44
Notre Dame Journal of Formal Logic
Volume 31, Number 1, Winter 1990

A Simplification of the Completeness Proofs for Guaspari and Solovay's R

FRANS VOORBRAAK*

Abstract Alternative proofs for Guaspari and Solovay's completeness theorems for R are presented. R is an extension of the provability logic L and was developed in order to study the formal properties of the provability predicate of PA occurring in sentences that may contain connectives for witness comparison. (The primary example of sentences involving witness comparison is the Rosser sentence.) In this article the proof of the Kripke model completeness theorems employs tail models, as introduced by Visser, instead of the more usual *finite* Kripke models. The use of tail models makes it possible to derive arithmetical completeness from Kripke model completeness by *literally* embedding Kripke models into PA. Our arithmetical completeness theorem differs slightly from the one proved by Guaspari and Solovay, and it also forms a solution to the problem (advanced by Smoryński) of obtaining a completeness result with respect to a variety of orderings.

Introduction This paper deals with the completeness proofs given in [2] for the theory R, which is an extension of the provability logic L introduced in [5]. R is formulated in a language containing $\leq$ and $<$ as connectives for witness comparison, which enables one to study the provability predicate as it occurs in formulas like $\mathrm{Pr}(\ulcorner A \urcorner) < \mathrm{Pr}(\ulcorner B \urcorner)$, defined as $\exists x(\mathrm{Proof}(x, \ulcorner A \urcorner) \wedge \forall y \leq x$ $\neg\mathrm{Proof}(y, \ulcorner B \urcorner))$. Rosser sentences are good examples of this kind of formula.

In Section 1 we describe the theories L and R and the alternative complete-

*The treatment of the Kripke model completeness theorem given in Section 2 has benefited much from [3]. Albert Visser's suggestions, corrections and remarks also made a substantial contribution to this paper.

Received June 30, 1986; revised December 8, 1988

ness proofs for L given by Visser. Our main object is to give similar alternative proofs for R. Sections 2 and 3 deal with the Kripke model, and arithmetical, completeness proofs, respectively.

Only elementary facts from modal logic and PA are used. Formally, no knowledge of provability logic is presupposed. For background knowledge, see [1] or the more up-to-date and up-tempo [4]. For that matter, the original articles referred to above are highly readable themselves. Some of the details omitted in this paper can be found in [10].

1 The theories **L** *and* **R** For convenience we summarize Solovay's completeness theorems from [5] and give definitions of some relevant notions.

1.1 Definition L is the following theory in $\mathcal{L}$, the language of propositional modal logic ($A, B, \ldots$ denote well-formed formulas of $\mathcal{L}$):

Axioms: Boolean tautologies
$\Box(A \to B) \to (\Box A \to \Box B)$
$\Box A \to \Box\Box A$
$\Box(\Box A \to A) \to \Box A$
Rules: modus ponens
$A/\Box A$.

(Some other names for this system found in the literature are PrL, G, and GL.)

1.2 Theorem (Kripke model completeness) *For any sentence A of $\mathcal{L}$:* L $\vdash$ *A iff A is valid in all finite, transitive, irreflexive (tree-ordered) Kripke models iff A holds at the root of all finite, transitive, irreflexive (tree-ordered) Kripke models.*

1.3 Definition An *arithmetical interpretation* * *of* $\mathcal{L}$ is a map from formulas of $\mathcal{L}$ to sentences of PA satisfying:

for all atomic p: p^* *is an arithmetical sentence*
$\bot^* = \underset{\sim}{0} = \underset{\sim}{1}$; $\top^* = \underset{\sim}{0} = \underset{\sim}{0}$
$(\ )^*$ commutes with Boolean connectives
$(\Box A)^* = \mathrm{Pr}(\ulcorner A^* \urcorner)$.

1.4 Theorem (Arithmetical completeness) *For any sentence A of $\mathcal{L}$*: L $\vdash A$ *iff for all* * (PA $\vdash A^*$).

In other words, L axiomatizes the schemata provable about the provability predicate $\mathrm{Pr}(x)$ in PA. We also have a system L^ω which axiomatizes the true schemata about $\mathrm{Pr}(x)$:

1.5 Definition L^ω is the following system of modal logic:

Axioms: All theorems of L
$\Box A \to A$
Rules: modus ponens.

1.6 Theorem *For any sentence A of $\mathcal{L}$:* $\mathrm{L}^\omega \vdash A$ *iff for all* * *A^* is true.*

Let $S(A) = \{B : \Box B$ is a subformula of $A\}$. Since $\mathrm{L}^\omega \vdash A$ iff $\mathrm{L} \vdash \bigwedge_{B \in S(A)} (\Box B \to B) \to A$, we also have a Kripke model completeness theorem for L^ω.

1.7 Definition An *A-sound* Kripke model is a Kripke model whose root satisfies $\bigwedge_{B \in S(A)} (\Box B \to B)$.

1.8 Theorem *For any sentence A of $\mathcal{L}$: $\mathrm{L}^\omega \vdash A$ iff A is forced at the root of all A-sound, finite, transitive, irreflexive (tree-ordered) Kripke models.*

The proof of the arithmetical completeness of L exploits the Kripke model completeness by "embedding" finite Kripke models "into PA": If $\mathrm{L} \nvdash A$, then we can find an interpretation * based on a countermodel of A such that $\mathrm{PA} \nvdash A^*$.

To justify the use of the phrase "embedding into PA" one would like to have a representation $[A]_K$ of the set of nodes of the Kripke model K in which A is forced such that

(1) $\mathrm{PA} \vdash [A]_K$ iff A is valid in K, and
(2) $\mathrm{PA} \vdash [A]_K \leftrightarrow A^*$, where * is an arithmetical interpretation based on K.

However, this is impossible for finite Kripke models K, since for such models there exists an n such that $\Box^n \bot$ and $\Box^{n+1} \bot$ are forced at the same (viz. all) nodes. But then (2) yields:

$$\mathrm{PA} \vdash (\Box^n \bot)^* \leftrightarrow (\Box^{n+1} \bot)^*$$

which is a contradiction.

In [8] the notion of tail model is introduced and it is shown that these tail models are really "embeddable into PA" in the above sense. Roughly, a tail model is a finite, transitive, irreflexive, tree-ordered Kripke model with an $\omega + 1$-tail attached to its bottom node, where the forcing of atomic formulas is defined constant on the tail (see Section 2 for an exact definition).

Since a tail model is automatically A-sound for every A in $\mathcal{L}$, we have the following alternative Kripke completeness theorem for L^ω:

1.8′ Theorem *For any sentence A of $\mathcal{L}$: $\mathrm{L}^\omega \vdash A$ iff A is forced at the root of all tail models.*

For a proof see [8] or the perhaps more accessible [9].

Tail models will be used extensively in the simplified arithmetical and Kripke model completeness proofs for the theories R and R^ω, which will be described below.

1.9 Definition The language $\mathcal{L}^+$ is obtained from $\mathcal{L}$ by adding two (partial) connectives for witness comparison: $\leq$ and $<$. We also have a new formation rule:

$\Box A \leq \Box B$ and $\Box A < \Box B$ are well-formed formulas whenever A and B are.

The *Σ-formulas* are those formulas of $\mathcal{L}^+$ which have $\leq$, $<$, or $\Box$ as their principal connective. The *boxed formulas* are those formulas of $\mathcal{L}^+$ which have $\Box$ as their principal connective.

