Notre Dame Journal of Formal Logic
Volume 31, Number 3, Summer 1990

Negative Membership

WAYNE D. BLIZARD*

Abstract Generalized sets whose characteristic functions may assume any integer value, positive or negative, are formalized in a first-order two-sorted theory MSTZ which contains an exact copy of ZFC and is relatively consistent.

Introduction By *negative membership* we mean the fact of belonging to a collection of objects a negative number of times. This concept extends the notion of belonging to a collection of objects any positive number of times, which has been formalized using multisets (see Blizard [1] and [2]). A *multiset*, or *mset*, is a collection of objects, called *elements*, in which elements are allowed to repeat. The number of times an element repeats in a multiset is called its *multiplicity*. The *cardinality* of a multiset is the sum of the multiplicities of its elements. In the multiset $[c,a,b,a,a,b]$ the element a has multiplicity 3, b has multiplicity 2, and c has multiplicity 1. We denote this multiset by $[a,b,c]_{3,2,1}$. The cardinality of $[a,b,c]_{3,2,1}$ is 6. A *set* is a multiset in which each element has multiplicity 1. We denote the set $[c,a,b]_{1,1,1}$ by $\{a,b,c\}$. It is assumed that elements of multisets have finite multiplicities but that the number of distinct elements in a multiset need not be finite. The concept of negative membership, or *negative multiplicity*, has been used and investigated in the literature (see [4], [6], [10], [16], [17], and [18]). We develop a first-order two-sorted theory MSTZ for multisets in which elements may have integer multiplicities (positive, negative, or zero). In MSTZ, $[a,b,c]_{-1,2,-4}$ denotes the unique multiset containing -1 copies of a, two copies of b, and -4 copies of c. The theory MSTZ is a generalization of the theory MST (see [1] and [2]). We show that MSTZ contains an exact copy of ZFC and that MSTZ is relatively consistent.

*This research was supported by the Natural Sciences and Engineering Research Council of Canada.

Received January 27, 1987, revised April 18, 1989

Negative multiplicity in the literature T. Hailperin [10] uses multisets to interpret the logical system in G. Boole's *Laws of Thought.* In his first edition (1976), Hailperin used the word "heap" instead of "multiset". (For reviews of both editions, see Gridgeman [8].) In Hailperin's notation, the multiset $\{(h_1)a_1, \ldots, (h_i)a_i, \ldots\}$ contains the element a_1 with multiplicity $h_1, \ldots,$ the element a_i with multiplicity $h_i, \ldots$. Hailperin needs additive inverses to interpret Boole's unrestricted subtraction. He therefore introduces *signed* multisets—multisets in which the multiplicities h_i can be *any* integer. He remarks, "While the notion of a signed multiset is not as intuitively simple as that of an unsigned multiset, a brief reflection on the history of the difficulties which were experienced until negative numbers were in good standing, should help one overcome resistance to the acceptance of signed multisets as a meaningful notion" ([10], p. 139).

The conceptual difficulties encountered when negative numbers were introduced into mathematics are discussed in M. Kline [11], pp. 252–253, 592–593 and E. Fischbein [7], pp. 97–102. It is interesting to note that negative numbers were introduced and accepted in China much earlier (by the second century B.C.) than in the West. The conceptual difficulties inherent in negative numbers are exactly the conceptual difficulties associated with negative membership (or negative multiplicity). One great advantage of negative numbers is that they permit unrestricted subtraction. In MSTZ, the existence of negative multiplicities allows for unrestricted complementation; that is, the mset $x - y$ is meaningfully defined for *all* msets x and y.

In [18] H. Whitney argues that the algebra of characteristic functions of sets is preferable to the usual algebra of sets since it can be expressed using operations on numbers. He investigates generalized characteristic functions: "Suppose we associate with each element of a set R any integer, positive, negative or zero, instead of merely one or zero. The resulting function will not in general be the characteristic function of a real set; but we may consider it as the characteristic function of a *generalized set*, where each element is counted any number of times" ([18], pp. 411–412). He remarks, ". . . 'generalized sets' . . . are useful in various mathematical theories" ([18], p. 405), citing ". . . chains in analysis situs [topology]" ([18], p. 412) as one example. Whitney develops the algebra of characteristic functions of "real" and "generalized" sets and establishes conditions on their normal forms for distinguishing between the two types of sets.

R. Rado [16] investigates properties of families of sets using multisets. Rado defines a multiset as a cardinal-valued function such that the nontrivial elements of the domain (elements whose image is nonzero) form a set. He calls the class of all such multisets the *cardinal module* since its structure most resembles that of a module over the semigroup of all cardinals. He defines a *signed-cardinal* as a pair (σ, λ) where λ is a cardinal and $\sigma \in \{-1, 1\}$. After describing the arithmetic of signed-cardinals, Rado investigates multisets that are certain signed-cardinal-valued functions ([16], pp. 139–140). This is exactly the notion of negative membership, or negative multiplicity, under consideration in this paper.

M. P. Schützenberger makes use of negative membership (positive and negative integer-valued characteristic functions) in his mathematical theory of elementary families of automata. S. Eilenberg ([4], p. 158) gives a brief description of Schützenberger's work with references. W. Reisig defines negative-valued multisets ([17], Section 9.2) in his introduction to relation nets. He states, ". . . we allow that some element d may also be contained in M 'negatively often'" ([17], p. 126). R. Feynman [6] employs negative-valued multisets (which occur at intermediate stages of computation) as a conceptual aid to introduce the concept of negative probability, which he then applies to two-state systems. The very idea of negative probability suggests the use of sample multisets containing both positive and negative numbers of occurrences of events.

In [14] Meyer and McRobbie are able to distinguish between the Anderson/Belnap system R of relevant implication and its Dunn/McCall extension RM. They find ([14], pp. 107–108) that in RM ". . . it is permissible to take premisses of arguments as being collected together into *sets* . . .", but in R *multisets* of premisses must be used since ". . . it matters in the theory of deduction for R but not in the theory of deduction for RM how often a premiss is *repeated* in the course of an argument". The details of the systems R and RM need not concern us here. Meyer and McRobbie, therefore, make use of multisets of premisses. Although they do not consider negative multiplicity, they come very close ([14], pp. 128, 130) to exactly the same concept. The *binary additive union* of two msets x and y is denoted by $x \uplus y$. The elements of $x \uplus y$ are all elements that belong either to x or to y. The multiplicity of an element in $x \uplus y$ equals the sum of its multiplicity in x and its multiplicity in y (where not being an element is interpreted as having multiplicity zero). For example, $[a,b]_{1,2} \uplus [b,c]_{2,3} = [a,b,c]_{1,4,3}$. Let $\varnothing$ be the mset with no elements. For two msets x and y, we say that x *divides* y, written $x|y$, iff $y = x \uplus z$ for some mset z. In fact, the mset z is unique. For any mset x, $x|x$ since $x = x \uplus \varnothing$; and $\varnothing|x$ since $x = \varnothing \uplus x$. Meyer and McRobbie note that if $x|y$ and $y = x \uplus z$, then an element with multiplicity n in x, and m in y, must have multiplicity $m - n$ in z. This is as far as they go. However, one can ask, for any mset x, does $x|\varnothing$? In other words, does there exist an mset z such that $\varnothing = x \uplus z$? This unique mset z must be such that an element has multiplicity n in x iff it has multiplicity $-n$ in z. This new type of mset is like Hailperin's signed multiset. In Hailperin's notation, if $x = \{(h_1)a_1, \ldots, (h_i)a_i, \ldots\}$ then the mset z is $\{(-h_1)a_1, \ldots, (-h_i)a_i, \ldots\}$. Hailperin's sum is equivalent to the additive union of msets. Therefore, although negative multiplicities do not arise in [14], Meyer and McRobbie's division operator leads directly to msets that are equivalent to Hailperin's signed multisets.

The relevance of multiset theory to physics has been discussed in [1] and [2]. There is an analogy between msets in MSTZ and certain (collections of) elementary particles in physics. M. Guillen has suggested that P. Dirac's 1930 prediction of the existence of the positron (confirmed in 1932) was an application of the concept of negative number to theoretical nuclear physics ([9], p. 65). Since 1932, negative counterparts (or antiparticles) have been identified for all other known subnuclear particles (including the proton and the neutron). Guillen concludes ". . . antimatter is now recognized to be as significant a part of the natural world as negative numbers are of the algebraic realm", ". . . scientific theories agree that antimatter is the embodiment of mathematical negativity . . .", and ". . . it is accurate to say that particles of antimatter are the mathematicians' negative numbers incarnate" ([9], pp. 65–66). One remarkable feature of the positron (with mass equal to the electron but with positive rather than negative charge) is that if it should ever be in the vicinity of an electron, both particles would be annihilated instantly ([9], p. 65). In MSTZ, we show that an mset and its unique 'shadow' mset 'annihilate' each other when joined in additive union.

***The formal theory* MSTZ** We now generalize the theory MST developed in [1] and [2] to obtain the theory MSTZ. The axioms for a vector space are stated in a *two-sorted language* with scalar variable symbols and vector variable symbols. The axioms for a vector space include the axioms for a field, stated in scalar variable symbols, and the axioms for vectors and scalars stated in both sorts of variable symbols. In exactly the same way, the language of MSTZ employs two sorts of variable symbols: *numeric variable symbols* $\dot{k}$, $\dot{l}$, $\dot{m}$, $\dot{n}$, ... used to denote multiplicities, and *mset variable symbols* x, y, z, ... used to denote msets and elements of msets. As we shall see, the axioms of MSTZ include axioms for the integers stated in numeric variable symbols, together with generalizations of ZF axioms stated in both numeric and mset variable symbols.

The theory MSTZ is formulated in the first-order predicate calculus with equality using the usual logical symbols: $\sim$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (if . . . then . . .), $\leftrightarrow$ (iff), $\exists$ (there exists), $\forall$ (for all), $=$ (equality), and enclosures (,), [,]. The first-order language of MSTZ is $L = \{<, e, \hat{}, +, \cdot, 0, 1, -1\}$, where $<$ is a binary predicate symbol, e is a ternary predicate symbol, $\hat{}$ is a unary function symbol, $+$ and $\cdot$ are binary function symbols, and 0, 1, and -1 are numeric constant symbols.

An *expression* of L is any finite sequence of symbols of L. The collection of *numeric terms* of L is the smallest collection of expressions of L that contains the numeric variable symbols $\dot{k}$, $\dot{l}$, $\dot{m}$, $\dot{n}$, . . . , the constant symbols 0, 1, and -1, and is *closed* under the function symbols $\cdot$ and $+$; that is, if s and t are any numeric terms of L, then $s \cdot t$ and $s + t$ are also numeric terms of L. The symbols s and t are metamathematical symbols used to denote numeric terms of L. We write $-s$ to stand for the numeric term $-1 \cdot s$. By $s - t$ we mean $s + (-t)$.

The *mset terms* of L are the mset variable symbols x, y, z, . . . together with all expressions of L of the form $\hat{s}$, where s is any numeric term. The numeric term $\hat{s}$ denotes the 'set' corresponding to the number denoted by s. (For nonnegative s, $\hat{s}$ is just the usual von Neumann numeral set corresponding to s.) We use the metamathematical symbols u, v, and w to denote mset terms.

The *atomic formulas* of L are all expressions of L of the forms $s < t$, $e(u, v, t)$, $s = t$, or $u = v$. These formation rules do not permit equality between mset and numeric terms. The numeric and mset universes of MSTZ are, therefore, disjoint.

The well-formed formulas (wffs) of L are defined as follows: all atomic formulas of L are wffs, and if ϕ and ψ are wffs, then so are $\sim\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, $\phi \leftrightarrow \psi$; and for all variable symbols x and $\dot{n}$, so are $\exists x\phi$, $\forall x\phi$, $\exists\dot{n}\phi$, and $\forall\dot{n}\phi$.

We adopt the convention of A. Levy ([13], p. 5) that $\phi(\ldots)$ means that the interesting cases of what is to be said are those where the variable symbols (mset or numeric) displayed in the list $(\ldots)$ occur free in ϕ. Therefore $\phi(x)$ (or $\phi(\dot{n})$) does *not* mean that x (or $\dot{n}$) is a free variable in ϕ, nor does it mean that ϕ has no free variables other than x (or $\dot{n}$).

