suppose $\boxdot\bot \in \Sigma$ (otherwise start with $\{\varphi | [\![\mathrm{BP}]\!]\varphi \in \Sigma\}$). Suppose, for all $n \in \omega$, that h_{n+1} and v_{n+1} are constants not in L_n and define, for all $n \in \omega$, $L_{n+1} = L_n \cup \{h_{n+1}, v_{n+1}\}$. Then by Claim 3.15.1, $\Sigma \cup \{\mathrm{hor}(h_1), \mathrm{ver}(v_1)\}$ is consistent, so it can be extended to an L_1-MCS Σ_1.

Recall that we have an enumeration $d_1, d_2, \ldots$ of all possible L_0-defects of lattices with elements in a countable set W. Defining $G_1 = (\{w\}, \emptyset, \{([w,w], \Sigma_1)\})$ for some $w \in W$, an iterative application of Lemma 3.15 yields the existence of a chain of maximally distinguishing lattices $G_1 \subseteq G_2 \subseteq \ldots$ such that $G_{n+1} = G_n$ if G_n has no defects; otherwise G_{n+1} is obtained by removing that defect of G_n having the lowest index. (If $G_{n+1} \neq G_n$ then G_{n+1} is an L_{n+1}-lattice of size $n + 1$.) One easily proves that for every defect of every G_n there is an $m > n$ such that G_m does not have this defect. Note that it is sufficient to remove only L_0-defects.

Now define the following valuation on the union F of the frames F_n: $V(p) = \{[s,t] | \, p \in U(s,t)\}$, where $U = \bigcup_{n\in\omega} U_n$.

Truth Lemma *For every L_0-formula φ:*

$$F, V \vDash \varphi[s,t] \qquad \textit{iff} \qquad \varphi \in U(s,t).$$

Proof: Using formula-induction:

(1) For atomic formulas the assertion is clear by definition of V.

(2) For $\varphi \equiv \neg\psi$ or $\varphi \equiv (\psi \wedge \chi)$, the proof is a routine check.

(3) Of the other possibilities we treat only the case $\varphi \equiv \diamondsuit\psi$.

$\Rightarrow$: Suppose $F, V \vDash \diamondsuit\psi \; [s,t]$. Then there is a $u \in T$ with $s \leq u < t$ and $F, V \vDash \psi \; [s,u]$. According to the induction hypothesis, $\psi \in U(s,u)$. As $T = \bigcup_{n\in\omega} T_n$ and $< \, = \bigcup_{n\in\omega} <_n$, this means that there must be an $n \in \omega$ such that $s,t,u \in T_n$ and $s \leq_n u <_n t$. Because U_n is coherent on G_n, this means that $U_n(s,u) <_\uparrow U_n(s,t)$. Then by definition of $<_\uparrow$, $\diamondsuit\psi \in U_n(s,t)$, so we can conclude that $\diamondsuit\psi \in U(s,t)$.

$\Leftarrow$: Suppose $\diamondsuit\psi \in U(s,t)$, and take a G_n for which $s,t \in T_n$. If there is no u in G_n with $s \leq_n u <_n t$ then this is an actual defect of G_n. In that case there must be an $m > n$ such that G_m does not have this defect, i.e., $G_m \supseteq G_n$ and there is a u in T_m satisfying $s \leq_m u <_m t$ and $\psi \in U_m(s,u)$.

Now $s \leq u < t$ and $\psi \in U(s,u)$. By the induction hypothesis, this yields $F, V \vDash \psi \; [s,u]$, so $F, V \vDash \diamondsuit\psi \; [s,t]$.

As it is not hard to prove that F (being the union of a chain of linear frames) is itself linear, we can conclude that for any consistent formula φ there is a linear model (F, V) and an interval $[s,t]$ in the frame such that $F, V \vDash \varphi \; [s,t]$.

This means that indeed φ is satisfiable in K_{lin}.

Definition 3.17 The interval tense logics L_{itv}, $L_{dis,}$ and L_Q consist of the same axioms and derivation rules as L_{itv}, with the following adaptations:

(1) L_{itv}, the (general) interval tense logic: Replace Axiom (I7) by Axiom (I7′):

(I7′) $(\langle\downarrow\rangle p \wedge \langle\downarrow\rangle q) \rightarrow$
$\{\langle\downarrow\rangle(p \wedge \langle\downarrow\rangle q) \vee \langle\downarrow\rangle(p \wedge q) \vee \langle\downarrow\rangle(\langle\downarrow\rangle p \wedge q)\}.$

(2) L_{dis}, the discrete interval tense logic: Add the following axiom to L_{itv}:

(D) DISC. (defined in Claim 1.6.3.2)

(3) L_{Q}: add the following axioms to L_{lin}:

(Ul) $\langle\leftarrow\rangle T$
(Ur) $\langle\uparrow\rangle T$
(D) DENSE. (defined in Claim 1.6.3.1)

Theorem 3.18
(1) L_{itv} *is sound and complete for* K_{itv}, *the class of all point-frames with linear intervals*;
(2) L_{dis} *is sound and complete for* K_{dis}, *the class of discrete frames*;
(3) L_{Q} *is sound and complete for* $\mathbb{Q}$.

Proof: Soundness is an easy matter in all cases.

The completeness proof for L_{itv}, which is conceptually analogous, yet technically more complex than the one for L_{lin}, is omitted here.

For L_{dis} one copies the proof for L_{itv}; arriving at the Truth Lemma, one observes that the formula DISC is true at every interval in the obtained model (F, V); as this formula has no propositional constants, this means that it is true at every model on F, whence $F \equiv$ DISC. So by Claim 1.6.3.2 F is discrete.

The completeness proof for L_{Q} begins exactly as the one for L_{lin}; in a similar way as with L_{dis}, one obtains a frame validating the formulas $\langle\leftarrow\rangle T$, $\langle\uparrow\rangle T$, and DENSE. So this (linear!) F must be unbounded to the left and to the right, and dense: $F \cong \mathbb{Q}$.

Question 3.19 The proof of Claim 3.15.1 is the only place in the completeness proof where the odd derivation rule (R) is used. It is an open question whether this rule is really needed to prove completeness; perhaps it can be omitted altogether or replaced by a (finite?) set of axioms.

Question 3.20 The facts presented in 3.1 seem to disqualify the use of the system HS for practical purposes; it may well be possible, however, that some *fragments* of the full language behave better. Are there natural and useful fragments of the language for which we can obtain completeness or decidability results with respect to their validity on the standard frames?

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