From now on $A, B, C, \ldots$ denote formulas of $\mathcal{L}^+$.

1.10 Definition R is the theory with the following rules and axioms:

Axioms: All schemas of L, for all $\mathcal{L}^+$-formulas
$A \to \Box A$, for all Σ-formulas A (*Σ-soundness*)

$$\left.\begin{array}{l} A \to (A \leq B) \vee (B \leq A) \\ (A \leq B) \to A \\ (A \leq B) \wedge (B \leq C) \to (A \leq C) \\ (A < B) \leftrightarrow (A \leq B) \wedge \neg(B \leq A) \end{array}\right\} \begin{array}{l} \text{for boxed formulas} \\ (\textit{the order axioms}) \end{array}$$

Rules: modus ponens
$A/\Box A$
$\Box A/A$.

The particular set of order axioms above, which is taken from [3], is somewhat simpler than but equivalent to the set given in [2]. It is easy to see that the order axioms say that $\leq$ is a weak pre-ordering of the boxed formulas and that $<$ is its associated strict pre-ordering, such that true formulas are witnessed before false ones and false formulas are totally unordered. Some of the following straightforward consequences of the order axioms will be used further on:

$$\begin{array}{l} A \to (A < B) \vee (B \leq A) \\ A \to (A \leq B) \vee (B < A) \\ (A \leq B) \wedge (B < C) \to (A < C) \\ (A < B) \wedge (B \leq C) \to (A < C) \\ A \wedge \neg B \to (A < B) \\ (A < B) \to (A \leq B) \\ (A < B) \to \neg(B < A). \end{array}$$

Here A, B, and C denote boxed formulas.

1.11 Definition The theory obtained from R by deleting the rule $\Box A/A$ is called R^-.

R^- is used to simplify the Kripke model completeness proof for R, which is the theory of all schemata provable in PA. Again, as in the case of L, we have a theory R^ω of all true schemata:

1.12 Definition R^ω is the following theory:

Axioms: All Theorems of R (in fact, R^- would do)
$\Box A \to A$

Rules: modus ponens.

In [2] it is noted that deductions in R cannot be normalized, since $\mathrm{R} \vdash \Box\top < \Box\bot$ and $\mathrm{R}^- \nvdash \Box\top < \Box\bot$, so any proof of $\Box\top < \Box\bot$ must use the rule $\Box A/A$. We will show that it is possible to replace the rule $\Box A/A$ by a rule with the subformula property. (It remains an open question whether a variant of the system so obtained satisfies a normalization theorem.)

1.13 Definition R′ is the theory obtained from R by replacing the rule $\Box A/A$ by the rule $A/\neg B \to (\Box A < \Box^2 B)$.

1.14 Theorem $R' \vdash A \Leftrightarrow R \vdash A$.

Proof: "$\Leftarrow$": Clearly it is sufficient to show that $R' \vdash \Box A$ implies $R' \vdash A$. Suppose therefore that $R' \vdash \Box A$, so for all formulas B, $R' \vdash \neg B \to (\Box^2 A < \Box^2 B)$. Substitution of "$A$" for "$B$" yields $R' \vdash A$.

"$\Rightarrow$": We show that $R \vdash A$ implies $R \vdash \neg B \to (\Box A < \Box^2 B)$. Suppose $R \vdash A$. Then $R \vdash \Box A$, so we have $R \vdash (\Box A < \Box^2 B) \vee (\Box^2 B \leq \Box A)$. Clearly $R \vdash (\Box A < \Box^2 B) \to \Box^2(\neg B \to (\Box A < \Box^2 B))$, and also $R \vdash (\Box^2 B \leq \Box A) \to \Box^2(\neg B \to (\Box A < \Box^2 B))$, since $R \vdash (\Box^2 B \leq \Box A) \to \Box^2 B$. We conclude that $R \vdash \Box^2(\neg B \to (\Box A < \Box^2 B)$ and hence $R \vdash \neg B \to (\Box A < \Box^2 B)$.

The rule $A/\neg B \to (\Box A < \Box^2 B)$ is similar to a rule of the system Z in [6], which results from R by replacing the axiom of Σ-soundness and the rule $\Box A/A$ by $A/\neg B \to (\Box A < \Box B)$ and the axiom $\Box A \to \Box(\neg B \to (\Box A < \Box B))$.

2 Kripke model completeness For the reasons stated above we employ tail models instead of the more usual finite, transitive, irreflexive Kripke models.

2.1 Definition A *tail model for L* is a transitive, irreflexive, tree-ordered Kripke model $K = \langle \omega, <, \Vdash \rangle$, which satisfies:

the forcing relation $\Vdash$ satisfies the usual clauses and is defined for all formulas of $\mathcal{L}$
if $m \neq 0$, then $0 < m$
if $n \neq 0$ and $n < m$, then $n > m$
for some $N \neq 0$: (i) for all $n, m \geq N$ $(n > m \Rightarrow n < m)$
(ii) for $n = 0$ or $n \geq N$:
for all p_i $(n \Vdash p_i \Leftrightarrow N \Vdash p_i)$.

An N which satisfies (i) and (ii) is called a *tail-element*. The subtree generated by a tail-element is called a *top part*. See Figure 1 for a representation of a typical tail model.

2.2 Definition A *tail model for R* is a tail model for L which in addition satisfies:

the forcing relation $\Vdash$ is defined for all formulas of $\mathcal{L}^+$
the forcing of the witness comparison formulas is persistent, i.e.:
- $k \Vdash A \leq B \Rightarrow$ for all $k' > k$ $k' \Vdash A \leq B$
- $k \Vdash A < B \Rightarrow$ for all $k' > k$ $k' \Vdash A < B$

all instances of the order axioms are forced at each node.

See Figure 2 for a typical tail model for R.

Note that in a tail model for R, for any Σ-formula A, $k \Vdash A$ implies for all $k' > k$ that $k' \Vdash A$. This feature is called *Σ-persistency*.

Unless explicitly stated otherwise, from now on "tail model" will mean "tail

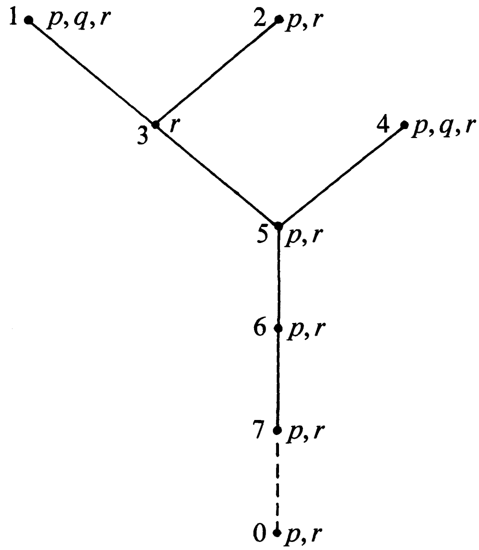


Figure 1.

model for R" and "finite Kripke tree" will mean "finite, transitive, irreflexive, tree-ordered Kripke model for R".

2.3 Tail Lemma *For every tail model K:*

$0 \Vdash A$ *iff for some M, for all* $n \geq M \quad n \Vdash A$
$0 \nVdash A$ *iff for some M, for all* $n \geq M \quad n \nVdash A$.