We adopt the following *convention for numeric quantifiers*: $\exists n\phi(n)$ stands for $\exists\dot{n}(\dot{n} \neq 0 \wedge \phi(\dot{n}))$ and $\forall n\phi(n)$ stands for $\forall\dot{n}(\dot{n} \neq 0 \rightarrow \phi(\dot{n}))$. Therefore, the bound numeric variable symbols k, l, m, $n, \ldots$ without dots are intended to range over nonzero numbers.

When $\phi(u)$ (or $\phi(t)$) is used after first using the notation $\phi(x)$ (or $\phi(\dot{n})$), we mean the wff obtained from $\phi(x)$ (or $\phi(\dot{n})$) by the *proper* substitution of the mset term u (or the numeric term t) for the free occurrences of x (or $\dot{n}$) if any. By *proper* substitution we mean that we assume that collisions between variable symbols are avoided. For example, if n is bound in $\phi(x)$, we assume that the bound occurrences of n are replaced by some other suitable numeric variable symbol before the mset term $\hat{n}$ is substituted for all free occurrences of x in $\phi(x)$ to obtain the wff $\phi(\hat{n})$. If we write $\phi(t)$ after first using the notation $\phi(n)$, the numeric term t is assumed to denote a nonzero number.

Whenever class terms of the forms $\{u \mid \phi(u)\}$ or $\{t \mid \psi(t)\}$ are used, the expressions in which they occur are always reducible to wffs of L (see, for example, A. Levy [13], 3.1, p. 9).

The intended interpretation of the atomic formula $e(u, v, t)$ is "u is an element of v with multiplicity t". We introduce the *dressed ternary epsilon predicate symbol* as follows: $u \in^t v$ stands for $e(u, v, t)$. From this point onward, we drop the predicate symbol e and use the dressed ternary epsilon predicate symbol instead. We also introduce a *binary epsilon predicate symbol* as follows: $u \in v$ stands for $\exists n \quad u \in^n v$. Recall that our convention for numeric quantifiers means that $\exists n \quad u \in^n v$ stands for $\exists\dot{n}(\dot{n} \neq 0 \wedge u \in^{\dot{n}} v)$. The intended interpretation of $u \in v$, therefore, is "u is an element of v with some nonzero multiplicity". We emphasize that $u \in v$ stands for a wff of L—it is *not* an atomic formula of L. Let $s \leq t$ stand for the wff $s < t \vee s = t$.

The nonlogical axioms of MSTZ include a collection of first-order axioms that characterize the integers stated in the numeric variable symbols $\dot{k}$, $\dot{l}$, $\dot{m}$, $\dot{n}, \ldots$ of L. For example, we could use the collection of axioms for an *ordered integral domain* (Pinter [15], pp. 170, 206) stated in the language $\{<, +, \cdot, 0, 1, -1\}$ together with the first-order axioms of *Peano Arithmetic* (Chang and Keisler [3], p. 42) stated in the language L restricted to nonnegative elements of the integral domain. The first six axioms of Peano arithmetic are numbered N1 through N6. The *induction schema* reads as follows: for every wff ϕ of L, the universal closure of

$$\mathbf{N7}_\phi \qquad \phi(0) \wedge \forall\dot{n}(0 \leq \dot{n} \rightarrow (\phi(\dot{n}) \rightarrow \phi(\dot{n}+1)) \rightarrow \forall\dot{n}(0 \leq \dot{n} \rightarrow \phi(\dot{n}))$$

is an axiom.

Equivalently, we could use the axioms for an *integral system* (Pinter [15], p. 207) where the second-order statement of the *well-ordering principle* (every nonempty set of nonnegative elements has a least element) is replaced by the following first-order axiom schema: for every wff ϕ of L, the universal closure of

$$\exists\dot{n}(0 \leq \dot{n} \wedge \phi(\dot{n})) \rightarrow \exists\dot{n}(0 \leq \dot{n} \wedge \phi(\dot{n}) \wedge \forall\dot{m}((0 \leq \dot{m} \wedge \dot{m} < \dot{n}) \rightarrow \sim\phi(\dot{m})))$$

is an axiom.

The remaining nonlogical axioms of MSTZ are stated in both numeric and mset variable symbols. They are generalizations of the axioms of MST which were formulated to resemble as much as possible the corresponding axioms in classical ZF set theory.

The *exact multiplicity* axiom of MSTZ is

I $\forall x \forall y \forall n \forall m((x \in^n y \wedge x \in^m y) \rightarrow n = m)$.

In other words, the multiplicity with which an element belongs to a multiset is unique.

The *axiom of extensionality* of MSTZ is

II $\forall x \forall y(\forall z \forall n(z \in^n x \leftrightarrow z \in^n y) \rightarrow x = y)$.

If two msets have exactly the same elements occurring with exactly the same multiplicities, then they are equal.

The *empty set* axiom of MSTZ is

III $\exists y \forall x \forall n \sim x \in^n y$.

In words, there exists an mset that contains no elements with nonzero multiplicity. By II this mset is unique and we denote it by $\varnothing$. Therefore, $\forall x \quad x \notin \varnothing$ or $\forall x \forall \dot{n}(x \in^{\dot{n}} \varnothing \rightarrow \dot{n} = 0)$.

Let $Set(u)$ stand for $\forall x \forall n(x \in^n u \rightarrow n = 1)$. Since $Set(\varnothing)$, we call $\varnothing$ the *empty set*. Let $u \subseteq v$ stand for $\forall z \forall n(z \in^n u \rightarrow \exists m(n \leq m \wedge z \in^m v))$. If $u \subseteq v$ we say that u is an *msubset* of v. If $u \subseteq v$ any element of u must have a nonzero multiplicity in v (by our convention for numeric quantifiers). Therefore, even though $-1 \leq 0$, for example, we do not have nonempty msubsets of $\varnothing$. Therefore, $[x,y]_{-1,-2} \nsubseteq \varnothing$, $\forall x(x \subseteq \varnothing \rightarrow x = \varnothing)$, $\forall x \quad \varnothing \subseteq x$, $[x]_{-1} \subseteq \{x\}$, $[x]_{-10} \subseteq [x]_{-7}$, and $[x,y]_{1,-2} \subseteq [x,y]_{2,-1}$. The relation $\subseteq$ is reflexive and transitive, and using Axioms I and II it is proved that $\subseteq$ is also anti-symmetric: $\forall x \forall y((x \subseteq y \wedge y \subseteq x) \rightarrow x = y)$.

The *elementary msets* axioms of MSTZ are

IV (i) $\forall x \forall n \exists y(x \in^n y \wedge \forall z(z \in y \leftrightarrow z = x))$
(ii) $\forall x \forall y(x \neq y \rightarrow \forall n \forall m \exists z(x \in^n z \wedge y \in^m z \wedge \forall z'(z' \in z \leftrightarrow (z' = x \vee z' = y))))$.

Axiom IV(i) states that for any mset x and any number n, there is a unique (by II) mset y containing exactly n copies of x and nothing else. Let $[u]_t$ denote the mset that contains exactly t copies of u and nothing else. We denote $[u]_1$ by $\{u\}$ and we call it the *singleton set* containing u since $Set([u]_1)$.

Axiom IV(ii) states that for any two *distinct* msets x and y and any numbers n and m, there exists a unique (by II) mset z containing exactly n copies of x, m copies of y, and nothing else. Let $[u,v]_{s,t}$ denote the mset that contains s copies of u, t copies of v, and nothing else. We denote $[u,v]_{1,1}$ by $\{u,v\}$ and we call it the *pair set* containing u and v since $Set([u,v]_{1,1})$. We require $x \neq y$ in Axiom IV(ii) since if $x = y$ but $n \neq m$, then IV(ii) would assert the existence of an mset that contradicts Axiom I. Therefore, we cannot write $\{x,x\}$ in the language of MSTZ since elements of pair sets must be *distinct*.

For msets u and v, we define the ordered pair $\langle u,v \rangle = \{\{u\}, \{u,v\}\}$ if $u \neq v$ and $\langle u,u \rangle = \{\{u\}, [u]_2\}$. We call $\langle u,v \rangle$ the *ordered pair set* since $Set(\langle u,v \rangle)$.

The *powerset* axiom of MSTZ is

V $\forall x \exists y(Set(y) \wedge \forall z(z \in y \leftrightarrow z \subseteq x))$.

For every mset x, there exists a *set* y whose elements are exactly the msubsets of x. The set y in Axiom V is unique by II. For any mset term u, we denote the set of all msubsets of u by $\mathbb{P}(u)$ and we call it the *powerset* of u since $Set(\mathbb{P}(u))$. The reasons we require $\forall x \quad Set(\mathbb{P}(x))$ are discussed in [1] and [2].

Powersets in MSTZ differ from powersets in MST since an mset in MSTZ has many more msubsets. For example, $\mathbb{P}(\{\varnothing\}) = \{\{\varnothing\}, \varnothing, [\varnothing]_{-1}, [\varnothing]_{-2}, \ldots\} \subseteq \mathbb{P}([\varnothing]_2)$. However, we can still prove in MSTZ:

Theorem $\quad \forall x \forall y (x \subseteq y \leftrightarrow \mathbb{P}(x) \subseteq \mathbb{P}(y))$.

Proof: The $\rightarrow$ direction follows by the transitivity of $\subseteq$. For the $\leftarrow$ direction, assume that $\mathbb{P}(x) \subseteq \mathbb{P}(y)$ and let $z \in^n x$. Since $z \in^n x$, $[z]_n \subseteq x$ and $[z]_n \in \mathbb{P}(x)$. By our assumption $[z]_n \in \mathbb{P}(y)$, so that $[z]_n \subseteq y$. But since $z \in^n [z]_n$, we must have $\exists m(n \leq m \wedge z \in^m y)$ and, therefore, $x \subseteq y$ as required.

The axiom of *foundation* of MSTZ is

VI $\quad \forall y(y \neq \varnothing \rightarrow \exists x(x \in y \wedge \forall z(z \in x \rightarrow z \notin y)))$.

In words, every nonempty mset of MSTZ contains an '$\in$-minimal' element (an element from which it is disjoint). Therefore, the defined binary epsilon relation $\in$ is *well-founded.* Axiom VI disallows infinitely descending $\in$-chains within an mset, $\in$-loops of the form $x_1 \in x_2 \in \ldots \in x_n = x_1$, and 'extraordinary' msets such that $x = \{x\}$.

The informal rules for binary union $\cup$, binary additive union $\uplus$, and binary intersection $\cap$ that must hold for their formal counterparts in MSTZ are as follows:

(i) Nonmembership is interpreted for this purpose as multiplicity zero;
(ii) For binary union, take the maximum of nonzero multiplicities;
(iii) For binary intersection, take the minimum multiplicity only when there are two nonzero multiplicities. Otherwise, the mset is not a common element and is *not* an element of the intersection;
(iv) For additive union, take the sum of multiplicities.

For example, in MSTZ we require that

$[x,y]_{-1,2} \cup [x,z]_{3,-1} = [x,y,z]_{3,2,-1}$

$[x,y]_{-1,2} \cap [x,z]_{3,-1} = [x]_{-1}$, and

$[x,y]_{-1,2} \uplus [x,z]_{3,-1} = [x,y,z]_{2,2,-1}$.

In this example we notice that the intersection is an msubset of both the union and the additive union, but the additive union (unlike in MST) is a proper msubset of the union (the sum of multiplicities is less than the maximum of multiplicities).

Axioms I through VI are stated exactly as in MST. However, for the *union axiom* we must make a change. In MST, the elements of $\cup x$ are the elements of elements of x. The multiplicity of an element $z \in \cup x$ is the maximum of the multiplicities with which z belongs to elements of x, if the maximum exists, and the minimum such multiplicity, otherwise. In MSTZ, however, neither the maximum nor the minimum such multiplicity need exist (consider the mset $x = \{[z]_1, [z]_{-1}, [z]_2, [z]_{-2}, \dots\}$).