Proof: Easy induction on A. We'll show one case: "$A = B \leq C$". Suppose $0 \Vdash B \leq C$. Then, by Σ-persistency, for all n, $n \Vdash B \leq C$. For the other direction, suppose for all $n \geq M$ that $n \Vdash B \leq C$. Then for all $n \geq M$, $n \Vdash B$, so, by the induction hypothesis $0 \Vdash B$. But then $0 \Vdash B \leq C$ or $0 \Vdash C < B$. From $0 \Vdash C < B$ we derive by Σ-persistency: for all n, $n \Vdash C < B$, which is a contradiction. So $0 \Vdash B \leq C$.

Next we'll list some straightforward definitions.

2.4 Definition Let K be a tail model. The *depth* $d(n)$ *of a node n* is defined as follows:

$$d(n) = 1 + \sup\{d(m) : n < m\}.$$

Note: $d(n) = 1$ if n is a top element of K and $d(0) = \omega$. In general we have $d(n) = \alpha$ iff $n \Vdash \Box^{\alpha}\bot$ and $n \nVdash \Box^{\beta}\bot$ for all $\beta < \alpha$, where $\Box^{\alpha}\bot$ is defined inductively by:

$\Box^{0}\bot = \bot$
$\Box^{k+1}\bot = \Box(\Box^{k}\bot)$
$\Box^{\omega}\bot = \top$.

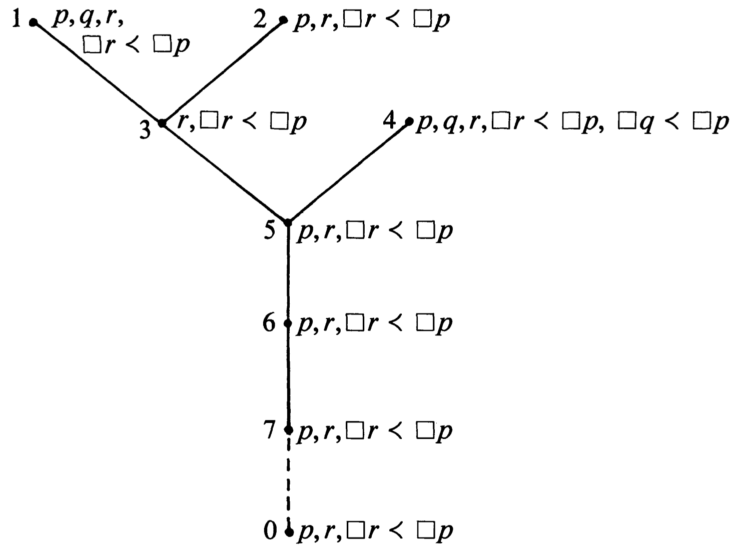


Figure 2.

If $A \not\equiv p$ or $B \not\equiv r$:

$i \Vdash \Box A < \Box B$ iff for some $j \leq i$, $j \Vdash \Box A$ and $j \nVdash \Box B$
$i \Vdash \Box A \leq \Box B$ iff for all $j \leq i$ $(j \Vdash \Box B \Leftrightarrow j \Vdash \Box A)$ and $i \Vdash \Box A$.

2.5 Definition A set of formulas S is called *adequate* if

S is closed under subformulas
$\Box A \in S$ and $\Box B \in S \Rightarrow (\Box A \leq \Box B) \in S$ and $(\Box A < \Box B) \in S$.

For any formula A, S_A denotes the *smallest adequate set containing* A.

2.6 Definition A *Kripke model restricted to a set* S of formulas is a Kripke model for which the forcing relation is required to be defined only for formulas from S.

One easily derives the Kripke model completeness of R^- from that of L once one has proved the Extension Lemma below. This lemma says that, given a Kripke model $\langle K, <, \Vdash \rangle$ restricted to an adequate set S, we can extend it to a Kripke model for R, i.e., we can define a forcing relation $\Vdash'$ such that $\langle K, <, \Vdash' \rangle$ is a Kripke model (for R) and $\Vdash'$ agrees with $\Vdash$ on formulas from S.

In [4] the extension lemma is proved by induction on the complexity of formulas, i.e., on the number of nestings of new witness comparisons, where "new" means "not already in S". The *complexity* $c(A)$ *of a formula* A is defined, relative to S, as follows:

if A is atomic or $A \in S$, $\mathrm{c}(A) = 0$
if $A \notin S$, $\mathrm{c}(A) = \mathrm{c}(B)$ if $A = \neg B$ or $A = \Box B$
$\mathrm{c}(A) = \max\{\mathrm{c}(B),\mathrm{c}(C)\}$ if $A = B \circ C$ for $\circ \in \{\vee,\wedge,\rightarrow\}$
$\mathrm{c}(A) = 1 + \max\{\mathrm{c}(B),\mathrm{c}(C)\}$ if $A = B \preceq C$ or $A = B \prec C$.

In [3] de Jongh introduced certain orderings of boxed formulas in nodes of a Kripke model which made it possible to define uniformly, for all n, an extension of the forcing relation on the set of formulas with complexity $\leq n$ to the set of formulas with complexity $\leq n + 1$. We will show that it is in fact possible to incorporate the induction on the complexity of formulas into the definition of the extended forcing relation.

2.7 Definition The *de Jongh orderings* $<_{k,\Vdash}$ and $\leq_{k,\Vdash}$ and the *de Jongh equivalence relation* $\equiv_{k,\Vdash}$ are defined (relative to a Kripke model $\langle K,<,\Vdash\rangle$) as follows:

$$A <_{k,\Vdash} B \Leftrightarrow \text{for some } k' \leq k,\ k' \Vdash A \text{ and } k' \nVdash B$$
$$A \leq_{k,\Vdash} B \Leftrightarrow k \Vdash A \text{ and for all } k' \leq k,\ (k' \Vdash B \Rightarrow k' \Vdash A)$$
$$A \equiv_{k,\Vdash} B \Leftrightarrow A \leq_{k,\Vdash} B \text{ and } B \leq_{k,\Vdash} A \text{ or } k \nVdash A \text{ and } k \nVdash B.$$

We will often write $A <_k B$, $A \leq_k B$, and $A \equiv_k B$ if it is clear from the context which forcing relation is relevant. It is a trivial exercise to show that $\leq_k$ and $<_k$ satisfy the order axioms and Σ-persistency.

The lemma below establishes the property of the de Jongh orderings which makes them useful: they correspond to the minimal structure imposed upon the forcing of boxed formulas by the requirements of a Kripke model for R.

2.8 Lemma *Let $\preceq$ and $\prec$ be orderings on the boxed formulas of an adequate set S which satisfy the order axioms and Σ-persistency on $\langle K,<,\Vdash\rangle$ restricted to S. Let $\sim_{k,\Vdash}$ be the equivalence relation defined as follows:*

$$A \sim_{k,\Vdash} B \Leftrightarrow k \Vdash A \preceq B \text{ and } k \Vdash B \preceq A \text{ or } k \nVdash A \text{ and } k \nVdash B.$$

Then for all $A,B \in S$ the following hold:

(i) $A <_k B \Rightarrow k \Vdash A \prec B$
(ii) $k \Vdash A \preceq B \Rightarrow A \leq_k B$
(iii) *$\sim_k$ is a refinement of $\equiv_k$.*

Proof: (i) Suppose $A <_k B$, i.e., for some $k' \leq k$, $k' \Vdash A$ and $k' \nVdash B$. Then $k' \Vdash A \wedge \neg B$. This implies $k' \Vdash A \prec B$ and, by Σ-persistency, $k \Vdash A \prec B$.

(ii) Suppose $k \Vdash A \preceq B$. Then $k \Vdash A$ and for all $k' \leq k$ $(k' \Vdash B \Rightarrow k' \Vdash A)$, for suppose for some $k' \leq k$ that $k' \Vdash B$ and $k' \nVdash A$, whence $k' \Vdash B \prec A$ and, by Σ-persistency, $k \Vdash B \prec A$, which yields a contradiction.

(iii) follows directly from (ii).

2.9 Extension Lemma
(i) *A tail model restricted to an adequate set S can be extended to a tail model for* R.

(ii) *A finite Kripke tree restricted to an adequate set S can be extended to a finite Kripke tree for* R.

Proof: (i) Let $\langle K,<,\Vdash\rangle$ be a tail model restricted to an adequate set S. Define a forcing relation $\Vdash'$ inductively as follows:

$k \Vdash' p_i \Leftrightarrow k \Vdash p_i$
the usual clauses for $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\Box$

$$k \Vdash' A \leq B \Leftrightarrow \begin{cases} k \Vdash A \leq B, \text{ if } A, B \in S \\ A <_{k,\Vdash'} B, \text{ if } A \notin S, B \in S \\ A \leq_{k,\Vdash'} B, \text{ if } B \notin S \end{cases}$$

$$k \Vdash' A < B \Leftrightarrow \begin{cases} k \Vdash A < B, \text{ if } A, B \in S \\ A \leq_{k,\Vdash'} B, \text{ if } A \in S, B \notin S \\ A <_{k,\Vdash'} B, \text{ if } A \notin S. \end{cases}$$

Notice that our use of the de Jongh orderings relative to $\Vdash'$ on the right is possible since $A <_{k,\Vdash'} B$ and $A \leq_{k,\Vdash'} B$ are defined whenever $\Vdash'$ is defined on A and B.

It is easy to verify that $\Vdash'$ extends $\Vdash$, that the forcing of witness comparison formulas is persistent, and that each instance of an order axiom is forced at each node. See below for an attempt to visualize the kind of ordering of boxed formulas the above definition of $\Vdash'$ produces.

The proof of (ii) is similar.

The rationale for the particular extension defined in the proof above can be extracted from the following considerations.

Suppose $\langle K,<,\Vdash\rangle$ is a tail model restricted to an adequate set S. If we want to extend $\Vdash$ to a forcing relation $\Vdash'$ defined on all formulas of $\mathcal{L}^+$, we have to choose which new comparison formulas are forced in which nodes, in such a way that the order axioms and Σ-persistency are satisfied. Σ-persistency is equivalent to the requirement that the choice of the order in k of boxed formulas which are forced in a node $k' < k$ has to agree with the choice made in k'. From Lemma 2.8(i) it follows that if $\Box A$ is forced in a node $k' < k$ and there is no $k' < k$ in which $\Box B$ is forced, then $\Vdash'$ has to satisfy $k \Vdash' \Box A < \Box B$. That means that we only have to choose, for each node k, the order of formulas within the $\equiv_{k,\Vdash'}$-equivalence class F_k of boxed formulas forced "for the first time" in k and the $\equiv_{k,\Vdash'}$-equivalence class N_k of boxed formulas not forced in k.

In view of the meaning of the order axioms it is easy to see (though somewhat tedious to prove) that these axioms are satisfied iff in addition to Σ-persistency the following conditions hold for all k:

- if $A \in N_k$, then $k \nVdash' A < B$ and $k \nVdash' A \leq B$ (this implies that all formulas (old and new) in N_k are $\sim_{k,\Vdash'}$-equivalent)
- if $A \in F_k$ and $B \in F_{k'}$ for some $k' < k$, then $k \Vdash' B < A$
- the $\sim_{k,\Vdash'}$-equivalence classes within F_k form a linear ordering.

The situation may be illustrated thus: Consider a tail model with 21 as top tail-element and with $0 < \ldots < 15 < 14 < 13 < 8 < 5$ as the chain of predecessors of 3. Then the ordering of $\equiv_{3,\Vdash}$-equivalence classes in 3 is pictured by:

$$[\quad]_{F_0} \cdots [\quad]_{F_{14}} \; [\quad]_{F_{13}} \; [\quad]_{F_8} \; [\quad]_{F_5} \; [\quad]_{F_3} \; [\quad]_{N_3}.$$

Notice that this ordering is independent of the forcing relation and only depends on the structure of the tail model below the node 3. Of course, the content of each class F_k does depend on the forcing relation.

If we represent a $\sim_{3,\Vdash}$-equivalence class of old formulas by $\bullet$, we get the following kind of picture of the ordering of those equivalence classes:

$$[\bullet\bullet]_{F_0} \cdots [\quad]_{F_{14}} \; [\quad]_{F_{13}} \; [\bullet\bullet\bullet]_{F_8} \; [\bullet]_{F_5} \; [\bullet\bullet]_{F_3} \; [\bullet]_{N_3}.$$

We now have to fill each class F_k with new boxed formulas such that the relative ordering of old formulas is respected and the $\sim_{k,\Vdash'}$-equivalence classes within F_k form a linear ordering. Thus, if we represent a $\sim_{3,\Vdash'}$-equivalence class by $\circ$, then the situation within each class F_k in the extended model will in general look something like $[\odot\odot\bullet\odot\bullet]$. (E.g., the occurrence of $\text{\textcircled{$\cdot\cdot$}}$ is excluded.) A possible choice is to put all new formulas of F_k into one $\sim_{k,\Vdash'}$-equivalence class and place this class at the end of the linear ordering of old $\sim_{k,\Vdash}$-equivalence classes within F_k (if there are any). In our example this choice has the following result:

$$[\bullet'\bullet'\circ]_{F_0} \cdots [\circ]_{F_{14}} \; [\circ]_{F_{13}} \; [\bullet'\bullet'\bullet'\circ]_{F_8} \; [\bullet'\circ]_{F_5} \; [\bullet'\bullet'\circ]_{F_3} \; [\odot]_{N_3}$$

(where $\bullet'$ represents a $\sim_{3,\Vdash'}$-equivalence class not containing any new formulas.)

The proof of the Extension Lemma shows that this method can be realized by means of an inductive definition of the extended forcing relation which exploits the properties of the de Jongh orderings.

2.10 Theorem *For any sentence A of $\mathcal{L}^+$, the following are equivalent:*

(i) $\mathrm{R}^- \vdash A$
(ii) *A is valid on all finite Kripke trees*
(iii) *A holds at the root of every finite Kripke tree.*

Proof: (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious. To prove (iii) $\Rightarrow$ (i), assume that $\mathrm{R}^- \nvdash A$. Let $D_0, \ldots, D_n$ be the formulas of S_A with principal connective $<$ or $\leq$ and let $p_0, \ldots, p_n$ be distinct atoms of $\mathcal{L}^+$ not occurring in S_A. For any B from S_A, let B' be the formula in $\mathcal{L}$ such that $B = B'[D_i/p_i]$. Define $X = \{\boxed{s}(p_i \to \Box p_i) : 0 \leq i \leq n\} \cup \{\boxed{s}B' : B \text{ is an order axiom involving only formulas of } S_A\}$. Then $\mathrm{L} \nvdash \bigwedge X \to A'$. By Theorem 1.2, there is a finite, transitive, irreflexive (tree-ordered) Kripke model for L $\langle K,<,\Vdash\rangle$ such that $\bigwedge X \wedge \neg A'$ is true at its root.