The *union* axiom of MSTZ is the formal L-sentence that states:

VII "For all msets x, there exists an mset z' such that, for all msets z and all integers n, z belongs to z' with multiplicity n iff z is an element of elements of x *and* if the multiplicities of z as an element of elements of x have a maximum m, then $n = m$; otherwise, if the multiplicities of z as an element of elements of x have a minimum k, then $n = k$; otherwise, $n = 1$."

Since this sentence is clearly expressible in L (see, for example, Axiom VII of MST in [1] and [2]), we do not explicitly write it out. We let $\bigcup x$ denote the unique mset z' in the above Axiom VII. If we define *binary union* by $u \cup v = \bigcup\{u, v\}$ if $u \neq v$ and $\bigcup\{u\}$ otherwise, then the formal binary union operation follows the informal rules discussed earlier.

The elements of the *additive union* $\biguplus x$ of x are exactly the elements of $\bigcup x$; that is, all elements of elements of x. To determine the multiplicity of an element z in $\biguplus x$ we must know whether z belongs to a *finite* number of elements of x. If so, then its multiplicity in $\biguplus x$ is a *finite* sum of products. If not, then its multiplicity is its multiplicity in $\bigcup x$.

The *additive union* axiom of MSTZ is the formal L-sentence that states:

VIII "For all msets x, there exists an mset z' such that, for all msets z and all integers n, z belongs to z' with multiplicity n iff z is an element of elements of x *and* if z belongs to a *finite* number of elements y of x such that $z \in^k y \in^l x$ holds (for each such y), then n equals the *finite* sum (over all such y) of the products $k \cdot l$; otherwise, n equals the multiplicity of z in $\bigcup x$."

The formal statement of this axiom is long and complex. It is exactly the same as the formal statement of Axiom VIII in MST ([1] and [2]) except for a small technical change. In MST, $\exists m \phi$ means $\exists m(m > 0 \wedge \phi)$, but in MSTZ, $\exists m \phi$ means $\exists m((m < 0 \vee m > 0) \wedge \phi)$. Because of this, $\exists m(\dots$ must be changed to $\exists m(m > 0 \dots$ and $\sim\exists m \exists f \dots$ must be changed to $\sim\exists m(m > 0 \wedge \exists f \dots$. Except for these minor changes, the formal statement of VIII in MSTZ is exactly the same as the formal statement of VIII in MST.

If we let $\biguplus x$ denote the unique mset z' in the above Axiom VIII, we can define *binary additive union* as follows: $u \uplus v = \biguplus\{u, v\}$ if $u \neq v$ and $\biguplus[u]_2$ otherwise. The formal binary operation $\uplus$ follows the informal rules discussed earlier.

The formal statement of the *separation schema* of MSTZ is exactly the same as in MST: for every wff $\phi(x, n)$ of L with free variables including x and n but excluding y and n', the universal closure of

$$\mathbf{IX}_\phi \quad \forall x \forall n \forall n' ((\phi(x,n) \wedge \phi(x,n')) \rightarrow n = n') \rightarrow \forall z \exists y \forall x \forall n (x \in^n y \leftrightarrow [x]_n \subseteq z \wedge \phi(x,n))$$

is an axiom of MSTZ.

The mset y is well-defined because we require that $\phi(x, n)$ be 'functional'. In all such axioms IX_ϕ we have $y \subseteq z$ since $\forall x \forall n (x \in^n y \rightarrow [x]_n \subseteq z)$.

For any mset z', let $\phi(x, n)$ be $x = z' \wedge x \in^n z$. $\phi(x, n)$ is 'functional' by Axiom I. We denote the mset $y \subseteq z$ in the consequent of Axiom IX_ϕ by $z_{z'}$. Therefore, $z_{z'} \subseteq z$ and $z_{z'} = [z']_n$ iff $z' \in^n z$ and $z_{z'} = \varnothing$ iff $z' \notin z$. The msubset $z_{z'}$ contains every copy of z' in z (if any) and nothing else. We define the numeric term $|z_{z'}|$ as follows: $|z_{z'}| = t$ iff $z' \in^t z$.

We need a general procedure whereby any mset containing elements with negative multiplicities is 'replaced' by an mset containing elements with positive multiplicities. The separation and replacement schemata of MST cannot *increase* multiplicities from negative to positive. We have adopted the separation schema of MST without change, but we modify the replacement schema to suit our needs. In any case it would have been necessary to modify the replacement schema, since a least multiplicity for elements of the 'domain' mset need not exist in MSTZ.

The *replacement schema* of MSTZ is: For every wff $\phi(x,y)$ of L with free variables including x and y but excluding y' and z', the universal closure of

$$\mathbf{X}_\phi \qquad \begin{aligned}&\forall x\forall y\forall y'((\phi(x,y) \wedge \phi(x,y')) \rightarrow y = y') \rightarrow\\ &\forall z\exists z'\forall y\forall n(y \in^n z' \leftrightarrow [\exists x(x \in z \wedge \phi(x,y)) \wedge\\ &[\forall x\forall m((x \in^n z \wedge \phi(x,y)) \rightarrow m > 0) \rightarrow\\ &(\exists x(x \in^n z \wedge \phi(x,y)) \wedge \forall x\forall m((x \in^m z \wedge \phi(x,y)) \rightarrow n \leq m))] \wedge\\ &(\exists x\exists m(x \in^m z \wedge \phi(x,y) \wedge m < 0) \rightarrow n = 1)])\end{aligned}$$

is an axiom of MSTZ.

In words, if the multiplicities of all elements x in z, such that $\phi(x,y)$, are positive, then the multiplicity of y in z' is the least multiplicity of all such elements x in z. If, on the other hand, there is some element x in z, such that $\phi(x,y)$, having a negative multiplicity in z, then the multiplicity of y in z' equals 1.

To define the mset $\bigcap x$ we use the following principle: If the multiplicities of z as an element of *every* element of x have a minimum, then the multiplicity n of z in $\bigcap x$ equals that minimum; otherwise, $n = 1$. Since a maximum multiplicity may exist that is less than 1, and a minimum multiplicity may not exist, in general $\bigcap x \nsubseteq \bigcup x$. For example, if $x = \{[z]_{-2}, [z]_{-3}, [z]_{-4}, \ldots\}$, then $\bigcap x = \{z\} \nsubseteq \bigcup x = [z]_{-2}$. The same example shows that, in general, $\bigcap x \nsubseteq \biguplus x$. If $x = \{\{z\}, [z]_{-1}, [z]_2, [z]_{-2}, \ldots\}$, then $\bigcup x = \biguplus x = \bigcap x = \{z\}$.

Since in general $\bigcap x \nsubseteq \bigcup x$, we cannot hope to define the mset $\bigcap x$ by separation on the mset $\bigcup x$ as we did in MST. Let x be an arbitrary mset. The mset $\bigcup x$ exists by Axiom VII. Elements of every element of x will be elements of $\bigcup x$, but there may not be enough copies of such elements in $\bigcup x$. We replace each element z in $\bigcup x$ by itself to obtain the mset y' that contains exactly the elements of $\bigcup x$ but with all multiplicities at least 1. We then invoke separation on the mset y' to obtain the msubset $\bigcap x \subseteq y'$. We use the wff $\phi(z,n)$ of L that states

> "z is an element of every element of x *and* if the multiplicities of z as an element of every element of x have a minimum k, then $n = k$; otherwise, $n = 1$."

If we define *binary intersection* by: $u \cap v$ equals $\bigcap\{u,v\}$ when $u \neq v$ and equals $\bigcap\{u\}$ otherwise, then the formal operation $\cap$ satisfies the informal requirements discussed earlier.

In MSTZ, we are able to preserve the identities $x \cup \varnothing = x \uplus \varnothing = x$ and $x \cap \varnothing = \varnothing$, and the implication $x \cap y = \varnothing \rightarrow x \uplus y = x \cup y$. Also, the distributive laws for multisets (as given by Knuth [12], p. 636) still hold in MSTZ. Specifically, binary $\cup$ and $\cap$ distribute over each other, $\uplus$ distributes over both $\cup$

and $\cap$, but neither $\cup$ nor $\cap$ distribute over $\uplus$. We say, therefore, that $\uplus$ is "distributively stronger" than both $\cup$ and $\cap$. For example,

$$[x]_{-1} \cup ([x]_{-2} \uplus [x]_{-3}) = [x]_{-1} \cup [x]_{-5} = [x]_{-1}, \text{ but}$$

$$([x]_{-1} \cup [x]_{-2}) \uplus ([x]_{-1} \cup [x]_{-3}) = [x]_{-1} \uplus [x]_{-1} = [x]_{-2};$$

$$[x]_1 \cap ([x]_{-8} \uplus [x]_2) = [x]_1 \cap [x]_{-6} = [x]_{-6}, \text{ but}$$

$$([x]_1 \cap [x]_{-8}) \uplus ([x]_1 \cap [x]_2) = [x]_{-8} \uplus [x]_1 = [x]_{-7}.$$

In MSTZ, we also preserve the relationships

$$x \cap y \subseteq x,\ y \subseteq x \cup y$$

since $\min(n,m) \leq n$, $m \leq \max(n,m)$ for any integers. We retain, as well, the equivalences

$$x \subseteq y \leftrightarrow x \cap y = x \leftrightarrow x \cup y = y.$$

A quick proof of the first equivalence: If $x \subseteq y$ and $z \in^n x$, then $\exists m \geq n$ and $z \in^m y$. Since $\min(n,m) = n$, $z \in^n x \cap y$. Therefore, $x \subseteq x \cap y$ and $x \cap y \subseteq x$ by the relationship above. Conversely, if $x \cap y = x$ and $z \in^n x$ then $z \in^n x \cap y$. Therefore, $\exists m \quad z \in^m y$ and $n = \min(m,n)$. So $\exists m \geq n$ and $z \in^m y$. Hence $x \subseteq y$.

The formal statements of the infinity and choice axioms of MSTZ are exactly the same as in MST. The *infinite mset* axiom of MSTZ is

XI $\quad \exists y(\varnothing \in y \wedge \forall x(x \in y \rightarrow x \cup \{x\} \in y)).$

As in MST, we obtain the *set* $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \ldots\}$ of (nonnegative) von Neumann numerals. We shall discuss their negative counterparts shortly.

The *choice mset* axiom of MSTZ is

XII
$$\begin{aligned}
&\forall y[[y \neq \varnothing \wedge \forall x(x \in y \rightarrow x \neq \varnothing) \wedge \\
&\forall x \forall z((x \in y \wedge z \in y \wedge x \neq z) \rightarrow x \cap z = \varnothing)] \rightarrow \\
&\exists y'(\forall x \forall n(x \in^n y \rightarrow \exists x'(x' \in^n y' \wedge x' \in x \wedge \\
&\forall x''((x'' \in x \wedge x'' \in y') \rightarrow x'' = x'))) \wedge \\
&\forall x' \forall n(x' \in^n y' \rightarrow \exists x(x \in^n y \wedge x' \in x)))].
\end{aligned}$$

The mset y' in Axiom XII is called a *choice mset* for y. The elements x' in y' occur with the same multiplicity, positive or negative, as do their corresponding elements x in the mset y. The multiplicity of the 'chosen' element x' in x is not taken into account. A choice mset y' for y is not unique unless every element x of y is a *simple* mset of the form $[x']_m$.

Exactly as in MST, we define the *transitive closure* $TC(y)$ of an mset y to be the smallest *set* containing the elements of y, the elements of elements of $y, \ldots$ etc. Informally, $TC(y) = y \cup (\bigcup y) \cup (\bigcup\bigcup y) \cup \ldots$. An mset y of MSTZ is called a *hereditary set* if $HSet(y)$, where $HSet(y)$ stands for $Set(y) \wedge \forall x(x \in TC(y) \rightarrow Set(x))$. The hereditary sets, or *hsets*, of MSTZ are the exact analogs of classical sets in ZFC.

For every mset y of MSTZ we prove the existence of a unique *set* of MSTZ that contains exactly the elements of y, but *normalizes* all multiplicities (positive or negative) of elements in y to 1. We call such a set the 'root set' of y.