Define $\Vdash'$ on S_A by $k \Vdash' B$ iff $k \Vdash B'$, for any B from S_A. Notice that $\bigwedge X$ is actually valid on $\langle K,<,\Vdash\rangle$, so $\langle K,<,\Vdash'\rangle$ is a finite Kripke tree restricted to S_A whose root satisfies $\neg A$. By the Extension Lemma $\langle K,<,\Vdash'\rangle$ can be extended to a finite Kripke tree whose root does not satisfy A.

The completeness proof for R is now very simple on account of the following lemma.

2.11 Lemma (de Jongh) R $\vdash A$ *iff for some* $n \in \mathbb{N}$, $\mathrm{R}^- \vdash \Box^n A$.

Proof: From right to left: Apply the rule $\Box A/A$ n times. For the other direction, apply induction to the length of the proof of A:

- For all axioms of R^- the result is trivial.
- (modus ponens) Assume that $\mathrm{R} \vdash B$ and $\mathrm{R} \vdash B \to A$. Then, by the induction hypothesis, $\mathrm{R}^- \vdash \Box^n B$ and $\mathrm{R}^- \vdash \Box^m(B \to A)$. Then we have $\mathrm{R}^- \vdash \Box^{\max(n,m)} B$ and $\mathrm{R}^- \vdash \Box^{\max(n,m)}(B \to A)$. So we conclude that $\mathrm{R}^- \vdash \Box^{\max(n,m)} A$.
- (necessitation) Assume that $A \equiv \Box B$ and $\mathrm{R} \vdash B$. Then, by the induction hypothesis, $\mathrm{R}^- \vdash \Box^n B$, for some $n \in \mathbb{N}$. So by necessitation, $\mathrm{R}^- \vdash \Box^n A$.
- ($\Box$A/A) Assume that $\mathrm{R} \vdash \Box A$. Then, by the induction hypothesis, $\mathrm{R}^- \vdash \Box^n \Box A$ for some $n \in \mathbb{N}$. So $\mathrm{R}^- \vdash \Box^{n+1} A$.

It is easy to see that, as remarked before, tail models and all their finite top parts are automatically A-sound for any formula A. The following lemma establishes that the validity of a formula A on all tail models is equivalent to the validity of A on all A-sound finite Kripke trees.

2.12 Prolongation Lemma
(i) *An A-sound finite Kripke tree restricted to S_A on which A is not valid can be prolonged to a tail model on which A is not valid*
(ii) *If A does not hold at the root of an A-sound finite Kripke tree restricted to S_A, then there is a tail model whose root does not satisfy A.*

Proof: (i) Suppose $\langle K,<,\Vdash\rangle$ is an A-sound finite Kripke tree restricted to S_A. Attach an $\omega + 1$-tail $k_{m+1}, k_{m+2}, \ldots, k_0$ to the bottom k_m of $\langle K,<,\Vdash\rangle$, while defining forcing for formulas from S_A on elements of the tail exactly as on k_m (this definition is possible by A-soundness). This results in a tail model restricted to S_A on which A is not valid. By the Extension Lemma, this tail model can be extended to a tail model for R on which A is not valid.

(ii) Notice that in the proof above $k_0 \nVdash A$ if $k_m \nVdash A$.

2.13 Theorem (Kripke model completeness)
(i) R $\vdash A$ *iff A is valid in all tail models K*
(ii) $\mathrm{R}^\omega \vdash A$ *iff* $0 \Vdash A$ *for all tail models K.*

Proof: (i) "if": Assume $\mathrm{R} \nvdash A$. Then by Lemma 2.11, for all $n \in \mathbb{N}$, $\mathrm{R}^- \nvdash \Box^n A$. In particular, $\mathrm{R}^- \nvdash \Box^n A$ with n greater than the number of subformulas of A. By Theorem 2.10, there exists a finite Kripke tree $K = \langle K,<,\Vdash\rangle$ with bottom node k_0 on which $\Box^n A$ is not valid. But then A is false at a node $k_n \in K$ such that there is a chain $k_0 < k_1 < k_2 < \ldots < k_n$ in K. By Σ-persistency, there exists an $m \leq n$ such that k_m and k_{m+1} force the same boxed formulas from S_A. But then $k_{m+1} \Vdash \Box B \to B$, for all $B \in S_A$, since $k_{m+1} \Vdash \Box B$ implies $k_m \Vdash \Box B$, which in turn implies $k_{m+1} \Vdash B$. By the Prolongation Lemma, the subtree generated by k_{m+1} can be prolonged to a tail model on which A is not valid.

"only if": Assume that K is a tail model on which A is not valid. Then for some $k \neq 0$, $k \nVdash A$. But then for each $n \in \mathbb{N}$ there exists a finite Kripke tree on which $\Box^n A$ is not valid, viz. the finite Kripke tree generated by the node $k_n <$ k with $\mathrm{d}(k_n) = \mathrm{d}(k) + n$. Therefore, for all $n \in \mathbb{N}$, $\mathrm{R}^- \nvdash \Box^n A$. So, by Lemma 2.11, $\mathrm{R} \nvdash A$.

(ii) "only if":

- 0 satisfies the theorems of R^-
- closure under modus ponens is trivial
- suppose $0 \Vdash \Box A$. Then, for all $k > 0$, $k \Vdash A$, hence for all $k \neq 0$, $k \Vdash A$. So, by the Tail Lemma, $0 \Vdash A$.

"if": Suppose $\mathrm{R}^\omega \nvdash A$. Then $\mathrm{R}^- \nvdash \bigwedge\!\!\!\bigwedge_{B \in S(A)} (\Box B \to B) \to A$. Hence there is a finite Kripke tree $K = \langle K, <, \Vdash \rangle$ such that the bottom node k of K satisfies $k \Vdash \bigwedge\!\!\!\bigwedge_{B \in S(A)} (\Box B \to B)$ and $k \nVdash A$. By the Prolongation Lemma, K can be extended to a tail model $K' = \langle \omega, <, \Vdash' \rangle$ with $0 \nVdash' A$.

2.14 Remark It follows from the proof of Theorem 2.13 that we may assume that the forcing of witness comparison formulas in a tail model is a primitive recursive function of nodes and boxed formulas, since if $\mathrm{R} \nvdash A$ we can construct a tail model in which A is not valid as follows:

(1) we start with a finite Kripke tree restricted to S_A
(2) we take the subtree generated by k_m as specified in the proof of 2.13
(3) we prolong this subtree to a tail model.

Clearly (3) is the only step which might be problematic, since in the proof of the Prolongation Lemma the Extension Lemma is used and there are clauses in the definition of the extended forcing relation at k which depend on the forcing relation at the nodes $k' < k$.

However, it is possible to give for each formula A a number *sd(A), the search depth of A*, such that if $\mathrm{d}(k) = \mathrm{sd}(A)$, then for all $k' < k$, $k' \Vdash A$ iff $k \Vdash A$. Therefore for each formula A we only have to search at most $\mathrm{sd}(A)$ nodes below k to know in which nodes below k A is forced.

$\mathrm{sd}(A)$ is defined as follows:

if A is atomic or $A \in S_A$: $\mathrm{sd}(A) = M$, with $M =$ the depth of the top tail-element;
if $A \notin S_A$: $\mathrm{sd}(A) = \mathrm{sd}(B)$, if $A = \neg B$
$\mathrm{sd}(A) = \max\{\mathrm{sd}(B), \mathrm{sd}(C)\}$, if $A = B \circ C$, with $\circ \in \{\wedge, \vee, \to, \leq, <\}$
$\mathrm{sd}(A) = 1 + \mathrm{sd}(B)$, if $A = \Box B$.

3 Arithmetical completeness While attempting to define what an arithmetical interpretation for $\mathcal{L}^+$ would be one encounters the following difficulty: circumstances extrinsic to the nature of the proof predicate $\mathrm{Pr}(x)$ decide whether, e.g., $\mathrm{Pr}(\ulcorner \underline{0} = \underline{0} \urcorner) < \mathrm{Pr}(\ulcorner \underline{1} = \underline{1} \urcorner)$ or $\mathrm{Pr}(\ulcorner \underline{1} = \underline{1} \urcorner) < \mathrm{Pr}(\ulcorner \underline{0} = \underline{0} \urcorner)$. To eliminate the arbitrariness that would result from fixing a particular proof predicate, Guaspari and Solovay allow arithmetical interpretations to be based on any "reasonable" proof predicate, i.e., any Σ_1^0-numeration of the theorems of PA in PA satisfying the derivability conditions and demonstrable Σ-completeness. Such a reasonable proof predicate is called "*standard*".

Remark Guaspari and Solovay actually prove that it is sufficient to allow interpretations to be based on Σ_1^0-formulas provably equivalent to $\Pr(x)$. In [4] "standard" refers to this kind of proof predicate.

An alternative way to abstract from accidental circumstances, and the one we are going to follow, is to fix a particular proof predicate $\Pr(x)$ and allow arithmetical interpretations to be based on reasonable (pre-)permutations of the natural numbers, where a (pre-)permutation is called reasonable if $\Pr(x)$ still satisfies demonstrable Σ-completeness when the witness comparison is based on the ordering of the (pre-)permuted natural numbers. In other words, we allow arithmetical interpretations to be based on any (pre-)ordering of the natural numbers such that $\Pr(\ulcorner A \urcorner) \circ \Pr(\ulcorner B \urcorner)$, with $\circ$ a connective for witness comparison based on the new ordering, is Σ_1^0. At the end of this section it will be shown that for our present purposes the latter alternative is to be preferred.

3.1 Definition Let $\leq^*$ be a weak pre-ordering on $\mathbb{N}$ and $<^*$ its associated strict pre-ordering such that $\Pr(x)$ satisfies demonstrable Σ-completeness when witness comparison is based on $\leq^*$ and $<^*$. An *arithmetical interpretation* * *of* $\mathcal{L}^+$ *based on* $\leq^*$ is a map from formulas of $\mathcal{L}^+$ to sentences of PA satisfying:

- for all atomic p, p^* is an arithmetical sentence
- $\perp^* = \underline{0} = \underline{1}$; $\top^* = \underline{0} = \underline{0}$
- $(\ )^*$ commutes with Boolean connectives
- $(\Box A)^* = \Pr(\ulcorner A^* \urcorner)$
- $(\Box A \leq \Box B)^* = (\Box A)^* \leq^* (\Box B)^*$, where $\exists x\, \varphi(x) \leq^* \exists x\, \psi(x) = \exists x(\varphi(x) \wedge \forall y <^* x\, \neg\psi(y))$
- $(\Box A < \Box B)^* = (\Box A)^* <^* (\Box B)^*$, where $\exists x\, \varphi(x) <^* \exists x\, \psi(x) = \exists x(\varphi(x) \wedge \forall y \leq^* x\, \neg\psi(y))$.

One easily verifies:

3.2 Theorem (Soundness)
(i) $\mathrm{R} \vdash A \Rightarrow$ *for all* * $\mathrm{PA} \vdash A^*$
(ii) $\mathrm{R}^\omega \vdash A \Rightarrow$ *for all* * A^* *is true.*

To prove the reverse implications, we define, for any tail model K and any formula A, a representation $[A]_K$ of the set of nodes in K in which A is forced, and an arithmetical interpretation $\langle A \rangle_K$ such that the following hold:

(i) $\mathrm{PA} \vdash [A]_K \leftrightarrow \langle A \rangle_K$
(ii) if A is not valid in K, $\mathrm{PA} + \neg[A]_K$ is consistent.

Fix a tail model $K = \langle \omega, <, \Vdash \rangle$.

3.3 Definition $[\![A]\!]_K = \{k \in \omega : k \Vdash A\}$. Define as a formula of the language of arithmetic:

$$x \in [\![A]\!]_K = \begin{cases} \bigvee\{x = i : i \Vdash A\}, \text{ if } [\![A]\!]_K \text{ is finite} \\ \bigwedge\{x \neq i : i \nVdash A\}, \text{ if } [\![A]\!]_K \text{ is cofinite.} \end{cases}$$

By convention, the empty disjunction is $\perp$ and the empty conjunction is $\top$.

It is easy to see that the following holds:

3.4 Lemma
(i) $\text{PA} \vdash x \in [\![A]\!]_K \wedge x \in [\![B]\!]_K \leftrightarrow x \in [\![A \wedge B]\!]_K$
(ii) $\text{PA} \vdash x \in [\![A]\!]_K \vee x \in [\![B]\!]_K \leftrightarrow x \in [\![A \vee B]\!]_K$
(iii) $\text{PA} \vdash x \notin [\![A]\!]_K \leftrightarrow x \in [\![\neg A]\!]_K$.

3.5 Definition $l = \lim_{n\to\infty} \text{h(n)}$, where h is defined as follows:

$$h(0) = 0$$

$$h(k+1) = \begin{cases} n, \text{ if for some } n > h(k), \text{Proof}(k+1, \ulcorner l \neq n \urcorner) \\ h(k), \text{ otherwise.} \end{cases}$$

The following lemma lists some properties of h and l. Proofs can be found in [5].

3.6 Lemma
(i) PA $\vdash$ "*h is weakly monotonic in* $<$"
(ii) PA $\vdash$ "*l exists*"
(iii) PA $\vdash$ "*if $l = i$ and $i < j$, then* PA + "$l = j$" *is consistent*"
(iv) PA $\vdash$ "*if $l = i$ and $i \nless j$, then PA* $\vdash$ "$l \neq j$""
(v) $l = 0$
(vi) for all i, PA + "$l = i$" is consistent.

Remark The arguments for (v) and (vi) cannot be formalized in PA.