Theorem (Existence of Root Sets) $\quad \forall y \exists z \forall x(x \in^1 z \leftrightarrow x \in y).$

Proof: Let y be an arbitrary mset. Let $\phi(x,y)$ be the wff $x = y$ of L. Applying replacement (Axiom X_ϕ) to the mset y gives an mset y' that is *similar* to y ($\forall x(x \in y \leftrightarrow x \in y')$ holds) and y' has all multiplicities at least 1 ($\forall x \forall n(x \in^n y' \rightarrow n \geq 1)$ holds). In fact, y is an msubset of y'. Let $\psi(x,n)$ be the wff $x = x \wedge n = 1$ of L. By separation (Axiom IX_ψ) on the mset y' we obtain the *subset* $z \subseteq y'$ such that $\forall x(x \in^1 z \leftrightarrow x \in y')$ holds.

We denote the unique mset z in the above theorem by y^*, and we call it the *root set* of y since $Set(y^*)$. Therefore, $\forall y \forall x(x \in^1 y^* \leftrightarrow x \in y)$. Since multiplicities of elements in y may be any integer, we could have $y = y^*$ or $Set(y)$, $y \subseteq y^*$, $y^* \subseteq y$, $y \nsubseteq y^*$ or $y^* \nsubseteq y$. As in MST, $\varnothing^* = \varnothing$ and $\forall x \forall y(x \subseteq y \rightarrow x^* \subseteq y^*)$.

From the structure (the existence of additive inverses) of the *numeric universe* $Z = \{\dot{n} | \dot{n} = \dot{n}\}$ of MSTZ, it is clear that there should correspond to every mset y a unique 'shadow' mset containing exactly the elements of y with multiplicities that are the additive inverses of their multiplicities in y.

Theorem (Existence of Shadow Msets) $\quad \forall y \exists z \forall x \forall n(x \in^n z \leftrightarrow x \in^{-n} y)$.

Proof: Let y be an arbitrary mset. The root set y^* exists by the previous theorem. For each $x \in y^*$, the simple msubset $y_x \subseteq y$ exists by separation. We replace each element x in the *set* y^* by the corresponding mset y_x to obtain the *set* $y' = \{y_x | x \in y^*\}$. For each $x \in y^*$, $y_x = [x]_n$ iff $x \in^n y$. Therefore, $y' = \{[x]_n | x \in y^* \wedge x \in^n y\}$. For each mset $[x]_n$ in y' the mset $[x]_{-n}$ exists by the existence of additive inverses in Z and Axiom IV(i). We replace each element $[x]_n$ in the *set* y' by the mset $[x]_{-n}$ to obtain the *set* $y'' = \{[x]_{-n} | x \in y^* \wedge x \in^n y\}$. Let z be the mset $\cup y''$ which exists by Axiom VII. Clearly, $\forall x \forall n(x \in^n z \leftrightarrow x \in^{-n} y)$ holds as required.

We denote the unique mset z in the above theorem by y^-, and call it the *shadow* mset of y. Therefore, $\forall y \forall x \forall n(x \in^n y \leftrightarrow x \in^{-n} y^-)$ and $\forall y(y \uplus y^- = \varnothing)$. We say that the msets y and y^- *annihilate* each other. Every mset y of MSTZ is the shadow of some mset (namely, y^-); that is, $\forall y((y^-)^- = y)$. Clearly, $\varnothing^- = \varnothing$, $\forall x(x = x^- \leftrightarrow x = \varnothing)$, and $\forall x \forall n(([x]_n)^- = [x]_{-n})$. Every mset of MSTZ is 'obtainable' from any other mset of MSTZ by a single application of additive union: $\forall x \forall y \exists z(x = y \uplus z)$. (*Proof:* Given msets x and y, let $z = y^- \uplus x$). Since the msets y and y^- are generally *distinct* (except when $y = \varnothing$), they can *coexist* as elements of the same mset, as in $\{y, y^-\}$. Although $\{y, y^-\} \neq \varnothing$, we have $\uplus\{y, y^-\} = \varnothing$. Since $\{x\} = [x]_1$, $\{x\}^- = [x]_{-1}$. Therefore, $\{x^-\} \neq \{x\}^-$ since $\{x^-\}$ contains the element x^- with multiplicity 1 but $\{x\}^-$ contains the element x with multiplicity -1.

Consider the mset $x = \{\{z\}, \{z\}^-\}$. By Axiom VII, $\cup x = \{z\}$. By Axiom VIII, however, $\uplus x = \varnothing$. Since a finite sum of products of multiplicities of z as an element of elements of x may equal zero (even though such multiplicities may have a nonzero maximum), it is possible for $z \in \cup x \wedge z \notin \uplus x$ to hold. As we have seen earlier, a sum of multiplicities may be less than a maximum of multiplicities. In MSTZ, therefore, in general $\cup x \nsubseteq \uplus x$.

The msets y and y^-, though distinct (when $y \neq \varnothing$), are similar since $\forall x(x \in y \leftrightarrow x \in y^-)$. They must, therefore, have the same root set ($y^* = (y^-)^*$). The msets $(y^-)^*$ and $(y^*)^-$ are generally distinct since all multiplicities in $(y^-)^*$ are

1 and all multiplicities in $(y^*)^-$ are -1. In fact, one can show that $\forall y((y^-)^* = (y^*)^- \leftrightarrow y = \varnothing)$.

If the mset y contains elements with only *positive* multiplicities (if $\forall x \forall n(x \in^n y \to n > 0)$ holds), our intuition is clear about the msets x that are elements of y and the numbers n that are their multiplicities in y. Since y and y^- contain exactly the same msets as elements, there is no mystery surrounding the nature of the elements in the shadow mset y^-. Multiplicity measures the extent to which an element occurs in a multiset. The multiplicity $-n$ of element x in y^- measures the extent to which (or lack thereof) x occurs in y^-, which depends entirely upon the extent n to which the same element x belongs to y (the greater the latter, the lesser the former).

Negative multiplicity loses much of its mystery when viewed as the natural result of *unrestricted complementation* in MSTZ. For *any* msets x and y, let $x - y$ stand for $x \uplus y^-$. Therefore, $x - x = \varnothing$, $x - \varnothing = x$, and $\varnothing - x = x^-$. We can think of the mset x^- as the mset $\varnothing - x$. In particular, $\{x\}^- = \varnothing - \{x\}$. The mset $\{x\}^- = [x]_{-1}$ is the result of removing a single copy of element x from $\varnothing$. From the definition of $x - y$, we have $x - x^- = x \uplus x$ and $x^- - x = x^- \uplus x^-$. Since shadow msets are unique, and since $(x - y) \uplus (y - x) = (x \uplus y^-) \uplus (y \uplus x^-) = \varnothing$, we have that $(x - y)^- = y - x$.

Since $(x \uplus y) \uplus (x^- \uplus y^-) = \varnothing$ and since shadow msets are unique, we must have $(x \uplus y)^- = x^- \uplus y^-$. However, in general, $(x \cup y)^- \neq x^- \cup y^-$ and $(x \cap y)^- \neq x^- \cap y^-$. For example, let $x = [z_1, z_2]_{-1,2}$ and $y = [z_2]_{-3}$. Then $x^- = [z_1, z_2]_{1,-2}$ and $y^- = [z_2]_3$, and $x \cup y = [z_1, z_2]_{-1,2}$ and $(x \cup y)^- = [z_1, z_2]_{1,-2}$, but $x^- \cup y^- = [z_1, z_2]_{1,3}$. Similarly, $x \cap y = [z_2]_{-3}$ and $(x \cap y)^- = [z_2]_3$ but $x^- \cap y^- = [z_2]_{-2}$.

We proved the existence of shadow msets having first proved the existence of root sets. To see that the reverse procedure is also possible, assume that the existence of shadow msets has been proved, and let x be an arbitrary mset. By separation, the msubsets $\bar{x}$ $(z \in^n \bar{x} \leftrightarrow z \in^n x \wedge n > 0)$ and $\underline{x}$ $(z \in^n \underline{x} \leftrightarrow z \in^n x \wedge n < 0)$ exist. If we apply separation to the mset $\bar{x} \cup (\underline{x})^-$, we obtain its root set. The root set of $\bar{x} \cup (\underline{x})^-$ is a subset of $\bar{x} \cup (\underline{x})^-$ since all multiplicities in $\bar{x} \cup (\underline{x})^-$ are at least 1. By Axiom II, the root set of $\bar{x} \cup (\underline{x})^-$ equals the required root set of x.

Exactly as in MST, we can define a *translation* $'$ from every wff ϕ of ZFC to a wff ϕ' of MSTZ as follows: $(x = y)'$ is $HSet(x) \wedge HSet(y) \wedge x = y$; $(x \in y)'$ is $HSet(x) \wedge HSet(y) \wedge x \in^1 y$; $(\sim\psi)'$ is $\sim\psi'$; $(\psi \vee \theta)'$ is $\psi' \vee \theta'$; $(\psi \wedge \theta)'$ is $\psi' \wedge \theta'$; $(\psi \to \theta)'$ is $\psi' \to \theta'$; $(\exists x\psi)'$ is $\exists x(HSet(x) \wedge \psi')$; and $(\forall x\psi)'$ is $\forall x(HSet(x) \to \psi')$. One can show that ([1] and [2]), for every wff ϕ of ZFC

$$\text{ZFC} \vdash \phi \text{ iff MSTZ} \vdash \phi'.$$

Therefore, the theory MSTZ contains a copy $\text{ZFC}' = \{\phi' \mid \phi \in \text{ZFC}\}$ of classical ZFC set theory.

With a copy of full set theory at hand, we have copies of the ordinals, the cardinals... etc. in MSTZ. The *mset universe* M of MSTZ is $\{x \mid x = x\}$ and the *hset universe* V′ of MSTZ is $\{x \mid HSet(x)\}$. Exactly as in ZFC, one can prove that $V' = \bigcup_{\alpha \in 0n} V'_\alpha$ where $V'_0 = \varnothing$, $V'_{\alpha+1} = \tilde{\mathbb{P}}(V'_\alpha)$, $V'_\lambda = \bigcup_{\alpha<\lambda} V'_\alpha$ if λ is a limit ordinal, and $\forall x \tilde{\mathbb{P}}(x) = \{x \in \mathbb{P}(x) \mid Set(x)\}$. The M_α hierarchy of MSTZ is defined as follows: $M_0 = \varnothing$; $M_{\alpha+1} = \{x \mid x^* \in V'_{\alpha+1} \cup \tilde{\mathbb{P}}(M_\alpha)\}$; and $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ whenever λ is a limit ordinal. Exactly as in MST using Axiom VI, one can prove that the mset universe $M = \bigcup_{\alpha \in On} M_\alpha$.

The first few levels of the M_α hierarchy of MSTZ are:

$M_0 = \varnothing$

$M_1 = \{\varnothing\}$

$M_2 = \{\varnothing, \{\varnothing\}, [\varnothing]_{-1}, [\varnothing]_2, [\varnothing]_{-2}, \dots\}$

$M_3 = \{\varnothing, \dots, [\varnothing]_n, \dots, \dots [\{\varnothing\}]_n, \dots, \dots [\varnothing, \{\varnothing\}]_{n,m} \dots, \dots [[\varnothing]_n]_m \dots, \dots \text{etc.}\}$

where n and m range over $\mathbb{Z}$. Included in M_3 are all msets of MSTZ whose root *sets* are *subsets* of M_2. In other words, $\{x \mid x^* \subseteq M_2\} \subseteq M_3$.

For any mset x in M, we define the *rank* of x in M, denoted by $r(x)$, to be the *least* ordinal α in On such that $x \in M_\alpha$.

We want to define the notion of the 'hereditary shadow' $y^=$ of an arbitrary mset y in M. The mset $y^=$ should have a root set $\{x^= \mid x \in y\}$ and be such that $\forall x \forall n(x^= \in^{-n} y^= \leftrightarrow x \in^n y)$. Therefore, $\varnothing^= = \varnothing$ and $([\varnothing]_n)^= = ([\varnothing^=]_n)^- = [\varnothing]_{-n}$. The mset $y^=$ is the result of taking the shadow of y, the shadow of elements of y, the shadow of elements of elements of $y, \dots$ through all msets in $TC(y)$.

Theorem (Existence of Hereditary Shadow Msets) $\forall y \exists z \forall x \forall n(x^= \in^{-n} z \leftrightarrow x \in^n y)$.

Proof: Let y be an arbitrary mset in M. We prove the existence of $y^=$ by induction on the rank of y in M. If $y = \varnothing$, then $\varnothing^= = \varnothing$ exists by Axiom III. If $y = [\varnothing]_n$ for some n in $\mathbb{Z}$, then $([\varnothing]_n)^= = [\varnothing]_{-n}$ exists by Axioms III and IV(i). If $r(y) > 2$, we assume that the mset $x^=$ exists for all msets x in M such that $r(x) < r(y)$. We now show that the mset $y^=$ exists. The mset y^- exists by the previous theorem. For each $x \in y^-$, $r(x) < r(y)$ since $\forall x(x \in y^- \leftrightarrow x \in y)$ holds. Therefore, for each $x \in y^-$, the mset $x^=$ exists by the induction hypothesis. We replace each x in y^- by the mset $x^=$ to obtain the mset y'. All multiplicities in y' are at least 1. Let $z \subseteq y'$ be the result of separation on the mset y', where $\Phi(x, n)$ is the wff of L that states "the multiplicity of $x^=$ in z equals the multiplicity of x in y^-". Then $\forall x \forall n(x^= \in^{-n} z \leftrightarrow x \in^n y)$ holds. Therefore, the mset $y^=$ exists for all msets y in M.

Remark Although perspicuous, the formal statement of the above theorem is not strictly correct since it contains the defined symbol $x^=$. What is needed is an L-sentence such as $\forall \alpha \exists f \forall y(r(y) < \alpha \to \exists z(\langle y, z\rangle \in f \wedge \forall x(x \in y \to \exists x'(\langle x, x'\rangle \in f \wedge \forall n(x' \in^{-n} z \leftrightarrow x \in^n y)))))$, where $\forall \alpha \dots$ is for all msets α, if "α is an ordinal" then ... and $\exists f \dots$ is there is an mset f, "f is an injective function" and. . . . So, for each $\alpha \in$ On and each mset y with $r(y) < \alpha$, $f_\alpha(y) = y^=$. We denote the unique mset z in the above theorem by $y^=$ and call it the *hereditary shadow* of the mset y. For shadows, we have $\forall y(y^-)^- = y$. We prove a similar result for hereditary shadows.

Theorem $\forall y(y^{=})^{=} = y$.

Proof: By induction on the rank of y in M. If $y = \varnothing$, $(y^{=})^{=} = (\varnothing^{=})^{=} = \varnothing^{=} = \varnothing = y$. If $y = [\varnothing]_n$ for some n in $\mathbb{Z}$, then $(y^{=})^{=} = (([\varnothing]_n)^{=})^{=} = ([\varnothing]_{-n})^{=} = [\varnothing]_n = y$. If $r(y) > 2$, we assume that $(x^{=})^{=} = x$ for all msets x such that $\mathrm{r}(x) < \mathrm{r}(y)$. Now $\forall x \forall n((x^{=} \in^{-n} y^{=} \leftrightarrow x \in^n y) \wedge ((x^{=})^{=} \in^n (y^{=})^{=} \leftrightarrow x^{=} \in^{-n} y^{=}))$ holds by the definitions of $y^{=}$ and $(y^{=})^{=}$. Therefore, $\forall x \forall n((x^{=})^{=} \in^n (y^{=})^{=} \leftrightarrow x \in^n y)$ holds. For each element $(x^{=})^{=}$ in $(y^{=})^{=}$ we have $\mathrm{r}(x) < \mathrm{r}(y)$ since $x \in y$. By the induction hypothesis, each such $(x^{=})^{=} = x$. Therefore, $\forall x \forall n(x \in^n (y^{=})^{=} \leftrightarrow x \in^n y)$ holds, and $(y^{=})^{=} = y$ by Axiom II (extensionality).

We note, in particular, that for hereditary sets y, $y^{=} = \{x^{=} | x \in y\}^{-}$ and $(y^{=})^{=} = (\{x^{=} | x \in y\}^{-})^{=} = \{x | x \in y\} = y$. For hereditary sets $y \neq \varnothing$, $y^{=}$ is not a hereditary set in the extreme—there are no nonempty *sets* in $TC(\{y^{=}\})$.

An example of the hereditary shadow of an mset that is not a hereditary set is $([\varnothing, [\{\varnothing\}]_n]_{m,k})^{=} = [\varnothing, [[\varnothing]_{-1}]_{-n}]_{-m,-k}$. It is easy to prove in MSTZ that $\forall y(y^{=} = y^{-} \leftrightarrow (y = \varnothing \vee \exists n\, y = [\varnothing]_n))$ holds (since $\forall x(x = x^{=} \leftrightarrow x = \varnothing)$ holds).

The last nonlogical axiom of MSTZ determines the syntax for the unary function symbol $\hat{}$ of L. The *numeric-mset correspondence* axiom of MSTZ is

N8 $\hat{0} = \varnothing \wedge \hat{1} = \{\varnothing\} \wedge \forall n((n > 0 \rightarrow$
$\widehat{n+1} = \hat{n} \cup \{\hat{n}\}) \wedge (n < 0 \rightarrow \hat{n} = (\widehat{-n})^{=}))$.

Axiom N8 defines the correspondence $\hat{}$ from integers in $\mathbb{Z}$ to certain msets in M. For example:

$$\hat{2} = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$$
$$\hat{3} = \{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$$
$$(\widehat{-1}) = (\hat{1})^{=} = \{\varnothing\}^{=} = \{\varnothing^{=}\}^{-} = \{\varnothing\}^{-}$$
$$(\widehat{-2}) = (\hat{2})^{=} = \{\varnothing, \{\varnothing\}\}^{=} = \{\varnothing, \{\varnothing\}^{-}\}^{-}$$
$$(\widehat{-3}) = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}^{=} = \{\varnothing, \{\varnothing\}^{-}, \{\varnothing, \{\varnothing\}^{-}\}^{-}\}^{-}.$$

In other words, for all integers $n < 0$, $\hat{n} = \{\hat{m} | n < m \leq 0\}^{-}$.

With this definition of the correspondence $\hat{}: \mathbb{Z} \rightarrow \mathrm{M}$, we have the following relationships:

$$\hat{0} \subseteq \hat{1} \subseteq \hat{2} \subseteq \ldots$$
$$\hat{0} \subseteq \widehat{-1} \subseteq \widehat{-2} \subseteq \ldots \quad \text{and}$$
$$\hat{0} \in \hat{1} \in \hat{2} \in \ldots$$
$$\hat{0} \in \widehat{-1} \in \widehat{-2} \in \ldots$$

where $\in$ in the third row is $\in^1$ and $\in$ in the fourth row is $\in^{-1}$. Thus the chain of $\subseteq$ relationships and the chain of $\in$ relationships to the right of $\hat{0}$ reverse themselves to the left of $\hat{0}$. Therefore, to the usual set-theoretic representation of the nonnegative integers 0, 1, 2,... by $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$,... corresponds an mset-theoretic representation of the negative integers -1, -2,... by $\{\varnothing\}^{-}$, $\{\varnothing, \{\varnothing\}^{-}\}^{-}$,... that preserves the nice relationships of the classical case. Traditionally, the negative integers are represented in ZFC by equivalence classes of ordered pairs of natural numbers. (See for example Enderton [5], p. 91, where the equivalence class $\{\langle n,n+1\rangle | n \in \mathbb{N}\}$ represents -1. The equivalence class to which the ordered pair $\langle n,m\rangle$ belongs is taken to represent the (ordered) difference $n - m$.)

We could have defined $\hat{n}$, for every $n < 0$, to be simply the *shadow* of $(-n)$; that is, $\forall n(n < 0 \to \hat{n} = (-n)^-$. With this definition, $(\widehat{-2}) = (\hat{2})^- = \{\varnothing, \{\varnothing\}\}^- = [\varnothing, \{\varnothing\}]_{-1,-1}$. However, we would not obtain the very nice relationships described above (here, $\widehat{-1} \notin \widehat{-2}$). With this simpler definition, we do have $\hat{n} \uplus (\widehat{-n}) = \hat{n} \uplus (\hat{n})^- = \varnothing$ corresponding to $n + (-n) = 0$. This does not hold for the hereditary shadow definition of $\hat{}$ above. However, because of the very nice $\subseteq$ and $\in$ relationships, we will stick to our hereditary shadow definition, that is, $\forall n(n < 0 \to \hat{n} = (\widehat{-n})^=)$. Thus, to every integer n in $\mathbb{Z}$ corresponds a unique mset $\hat{n}$ in M.

The mset $[\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \dots]_{-1,-2,-3,\dots}$ satisfies the conditions in the *infinite mset* Axiom XI of MSTZ. The collection of (nonnegative) von Neumann numerals is that unique set $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \dots\}$ that is a subset of the root set y^* of every mset y of MSTZ that satisfies Axiom XI. By separation (the wff $\Phi(x,n)$ is $x \neq \varnothing \wedge n = 1$) we obtain the subset $y' = \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \dots\}$. We replace each element x in y' by the mset $x^=$ to obtain the *set* $y'' = \{\{\varnothing\}^=, \{\varnothing, \{\varnothing\}\}^=, \dots\}$. The *set* $y' \cup y''$ is $\{\hat{n} | n \in \mathbb{Z} \wedge n \neq 0\}$. Let z be an arbitrary mset. We replace each element $\hat{n}$ in the *set* $y' \cup y''$ by the mset $[z]_n$ (which exists by Axiom IV(i)) to obtain the set $\{\{z\}, [z]_{-1}, [z]_2, [z]_{-2}, \dots\}$. This is the set that provided many useful examples in our discussion of the msets $\cup x$, $\uplus x$, and $\cap x$. We can also show that $\{\hat{n} | n \in \mathbb{Z}\}$ is a *set* in M. Since $\{\hat{n} | n \in \mathbb{Z} \wedge n \neq 0\}$ is a set ($y' \cup y''$ above), $\{\hat{n} | n \in \mathbb{Z}\} = \{\hat{n} | n \in \mathbb{Z} \wedge n \neq 0\} \cup \{\varnothing\}$ is a set of MSTZ.

The definitions of function, injection, surjection, and bijection between msets are exactly as in MST ([1] and [2]). The function $f\colon x^* \to y^*$ (defined in the classical sense) is automatically a *function* from x to y, denoted by $f\colon x \to y$. The function $f\colon x \to y$ is an *injection* if $f\colon x^* \to y^*$ is an injection (defined classically) and $\forall z(z \in x^* \to |x_z| \leq |y_{f(z)}|)$, recalling that $|x_z| = n$ iff $z \in^n x$ and $|y_{f(z)}| = m$ iff $f(z) \in^m y$. The function $f\colon x \to y$ is a *surjection* if $f\colon x^* \to y^*$ is a surjection (defined classically) and $\forall z(z \in x^* \to |x_z| \geq |y_{f(z)}|)$. The function $f\colon x \to y$ is a *bijection* if it is an injection and a surjection; that is, if $f\colon x^* \to y^*$ is a bijection (defined classically) and $\forall z(z \in x^* \to |x_z| = |y_{f(z)}|)$. The numeric terms $|x_z|$ and $|y_{f(z)}|$ may be positive or negative in MSTZ. For example, the function f: $\{x,y\} \to \{z_1,z_2\}$ given by $f = \{\langle x,z_1\rangle,\langle y,z_2\rangle\}$ is an injection from $[x,y]_{-2,1}$ *to* $[z_1,z_2]_{-1,5}$, *a surjection from* $[x,y]_{-2,1}$ *to* $[z_1,z_2]_{-3,-2}$, *and neither from* $[x,y]_{-1,1}$ to $[z_1,z_2]_{-2,2}$. If $x \subseteq y$, there is the natural injection $f\colon x \to y$ that embeds x into y. For example, if $x^- \subseteq x$ and $x \neq \varnothing$, then the identity map on $x^* = (x^-)^*$ is the natural embedding of x^- into x.