Now we are able to define a representation $[A]_K$ of $[\![A]\!]_K$ in PA such that PA + $\neg [A]_K$ is consistent if and only if A is not valid in K:

3.7 Definition $[A]_K = l \in [\![A]\!]_K$.

Our next problem is to find an ordering $\leq_K$ on N such that there exists an arithmetical interpretation $\langle\ \rangle_K$ based on $\leq_K$ satisfying $\text{PA} \vdash \langle A\rangle_K \leftrightarrow [A]_K$. That means that we have as constraints on $\langle\ \rangle_K$:

- $\text{PA} \vdash [\Box A < \Box B]_K \leftrightarrow \exists x(\text{Proof}(x, \ulcorner\langle A\rangle_K\urcorner) \wedge \forall y \leq_K x\, \neg\text{Proof}(y, \ulcorner\langle B\rangle_K\urcorner))$
- $\text{PA} \vdash [\Box A \leq \Box B]_K \leftrightarrow \exists x(\text{Proof}(x, \ulcorner\langle A\rangle_K\urcorner) \wedge \forall y <_K x\, \neg\text{Proof}(y, \ulcorner\langle B\rangle_K\urcorner))$.

Let $\text{Minproof}(x,n) :\leftrightarrow \text{Proof}(x,n) \wedge \forall y \leq x\, \neg\text{Proof}(y,n)$ and define MINPROOFS $= \{x : \text{Minproof}(x, \ulcorner\langle A\rangle_K\urcorner), \text{ for some } A\}$. $\langle A\rangle_K$ is of course dependent on $\leq_K$, the ordering we still have to define. But because $\langle A\rangle_K$ occurs at the level of *codes*, we can implement this innocent circularity by the recursion theorem.

We obtain our new ordering $\leq_K$ by pushing the elements of MINPROOFS exactly as much forward as is needed to satisfy the above constraints. That is, we change the order of m and a larger n if and only if n is a minimal proof of $\langle A\rangle_K$ and for some i, $h(i) \Vdash \Box A \leq \Box B$ for some B such that either m or, if $m \notin$ MINPROOFS, some $p < m$ is a minimal proof of $\langle B\rangle_K$.

If for some i, $h(i) \Vdash \Box A \leq \Box B \wedge \Box B \leq \Box A$, and m and n are minimal proofs of $\langle A\rangle_K$ and $\langle B\rangle_K$ respectively, then m and n will be equivalent in our new ordering. We therefore discriminate between pairs which are weakly permuted in our new ordering and those which are strictly permuted. By the recursion theorem we can define the following permutation relations of pairs which are weakly, respectively strongly, permuted:

$\mathrm{WP}_K(m,n) \Leftrightarrow m < n$, $\mathrm{Minproof}(n, \ulcorner\langle A\rangle_K\urcorner)$, and for some i, $h(i) \Vdash \Box A \leq \Box B$ for some B, such that $\mathrm{Minproof}(m, \ulcorner\langle B\rangle_K\urcorner)$ or, if $m \notin$ MINPROOFS, $\mathrm{Minproof}(p, \ulcorner\langle B\rangle_K\urcorner)$, for some $p < m$.

$\mathrm{SP}_K(m,n) \Leftrightarrow m < n$, $\mathrm{Minproof}(n, \ulcorner\langle A\rangle_K\urcorner)$, and for some i, $h(i) \Vdash \Box A < \Box B$ if $\mathrm{Minproof}(m, \ulcorner\langle B\rangle_K\urcorner)$ and $h(i) \Vdash \Box A \leq \Box B$ if $m \notin$ MINPROOFS, and for some $p < m$, $\mathrm{Minproof}(p, \ulcorner\langle B\rangle_K\urcorner)$.

Notice that WP_K and SP_K are recursive. For instance,

$\neg\mathrm{WP}_K(m,n) \Leftrightarrow m \geq n$ or $n \notin$ MINPROOFS or ($m < n$ and $\mathrm{Minproof}(n, \ulcorner\langle A\rangle_K\urcorner)$ and for some i, $h(i) \Vdash \Box B < \Box A$ for some B, such that $\mathrm{Minproof}(m, \ulcorner\langle B\rangle_K\urcorner)$ or, if $m \notin$ MINPROOFS, $\mathrm{Minproof}(p, \ulcorner\langle B\rangle_K\urcorner)$, for some $p < m$).

The fact that we restrict our action to the set of *minimal* proofs is not relevant to our present purposes. This restriction is used in [10] to derive a result of Tuttas [7]. Let us now give the formal definition of $\leq_K$ and $\langle\ \rangle_K$:

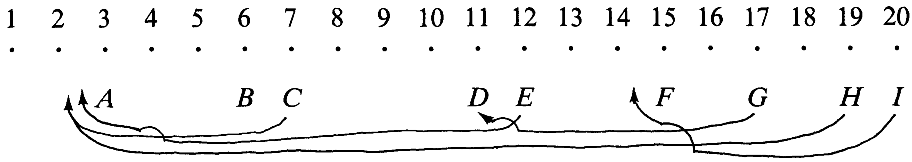
3.8 Definition

(i) $n \leq_K m \Leftrightarrow \mathrm{WP}_K(m,n)$ or ($n \leq m$ and not $\mathrm{SP}_K(n,m)$)
(ii) $n <_K m \Leftrightarrow \mathrm{SP}_K(m,n)$ or ($n < m$ and not $\mathrm{WP}_K(n,m)$)
(iii) $\langle\ \rangle_K$ is the arithmetical interpretation based on $\leq_K$ induced by:

- $\langle p_i\rangle_K = l \in [\![p_i]\!]_K \wedge \underline{i} = \underline{i}$
- $\langle \Box A < \Box B\rangle_K = \langle \Box A\rangle_K <_K \langle \Box B\rangle_K$, where $\exists x\, \varphi(x) <_K \exists x\, \psi(x) = \exists x(\varphi(x) \wedge \forall y \leq_K x\, \neg\psi(y))$
- $\langle \Box A \leq \Box B\rangle_K = \langle \Box A\rangle_K \leq_K \langle \Box B\rangle_K$, where $\exists x\, \varphi(x) \leq_K \exists x\, \psi(x) = \exists x(\varphi(x) \wedge \forall y <_K x\, \neg\psi(y))$.

In the appendix it is proved that $\leq_K$ is a weak pre-ordering and $<_K$ is its associated strict pre-ordering.

Example Suppose for some i that $h(i) \Vdash \Box H \sim \Box C < \Box E < \Box A < \Box B < \Box D \sim \Box G < \Box I < \Box F$ and suppose that $\mathrm{Minproof}(n, \ulcorner\langle\Phi\rangle_K\urcorner)$ iff Φ is appended to n in the following picture:



Then (1,2,...,20) gets permuted into (1,2,[7,19],12,3,4,5,6,8,9,10,[11,17],13,14,20,15,16,18).

We admit that it is not immediately clear that $\langle\ \ \rangle_K$ is indeed an arithmetical interpretation. However, before verifying the Σ-completeness for witness comparison based on $\leq_K$, it will be convenient to show the provable equivalence of $[A]_K$ and $\langle A\rangle_K$:

3.9 Theorem $\quad \mathrm{PA} \vdash [A]_K \leftrightarrow \langle A\rangle_K$.

Proof: By induction on A.

- The cases of atoms and Boolean connectives are easy (use PA $\vdash$ "l exists" and Lemma 3.4).
- Suppose $A \equiv \Box B$.