A function f of MSTZ is a *set* $(f = f^*)$ of ordered-pair sets. What is the shadow of f? The mset f^- is not a function of MSTZ when $f \neq \varnothing$ since $\sim Set(f^-)$. The elements of f^- are exactly the ordered-pair sets in f but their multiplicity in f^- is -1. As with all msets, $f \uplus f^- = \varnothing$ and $f^* = (f^-)^*$. However, since $Set(f)$, we also have for functions that $f = f^* = (f^-)^*$.

Before we define the cardinality of msets in MSTZ, we must distinguish between various types of msets. An mset y is called M^+ iff $\forall x \forall n(x \in^n y \to n > 0)$

holds. An mset y is called M^- iff $\forall x \forall n(x \in^n y \to n < 0)$ holds. An mset that is neither M^+ nor M^- is called *hybrid.* For example, $\varnothing$ is M^+ and M^- but not hybrid, $[x]_3$ is M^+, $[y]_{-7}$ is M^-, and $[x,y]_{-1,1}$ is hybrid. From the definitions, we get immediately this result:

Lemma x is $M^+ \leftrightarrow x^- \subseteq x$
x is $M^- \leftrightarrow x \subseteq x^-$
x is $M^+ \leftrightarrow x^-$ is M^-
x is hybrid $\leftrightarrow x^- \nsubseteq x \wedge x \nsubseteq x^-$.

If x is M^+, then the identity map on $x^* = (x^-)^*$ is an injection from x^- into x (the natural embedding $x^- \subseteq x$) and a surjection from x onto x^-. There is no bijection from x to x^- since multiplicities do not agree. If x is hybrid, then x^- is hybrid and the identity map on $x^* = (x^-)^*$ is a function from x to x^- and from x^- to x but neither function can be injective, surjective, or bijective.

For all msets x, $x \uplus x^- = \varnothing$. What are the msets $x \cup x^-$ and $x \cap x^-$? The above lemma, together with the equivalences $x \subseteq y \leftrightarrow x \cup y = y \leftrightarrow x \cap y = x$, give us this:

Lemma x is $M^+ \leftrightarrow x \cup x^- = x \leftrightarrow x \cap x^- = x^-$
x is $M^- \leftrightarrow x \cup x^- = x^- \leftrightarrow x \cap x^- = x$.

As mentioned earlier, by applying the separation schema to any mset x, we can define two msubsets $\bar{x}$ and $\underline{x}$ of x such that $\forall z \forall n(z \in^n \bar{x} \leftrightarrow (z \in^n x \wedge n > 0))$ and $\forall z \forall n(z \in^n \underline{x} \leftrightarrow (z \in^n x \wedge n < 0))$. The msubsets $\bar{x}$ and $\underline{x}$ are disjoint by Axiom I.

Lemma *$\bar{x}$ is the "largest" M^+ msubset of x; that is, $\bar{x} \subseteq x \wedge \bar{x}$ is $M^+ \wedge \forall y((y \subseteq x \wedge y$ is $M^+) \to y \subseteq \bar{x})$.*

Proof: If $z \in^n y \subseteq x \wedge y$ is M^+ then $\exists m \geq n > 0$ such that $z \in^m x$. Hence, $z \in^m \bar{x}$ and $y \subseteq \bar{x}$.

Lemma *Every mset x can be "separated" into an M^+ part $\bar{x}$ and an M^- part $\underline{x}$ such that $x = \bar{x} \uplus \underline{x} = \bar{x} \cup \underline{x}$.*

Proof: By Axiom II and the fact $\bar{x} \cap \underline{x} = \varnothing$. (Recall that $\forall x \forall y(x \cap y = \varnothing \to x \uplus y = x \cup y)$.)

Lemma *x is $M^+ \leftrightarrow x = \bar{x}$ (or $\underline{x} = \varnothing$)*
x is $M^- \leftrightarrow x = \underline{x}$ (or $\bar{x} = \varnothing$)
x is hybrid $\leftrightarrow \bar{x} \neq \varnothing \wedge \underline{x} \neq \varnothing$.

Proof: Obvious from the definitions of $\bar{x}$ and $\underline{x}$.

Lemma *For any mset x, $\bar{\bar{x}} = \bar{x} \wedge \underline{\underline{x}} = \underline{x} \wedge \underline{\bar{x}} = \varnothing$.*

Proof: Since $\bar{x}$ is M^+, $\bar{\bar{x}} = \bar{x}$; since $\underline{x}$ is M^-, $\underline{\underline{x}} = \underline{x}$; and since $\bar{x}$ is M^+, $\underline{\bar{x}} = \varnothing$ (all by the above lemma).

Lemma *For any mset x, $x - \bar{x} = \underline{x}$ and $x - \underline{x} = \bar{x}$.*

Proof: $x - \bar{x} = (\bar{x} \uplus \underline{x}) \uplus (\bar{x})^- = \underline{x}$; $x - \underline{x} = (\bar{x} \uplus \underline{x}) \uplus (\underline{x})^- = \bar{x}$.

In [1] and [2] we defined the cardinality of multisets in such a way that the multiplicity of each element in an mset contributes to the cardinality of the mset as a whole. It is consistent with this philosophy of cardinality, therefore, that in MSTZ the multiplicity of an element in an mset should augment or diminish the cardinality of the mset as a whole, depending upon whether it is positive or negative. The positive extent to which some elements occur in a mset x and the negative extent to which other elements occur in x should cancel each other out (either wholly or partly) when counted in the cardinality of x as a whole.

Since MSTZ contains a copy ZFC′ of ZFC, the classical (positive) cardinal numbers are particular hereditary sets in M. They are the hset copies κ' in $\mathrm{V}' = \{x \mid HSet(x)\}$ of cardinal numbers κ in V. A *negative cardinal number* of MSTZ is an mset in M of the form $(\kappa')^{=}$ (the hereditary shadow of the hset κ' that corresponds to the cardinal number κ). From this point on, we do not distinguish between cardinal numbers of ZFC and their hset associates of MSTZ. We use variable symbols α, β, λ, $\kappa, \ldots$ for msets that are cardinal numbers of MSTZ. The mset $\aleph_0^{=} = \{\hat{n} \mid n \geq 0\}^{=} = \{(\hat{n})^{=} \mid n \geq 0\}^{-} = \{(\widehat{-n}) \mid n \geq 0\}^{-}$ is the mset whose elements are exactly the nonpositive finite cardinal numbers of MSTZ, each of which belongs to $\aleph_0^{=}$ with multiplicity -1. We want to define the cardinality $\mathrm{C}(u)$ of an mset u in such a way that $\mathrm{C}(u^{-}) = \mathrm{C}(u)^{=}$.

In order to define the cardinality of msets in MST we required an operation H with the properties: $\forall x\, HSet(H(x))$, $\forall x \forall y (x \neq y \rightarrow H(x) \neq H(y))$, and $\forall x$ $x^* \approx H(x)$, where $x \approx y$ stands for "there is a bijection from x to y". Since we only apply the H operation to M^+ msets of MSTZ, the definition of H in MST can be used in MSTZ. Hence, for any M^+ mset v, we define

$$H(v) = \{\langle H(u), \hat{n} \rangle \mid u \in^n v \wedge n > 0\}.$$

For the proofs that H is well-defined and has the required properties, see [1] and [2]. Therefore, the operation H takes arbitrary distinct M^+ msets to distinct hsets such that $H(v)$ is 'equinumerous' to the root set v^*.

Given an arbitrary mset $u = \bar{u} \uplus \underline{u}$, we look at $\mathrm{C}(\bar{u})$ and $\mathrm{C}((\underline{u})^{-})$ in order to determine $\mathrm{C}(u)$. If the mset u is M^+, we define the *cardinality* $\mathrm{C}(u)$ of u exactly as we did in MST. ($\mathrm{C}(u^*)$ is the cardinality of $H(u^*)$ in ZFC′. If $\mathrm{C}(u^*)$ is infinite in ZFC′, then define $\mathrm{C}(u) = \mathrm{C}(u^*)$. If $\mathrm{C}(u^*)$ is finite in ZFC′ and $t = \sum_{v \in u^*} |u_v|$ then we define $\mathrm{C}(u) = \hat{t}$.) If, on the other hand, the mset u is *not* M^+, we proceed by cases:

(1) If $\bar{u}$ and $\underline{u}$ are both finite; that is, if $\mathrm{C}(\bar{u}) = \hat{s}$ and $\mathrm{C}((\underline{u})^{-}) = \hat{t}$, then we define $\mathrm{C}(u) = (\widehat{s - t})$.

(2) If $\bar{u}$ is infinite and $\underline{u}$ is finite; that is, if $\mathrm{C}(\bar{u}) = \kappa \geq \aleph_0$ and $\mathrm{C}((\underline{u})^{-}) = \hat{t}$, then we define $\mathrm{C}(u) = \kappa$.

(3) If $\bar{u}$ is finite but $\underline{u}$ is infinite; that is, if $\mathrm{C}(\bar{u}) = \hat{s}$ and $\mathrm{C}((\underline{u})^{-}) = \kappa \geq \aleph_0$, then we define $\mathrm{C}(u) = \kappa^{=}$.

(4) If $\bar{u}$ and $\underline{u}$ are both infinite; that is, if $\mathrm{C}(\bar{u}) = \kappa \geq \aleph_0$ and $\mathrm{C}((\underline{u})^{-}) = \lambda \geq \aleph_0$, then
   (i) if $\kappa > \lambda$, we define $\mathrm{C}(u) = \kappa$;
   (ii) if $\lambda > \kappa$, we define $\mathrm{C}(u) = \lambda^{=}$;
   (iii) if $\kappa = \lambda$, we define $\mathrm{C}(u) = \hat{0} = \varnothing$.

An mset u is *finite* if Case (1) holds; that is, if there are numeric terms s and t such that $C(\bar{u}) = \hat{s}$ and $C((\underline{u})^-) = \hat{t}$. In all other cases, the mset u is called *infinite*. In MSTZ, there are many nonempty msets y with cardinality $C(y) = \hat{0} = \varnothing$. For example, $[x,y,z]_{-3,4,-1} \neq \varnothing$ and $C([x,y,z]_{-3,4,-1}) = \hat{0} = \varnothing$. There are infinite msets of MSTZ with cardinality $\varnothing$. For example, if $\bar{y} = [x_1,x_2,x_3,\ldots]_{1,2,3,\ldots}$ and $\underline{y} = [z_1,z_2,z_3,\ldots]_{-2,-4,-6,\ldots}$ then $C(y) = C(\bar{y} \uplus \underline{y}) = \varnothing$. In this case, $C(\underline{y}) = C(\bar{y})^= = \aleph_0^=$.

Lemma $\forall x \forall y (x_y)^- = (x^-)_y$.

Proof: If $y \notin x$ then $x_y = \varnothing$, $(x_y)^- = \varnothing^- = \varnothing$ and $y \notin x^-$ so that $(x^-)_y = \varnothing$. If $y \in^n x$ then $x_y = [y]_n$, $(x_y)^- = [y]_{-n}$, and $y \in^{-n} x^-$ so that $(x^-)_y = [y]_{-n}$ as required.

Therefore, $C(x_y) = \hat{t}$ iff $|x_y| = t$ iff $y \in^t x$ iff $y \in^{-t} x^-$ iff $|(x^-)_y| = |(x_y)^-| = -t$ iff $C((x_y)^-) = \widehat{-t}$. Also, $\forall x \forall y \forall n((y \notin x \to C(x_y) = \hat{0}) \wedge (y \in^n x \to C(x_y) = \hat{n}))$ holds in MSTZ.

In MSTZ, $\forall x \forall y (x \subseteq y \to C(x) \subseteq C(y))$ does *not* hold. Consider the case $[x]_{-2} \subseteq [x]_2$ where $C([x]_{-2}) = \widehat{-2} = \hat{2}^= = \{\varnothing,\{\varnothing\}^-\}^-$ and $C([x]_2) = \hat{2} = \{\varnothing,\{\varnothing\}\}$. Since $\{\varnothing\}^- \in \widehat{-2}$ but $\{\varnothing\}^- \notin \hat{2}$, $C([x]_{-2}) \nsubseteq C([x]_2)$.

Let Card be the class of cardinal numbers in ZFC and Card′ the corresponding class of ZFC′. Let CARD $= \{C(x) | x \in M\}$ be the class of cardinal numbers in MSTZ. Therefore, CARD $=$ Card′ $\cup \{\kappa^= | \kappa \in \text{Card}'\}$. The well-order $\leq'$ in Card′ induces a *linear order* $\leq$ in CARD as follows. Let α and β be msets in CARD. The class CARD contains no hybrid msets. If α and β are both M^+, then we define $\alpha \leq \beta$ iff $\alpha \leq' \beta$. If α is M^- and β in M^+, then define $\alpha \leq \beta$. If α is M^+ and β is M^-, then define $\beta \leq \alpha$. If both α and β are M^-, then we define $\alpha \leq \beta$ iff $\beta^= \leq' \alpha^=$. We note that $\leq$ does *not* well-order CARD.

In MSTZ, $\forall x \forall y (x \subseteq y \to C(x) \leq C(y))$ does *not* hold. For example, $\{z\} \subseteq [z,z']_{1,-2}$ but $\hat{1} \nleq \widehat{-1}$. In MSTZ, $\forall x \forall y (C(x \cap y) \leq C(x \cup y))$ does *not* hold. For example, if $x = \{z_1,z_2\}$ and $y = [z_1,z_3]_{1,-3}$ and $z_1 \neq z_2$, then $x \cap y = \{z_1\}$ and $C(x \cap y) = \hat{1}$, but $x \cup y = [z_1,z_2,z_3]_{1,1,-3}$ and $C(x \cup y) = \widehat{-1}$. The same example also shows that $\forall x \forall y (C(x \cap y) \leq C(x \uplus y))$ does *not* hold in MSTZ since $\hat{1} \nleq \hat{0}$. If $x = y = [z]_{-1}$ then $x \cup y = [z]_{-1}$ and $x \uplus y = [z]_{-2}$. Therefore, in general, $C(x \cup y) \nleq C(x \uplus y)$ in MSTZ.

One could develop a cardinal arithmetic (with additive inverses) in CARD based on the (positive) cardinal arithmetic in Card′. For an example see Rado [16], pp. 139–140.

Remark We proved earlier that $(x \uplus y)^- = x^- \uplus y^-$. Since $x = \bar{x} \uplus \underline{x}$, we have $x^- = (\bar{x} \uplus \underline{x})^- = (\bar{x})^- \uplus (\underline{x})^-$. However, $x^- = (\overline{x^-}) \uplus (\underline{x^-})$. Clearly, $(\bar{x})^- = (\underline{x^-})$ and $(\underline{x})^- = (\overline{x^-})$. We use these identities in the following result:

Theorem $\forall x\, C(x^-) = C(x)^=$

Proof: We proceed by cases.

Case 1: If $C((\overline{x^-})) = \hat{s}$ and $C((\underline{x^-})^-) = \hat{t}$ then $C(x^-) = (s \dot{-} t)$. However, as we observed above $(\underline{x^-})^- = ((\overline{x})^-)^-) = \bar{x}$ and $(\overline{x^-}) = (\underline{x})^-$ so that $C(\bar{x}) = \hat{t}$,

$C((\underline{x})^-) = \hat{s}$, and $C(x) = (\widehat{t - s})$. Therefore, $C(x^-) = (\widehat{s - t}) = (\widehat{-(t - s)}) = C(x)^=$.

Case 2: If $C((\overline{x^=})) = \kappa \geq \aleph_0$ nd $C(((\underline{x^-}))^-) = \hat{t}$ then $C(x^-) = \kappa$. However, $C(\bar{x}) = \hat{t}$ and $C((\underline{x})^-) = \kappa \geq \aleph_0$ so that $C(x) = \kappa^=$. Therefore, $C(x^-) = \kappa = (\kappa^=)^= = C(x)^=$.

Case 3: Case 3 proceeds exactly as Case 2.

Case 4: If $C((\overline{x^=})) = \kappa \geq \aleph_0$ and $C(((\underline{x^-}))^-) = \lambda \geq \aleph_0$, then $C((\underline{x})^-) = \kappa \geq \aleph_0$ and $C(\bar{x}) = \lambda \geq \aleph_0$.

(i) If $\kappa > \lambda$ then $C(x^-) = \kappa$, but $C(x) = \kappa^=$ so that $C(x^-) = \kappa = (\kappa^=)^= = C(x)^=$.
(ii) If $\kappa < \lambda$ then $C(x^-) = \lambda^=$ and $C(x) = \lambda$. Therefore, $C(x^-) = \lambda^= = C(x)^=$.
(iii) If $\kappa = \lambda$ then $C(x^-) = \hat{0} = \varnothing$ and $C(x) = \hat{0} = \varnothing$. Therefore, $C(x^-) = \varnothing = \varnothing^= = C(x)^=$ as required.

This completes the proof of the theorem.

Corollary $\forall x\, C(x) = C(x^-)^=$.

Proof: Since $\forall x(x = (x^-)^-)$ holds in MSTZ, $C(x) = C((x^-)^-) = C(x^-)^=$ by the theorem.

Corollary $\forall x \forall y\, C(x - y) = C(y - x)^=$.

Proof: We proved earlier that $x - y = (y - x)^-$. Therefore, $C(x - y) = C((y - x)^-) = C(y - x)^=$ by the theorem.

A theory T' with language $L(T')$ is a *conservative extension* of a theory T with language $L(T)$ if

(i) $L(T) \subseteq L(T')$, and
(ii) for every wff ϕ of $L(T)$, $T \vdash \phi$ iff $T' \vdash \phi$.

The theory MSTZ is *not* a conservative extension of the theory MST ([1] and [2]). In fact, MSTZ is not even an extension of MST. For example, the theorem $\forall x \forall y((x \subseteq y \wedge Set(y)) \rightarrow Set(x))$ of MST is not a theorem of MSTZ (consider $[x]_{-1} \subseteq \{x\}$). The same type of example shows that the sentence $\forall x \exists y(y \subseteq \{x\} \wedge y \neq \{x\} \wedge y \neq \varnothing)$ is a theorem of MSTZ. It is *not*, however, a theorem of MST (for such an mset y to exist in MST would imply $\exists n(0 < n \wedge n < 1)$ holds in Peano arithmetic).

Although theorems about hsets in MSTZ are the exact analogs of theorems about sets in ZFC (for every wff ϕ of ZFC, ZFC $\vdash \phi$ iff MSTZ $\vdash \phi'$), MSTZ is *not* a conservative extension of ZFC in the strict sense above. The nonlogical symbol $\in$ of ZFC is not in $L = \{<, e, \hat{}, +, \cdot, 0, 1, -1\}$. The languages of ZFC and MSTZ are disjoint. However, MSTZ is a conservative extension of our copy ZFC$'$ of ZFC (for every wff ϕ of ZFC, ZFC $\vdash \phi$ iff ZFC$' \vdash \phi'$ iff MSTZ $\vdash \phi'$).

A model of MSTZ We assume ZFC.

Let $\mathbb{Z}$ be the set of integers as usually constructed in ZFC, and let $\bar{\mathbb{Z}}$ be the set $\mathbb{Z}$ with the set representing zero removed. Let F be a hierarchy of $\bar{\mathbb{Z}}$-valued functions in ZFC defined as follows:

$$F_0 = \varnothing,$$

$$F_{\alpha+1} = \{x\colon \operatorname{dom} x \to \bar{\mathbb{Z}} \mid \operatorname{dom} x \subseteq F_\alpha\},$$

$$F_\lambda = \bigcup_{\alpha<\lambda} F_\alpha \text{ if } \lambda \text{ is a limit ordinal, and}$$

$$F = \bigcup_{\alpha\in\mathrm{On}} F_\alpha.$$

We define an L-structure $\mathbb{F}$ with two domains: the set $\mathbb{Z}$ over which the numeric variable symbols range, and the class F over which the mset variable symbols range. We are, therefore, modeling multisets of MSTZ by (nonzero) integer-valued functions of ZFC.

The language of MSTZ is $\{<, e, \hat{}, +, \cdot, 0, 1, -1\}$. Let the nonlogical symbols $<$, $+$, $\cdot$, 0, 1, and -1 be interpreted in $\mathbb{F}$ by the strict order relation in $\mathbb{Z}$, binary addition in $\mathbb{Z}$, binary multiplication in $\mathbb{Z}$, and the sets in $\mathbb{Z}$ representing zero, one, and minus one, respectively. Let the ternary predicate $e(x,y,n)$, or $x \in^n y$, be interpreted in $\mathbb{F}$ by $y(x) = n$ (or $\langle x,n\rangle \in y$).

To interpret the unary function symbol $\hat{}$ of L in $\mathbb{F}$ we define a function $\hat{}$: $\mathbb{Z} \to F$ as follows:

$$\hat{0} = \varnothing, \text{ the empty function in } F_1\text{; and}$$

$$\widehat{n+1} = \hat{n} \cup \{\langle \hat{n},1\rangle\} \text{ for all } n \geq 0.$$

So, for example,

$$\hat{1} = \varnothing \cup \{\langle\varnothing,1\rangle\} = \{\langle\varnothing,1\rangle\} \text{ in } F_2\text{; and}$$

$$\hat{2} = \{\langle\varnothing,1\rangle\} \cup \{\langle\{\langle\varnothing,1\rangle\}, 1\rangle\}$$
$$= \{\langle\varnothing,1\rangle, \langle\{\langle\varnothing,1\rangle\}, 1\rangle\}$$

which is in F_3 since $\operatorname{dom} \hat{2} = \{\varnothing,\{\langle\varnothing,1\rangle\}\} \subseteq F_2$. In order to define $\hat{n}$ for all $n < 0$, we need the interpretation of shadows and hereditary shadows in $\mathbb{F}$. Let y be an arbitrary function in F. The 'shadow' function y^- of y in $\mathbb{F}$ is such that $\operatorname{dom} y^- = \operatorname{dom} y$ and $\forall x \in \operatorname{dom} y^- = \operatorname{dom} y$, $y^-(x) = -y(x)$, where $-y(x)$ is the additive inverse of $y(x)$ in $\mathbb{Z}$. The function y^- is in F since the rank of y^- in F equals the rank of y in F (since $\operatorname{dom} y^- = \operatorname{dom} y$). Therefore, $\varnothing^- = \varnothing$, $\{\langle x,n\rangle\}^- = \{\langle x,-n\rangle\}$ for all $n \neq 0$, and $\forall x\forall n(\langle x,n\rangle \in y \leftrightarrow \langle x,-n\rangle \in y^-)$.

The 'hereditary shadow' function $y^=$ of y in $\mathbb{F}$ is such that $y^= = \{\langle x^=, -n\rangle | \langle x,n\rangle \in y\}$. Equivalently, using the 'shadow' function, $y^= = \{\langle x^=, n\rangle | \langle x,n\rangle \in y\}^-$, or $y^= = \{\langle x^=,n\rangle | \langle x,n\rangle \in y^-\}$. The function $y^=$ is in F since the rank of $y^=$ in F equals the rank of y (by induction on rank). Therefore, $\varnothing^= = \varnothing$, $\{\langle x,n\rangle\}^= = \{\langle x^=,-n\rangle\}$ for all $n \neq 0$, and $\forall x\forall n(\langle x,n\rangle \in y \leftrightarrow \langle x^=,-n\rangle \in y^=)$. We now complete our definition of the function $\hat{}$: $\mathbb{Z} \to F$. For all $n < 0$, define $\hat{n} = (\widehat{-n})^=$. For example, we have

$$\widehat{-1} = (\hat{1})^= = \{\langle\varnothing,1\rangle\}^= = \{\langle\varnothing^=,-1\rangle\} = \{\langle\varnothing,-1\rangle\}$$

which is in F_2, and

$$\widehat{-2} = (\hat{2})^{=} = \{\langle\varnothing,1\rangle, \langle\{\langle\varnothing,1\rangle\}, 1\rangle\}^{=}$$
$$= \{\langle\varnothing^{=},-1\rangle, \langle\{\langle\varnothing,1\rangle\}^{=}, -1\rangle\}$$
$$= \{\langle\varnothing,-1\rangle, \langle\{\langle\varnothing,-1\rangle\}, -1\rangle\}$$

which is in F_3 since dom $\widehat{-2} = \{\varnothing, \{\langle\varnothing,-1\rangle\}\} \subseteq F_2$.

For any wff ϕ of L, we define the *interpretation* of ϕ in $\mathbb{F}$, denoted by $\mathbb{F}(\phi)$, by induction on the logical complexity of ϕ as follows:

$\mathbb{F}(\dot{n} < \dot{m})$ is $\dot{n} \in \mathbb{Z} \wedge \dot{m} \in \mathbb{Z} \wedge \dot{n} < \dot{m}$;

$\mathbb{F}(e(x,y,\dot{n}))$ is $x \in F \wedge y \in F \wedge \dot{n} \in \mathbb{Z} \wedge y(x) = \dot{n}$;

$\mathbb{F}(\dot{n} = \dot{m})$ is $\dot{n} \in \mathbb{Z} \wedge \dot{m} \in \mathbb{Z} \wedge \dot{n} = \dot{m}$; and

$\mathbb{F}(x = y)$ is $x \in F \wedge y \in F \wedge x = y$.

(Equality of functions in F is just equality of sets of ZFC.) If ϕ is nonatomic, $\mathbb{F}(\sim\psi)$ is $\sim\mathbb{F}(\psi)$; $\mathbb{F}(\psi \vee \theta)$ is $\mathbb{F}(\psi) \vee \mathbb{F}(\theta)$; $\mathbb{F}(\psi \wedge \theta)$ is $\mathbb{F}(\psi) \wedge \mathbb{F}(\theta)$; $\mathbb{F}(\psi \rightarrow \theta)$ is $\mathbb{F}(\psi) \rightarrow \mathbb{F}(\theta)$; $\mathbb{F}(\psi \leftrightarrow \theta)$ is $\mathbb{F}(\psi) \leftrightarrow \mathbb{F}(\theta)$; $\mathbb{F}(\exists n\psi)$ is $\exists n(n \in \bar{\mathbb{Z}} \wedge \mathbb{F}(\psi))$; $\mathbb{F}(\forall n\psi)$ is $\forall n(n \in \bar{\mathbb{Z}} \rightarrow \mathbb{F}(\psi))$; $\mathbb{F}(\exists x\psi)$ is $\exists x(x \in F \wedge \mathbb{F}(\psi))$; and $\mathbb{F}(\forall x\psi)$ is $\forall x(x \in F \rightarrow \mathbb{F}(\psi))$.

For every wff ϕ of L, $\mathbb{F}(\phi)$ is a wff of ZFC. By "ϕ holds in $\mathbb{F}$" we mean ZFC $\vdash \mathbb{F}(\phi)$. To show that $\mathbb{F}$ is a model of MSTZ we must show that every theorem of MSTZ holds in $\mathbb{F}$; that is, if MSTZ $\vdash \phi$ then ZFC $\vdash \mathbb{F}(\phi)$.

If ϕ is a numeric axiom of MSTZ (for example, an axiom of an 'integral system' or Axiom N8), then our interpretations of $<$, $+$, $\cdot$, 0, 1, and -1 in $\mathbb{Z}$, and our definition of the function $\hat{}: \mathbb{Z} \rightarrow F$ in $\mathbb{F}$, are such that ZFC $\vdash \mathbb{F}(\phi)$. If ϕ is a nonnumeric axiom of MSTZ (one of Axioms I through XII), then the proof of ZFC $\vdash \mathbb{F}(\phi)$ is identical (in most cases) to the proof in [1] and [2] for the corresponding axiom of MST. Since the union, additive union, and replacement axioms of MSTZ differ from those in MST, slight modifications must be made to the 'union', 'additive union', and 'range of replacement' functions in F. The changes needed for the 'union' and 'additive union' function definitions in F are obvious. For the replacement axioms, we give a detailed proof.

For any replacement axiom X_ϕ of MSTZ, consider its interpretation $\mathbb{F}(X_\phi)$ in $\mathbb{F}$ (we assume variables are restricted to the class F, or the set $\bar{\mathbb{Z}}$, as appropriate):

$$\begin{aligned}
&\forall x\forall y\forall y'((\phi'(x,y) \wedge \phi'(x,y') \rightarrow y = y') \rightarrow \\
&\forall z\exists z'\forall y\forall n(z'(y) = n \leftrightarrow [\exists x(x \in \text{dom } z \wedge \phi'(x,y)) \wedge \\
&[\forall x\forall m((z(x) = m \wedge \phi'(x,y)) \rightarrow m > 0) \rightarrow \\
&(\exists x(z(x) = n \wedge \phi'(x,y)) \wedge \forall x\forall m((z(x) = m \wedge \phi'(x,y)) \rightarrow n \leq m))] \wedge \\
&(\exists x\forall m(z(x) = m \wedge \phi'(x,y) \wedge m < 0) \rightarrow n = 1)])
\end{aligned}$$

where $\phi'(x,y)$ is $\mathbb{F}((\phi(x,y))$, a wff of ZFC with free variables including x and y but excluding y' and z'. Let $\phi'(x,y)$ be such that the antecedent of $\mathbb{F}(X_\phi)$ holds, and let z be an arbitrary function in F with domain dom z. We apply ZFC replacement to the set dom z using the wff $\phi'(x,y)$ to obtain a new set (the elements of which are restricted to the class F), which we shall call dom z'. Therefore,

$$y \in \text{dom } z' \leftrightarrow \exists x(x \in \text{dom } z \wedge \phi'(x,y)).$$

We define a function z' with domain dom z' as follows: For every $y \in$ dom z',

$z'(y) = 1$, if $\exists x(x \in \text{dom } z \wedge \phi'(x,y) \wedge z(x) < 0)$ holds, and
$z'(y) = n$, if $[\exists x(z(x) = n \wedge \phi'(x,y)) \wedge \forall x \forall m((z(x) = m \wedge \phi'(x,y)) \rightarrow n \leq m)]$ holds, otherwise.

Clearly, the function z' satisfies the consequent of $\mathbb{F}(X_\phi)$. To establish that the function z' is in fact in F let β be the union of the ranks of all y in F that are functions in dom z'. The F_α-hierarchy is such that $F_\alpha \subseteq F_\beta$ for all $\alpha \leq \beta$. Therefore, if $y \in$ dom z', then $y \in F_\beta$. Hence, dom $z' \subseteq F_\beta$ and $z' \in F_{\beta+1} \subseteq F$ as required. Therefore, Axiom X_ϕ holds in $\mathbb{F}$; that is, ZFC $\vdash \mathbb{F}(X_\phi)$.

Since ZFC $\vdash \mathbb{F}(\phi)$ for every axiom ϕ in MSTZ, we obtain the result:

Theorem *For every wff ϕ in L, if* MSTZ $\vdash \phi$ *then* ZFC $\vdash \mathbb{F}(\phi)$.

The L-structure $\mathbb{F}$ is a model of MSTZ. It follows that MSTZ is *relatively consistent*; that is, if ZFC is consistent, then MSTZ is consistent.[1]

NOTE

1. The positive integers can be identified with the set of all finite multisets of prime numbers; that is, every positive integer n is associated with the unique multiset N of its prime factors (1 is associated with the empty multiset $\varnothing$). Therefore,

$$p \in^\alpha N \text{ iff } p^\alpha | n \wedge p^{\alpha+1} \nmid n.$$

With this identification, the multiplicative properties of positive integers translate into properties of multisets of primes. For example, the property 'prime' corresponds to 'singleton', the property 'relative prime' to 'disjoint', 'product of distinct primes' (or, 'square-free') to 'set', 'power of a single prime' to 'simple multiset', 'divides' to 'is an msubset of', 'multiplication' to 'additive union', 'lowest common multiple' to 'union' and 'greatest common divisor' to 'intersection'. I am grateful to Tom Etter for reminding me of a very nice example of negative membership using the above identification generalized to positive rational numbers. The positive rationals can be identified with the set of all finite multisets of prime numbers in which multiplicities may be positive or negative; that is, a positive rational $q = n/m$ is associated with a multiset Q such that

$$p \in^\alpha Q \wedge \alpha > 0 \text{ iff } p^\alpha | n \wedge p^{\alpha+1} \nmid n \text{ and}$$
$$p \in^\alpha Q \wedge \alpha < 0 \text{ iff } p^{-\alpha} | m \wedge p^{-\alpha+1} \nmid m.$$

In other words, a negative multiplicity α of an element p in Q indicates that $-\alpha$ is the highest power of p that divides the denominator of q. To avoid having distinct multisets for numerically equal rationals, we define $Q = N \uplus M^-$ (where M^- is the shadow of M). If n and m are not relative prime (if $N \cap M \neq \varnothing$), the addition of negative and positive multiplicities of common prime factors in $N \uplus M^-$ is equivalent to cancellation to lowest terms. With this interpretation of negative membership, the shadow operation $^-$ is simply reciprocation $^{-1}$. Therefore, $n/1$ is $N \uplus \varnothing^- = N$ and $1/n$ is $\varnothing \uplus N^- = N^-$ and, in general, $(n/m)^{-1} = m/n = m \cdot 1/n$ is $M \uplus N^- = (N \uplus M^-)^-$. The annihilation $N \uplus N^- = \varnothing$ corresponds to the simple fact $n/n = 1$.

REFERENCES

[1] Blizard, W., *Generalizations of the Concept of Set: A Formal Theory of Multisets*, Doctoral thesis, Mathematical Institute, University of Oxford, 1986.

[2] Blizard, W., "Multiset theory," *Notre Dame Journal of Formal Logic*, vol. 30 (1989), pp. 36–66.

[3] Chang, C. and H. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973.

[4] Eilenberg, S., *Automata, Languages and Machines*, Volume A, Academic Press, New York, 1974.

[5] Enderton, H., *Elements of Set Theory*, Academic Press, New York, 1977.

[6] Feynman, R. P., "Negative probability," pp. 235–248 in *Quantum Implications: Essays in Honour of David Bohm*, edited by B. J. Hiley and F. David Peat, Routledge and Kegan Paul, London, 1987.

[7] Fischbein, E., *Intuition in Science and Mathematics*, Reidel, Dordrecht, 1987.

[8] Gridgeman, N. T., reviews of Hailperin [10], *The Journal of Symbolic Logic*, vol. 50 (1985), pp. 851–852, vol. 53 (1988), pp. 1253–1254.

[9] Guillen, M., *Bridges to Infinity: The Human Side of Mathematics*, Rider, London, 1983.

[10] Hailperin, T., *Boole's Logic and Probability*, Second Edition, North-Holland, Amsterdam, 1986.

[11] Kline, M., *Mathematical Thought From Ancient to Modern Times*, Oxford University Press, New York, 1972.

[12] Knuth, D., *The Art of Computer Programming: Volume 2 Seminumerical Algorithms*, Second Edition, Addison-Wesley, Reading, Massachusetts, 1981.

[13] Levy, A., *Basic Set Theory*, Springer-Verlag, Berlin, 1979.

[14] Meyer, R. and M. McRobbie, "Multisets and relevant implication, I and II," *Australasian Journal of Philosophy*, vol. 60 (1982), pp. 107–139, 265–281.

[15] Pinter, C., *A Book of Abstract Algebra*, McGraw-Hill, New York, 1982.

[16] Rado, R., "The cardinal module and some theorems on families of sets," *Annali di Matematica pura ed applicata*, vol. 102 (1975), pp. 135–154.

[17] Reisig, W., *Petri Nets: An Introduction*, Springer-Verlag, Berlin, 1985.

[18] Whitney, H., "Characteristic functions and the algebra of logic," *Annals of Mathematics*, vol. 34 (1933), pp. 405–414.

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
16 Mill Lane
Cambridge CB2 1SB, England