  (a) In case $[\![A]\!]_K$ is cofinite we have, by the Tail Lemma, that $[\![\Box B]\!]_K = [\![B]\!]_K = \omega$. By the induction hypothesis $\mathrm{PA} \vdash \top \leftrightarrow \langle B\rangle_K$. Hence $\mathrm{PA} \vdash \langle B\rangle_K$ which implies $\mathrm{PA} \vdash \mathrm{Pr}(\ulcorner\langle B\rangle_K\urcorner)$, i.e., $\mathrm{PA} \vdash \langle\Box B\rangle_K \leftrightarrow \top$. So we conclude that $\mathrm{PA} \vdash [\Box B]_K \leftrightarrow \langle\Box B\rangle_K$.

  (b) Suppose A_K is finite. Let $j_0,\ldots,j_s$ be all the j such that $j \Vdash \Box B$ and $j \nVdash B$. Notice that for each i with $i \nVdash \Box B$ there is a j_k such that $i < j_k$. Clearly it is sufficient to prove that $\mathrm{PA} \vdash \mathrm{Pr}(\ulcorner[B]_K\urcorner) \leftrightarrow [\Box B]_K$. To prove this, reason in PA as follows:

  "$\rightarrow$": Suppose $\mathrm{Pr}(\ulcorner[B]_K\urcorner)$. Let j be any element of $\{j_0,\ldots,j_s\}$. By the definition of $[\ \ ]_K$ we have $\mathrm{Pr}(\ulcorner l \neq j\urcorner)$. Assume $\mathrm{Proof}(p+1, \ulcorner l \neq j\urcorner)$. Suppose $l < j$, then $h(p) < j$. But then $h(p+1) = j$, which yields a contradiction. So we have $l \not< j$, through which we obtain $\mathbb{W}\{l = i : i \Vdash \Box B\}$.

  "$\leftarrow$": Suppose $l = i$ and $i \Vdash \Box B$. Since $[\![\Box B]\!]_K$ is supposed to be finite, $0 \nVdash \Box B$ and thus $i \neq 0$. So, by the definition of h, $\mathrm{Pr}(\ulcorner l \neq i\urcorner)$. Moreover, since for some x, $h(x) = i$, by Σ_1^0-completeness $\mathrm{Pr}(\ulcorner\exists x\, h(x) = i\urcorner)$. Hence $\mathrm{Pr}(\ulcorner l > i\urcorner)$. Since $j > i$ implies $j \Vdash B$, we have $\mathrm{Pr}(\ulcorner\mathbb{W}\{l = j : j \Vdash B\}\urcorner)$.

- Suppose $A \equiv \Box B \leq \Box C$. Reason in PA as follows:

  "$\leftarrow$": Suppose $\exists x(\mathrm{Proof}(x, \ulcorner\langle B\rangle_K\urcorner) \wedge \forall y <_K x\, \neg\mathrm{Proof}(y, \ulcorner\langle C\rangle_K\urcorner))$. Then $\mathrm{Pr}(\ulcorner\langle B\rangle_K\urcorner)$ and, by the induction hypothesis, for some i, $h(i) \Vdash \Box B$. But then $h(i) \Vdash (\Box B \leq \Box C) \vee (\Box C < \Box B)$. Suppose $h(i) \Vdash \Box C < \Box B$. Then $h(i) \Vdash \Box C$ and thus for some y $\mathrm{Minproof}(y, \ulcorner\langle C\rangle_K\urcorner)$. But then $y <_K x$ for all x such that $\mathrm{Proof}(x, \ulcorner\langle B\rangle_K\urcorner)$, so we would have for all x $(\mathrm{Proof}(x, \ulcorner\langle B\rangle_K\urcorner) \rightarrow \exists y <_K x\, \mathrm{Proof}(y, \ulcorner\langle C\rangle_K\urcorner))$, which yields a contradiction. We conclude that $h(i) \Vdash \Box B \leq \Box C$, so by Σ-persistency, $l \in [\![A]\!]_K$.

"$\rightarrow$": Suppose $l \in [\![\Box B \leq \Box C]\!]_K$. Then for some i, $h(i) \Vdash \Box B \leq \Box C$, and thus $h(i) \Vdash \Box B$. By the induction hypothesis, for some x $\mathrm{Minproof}(x, \ulcorner \langle B \rangle_K \urcorner)$. Suppose $\exists y <_K x\ \mathrm{Proof}(y, \ulcorner \langle C \rangle_K \urcorner)$, then for some j, $h(j) \Vdash \Box C < \Box B$, which yields a contradiction. So we have for some x $(\mathrm{Proof}(x, \ulcorner \langle B \rangle_K \urcorner) \wedge \forall y <_K x\ \neg \mathrm{Proof}(y, \ulcorner \langle C \rangle_K \urcorner))$.

- The case of $A = \Box B < \Box C$ is similar to the one above.

3.10 Definition Σ_K is the set of arithmetical formulas which consists of
- $\mathrm{Pr}(\ulcorner A \urcorner)$, for some arithmetical formula A
- $\mathrm{Pr}(\ulcorner A \urcorner) <_K \mathrm{Pr}(\ulcorner B \urcorner)$, for arithmetical formulas A and B
- $\mathrm{Pr}(\ulcorner A \urcorner) \leq_K \mathrm{Pr}(\ulcorner B \urcorner)$, for arithmetical formulas A and B.

3.11 Lemma (Σ-completeness) *Any formula from Σ_K is provably equivalent to a Σ_1^0-formula.*

Proof:

- $A = \mathrm{Pr}(\ulcorner B \urcorner)$: Trivial.
- $A = \mathrm{Pr}(\ulcorner B \urcorner) <_K \mathrm{Pr}(\ulcorner C \urcorner)$, i.e., $A = \exists x (\mathrm{Proof}(x, \ulcorner B \urcorner) \wedge \forall y \leq_K x\ \neg \mathrm{Proof}(y, \ulcorner C \urcorner))$: Reason in PA. Define $I = \{x : x = \ulcorner \langle A \rangle_K \urcorner$, for some $A\}$.

  Case 1: $\ulcorner B \urcorner, \ulcorner C \urcorner \notin I$: Then by the definition of $\leq_K$, $A \leftrightarrow \exists x (\mathrm{Proof}(x, \ulcorner B \urcorner) \wedge \forall y \leq x\ \neg \mathrm{Proof}(y, \ulcorner C \urcorner))$.

  Case 2: $\ulcorner B \urcorner = \ulcorner \langle D \rangle_K \urcorner, \ulcorner C \urcorner = \ulcorner \langle E \rangle_K \urcorner$: Then by Theorem 3.9, $A \leftrightarrow [\Box D < \Box E]_K$.

  Case 3: $\ulcorner B \urcorner = \ulcorner \langle D \rangle_K \urcorner, \ulcorner C \urcorner \notin I$: Let $m(0) = 0$ and, for $x \neq 0$, $m(x) = \max\{y : y < x$ and not $\mathrm{WP}_K(y,x)\}$. Then m is recursive and by the definition of $\leq_K$, $A \leftrightarrow \exists x (\mathrm{Proof}(x, \ulcorner B \urcorner) \wedge \forall y \leq m(x)\ \neg \mathrm{Proof}(y, \ulcorner C \urcorner))$.

  Case 4: $\ulcorner B \urcorner \notin I$, $\ulcorner C \urcorner = \ulcorner \langle E \rangle_K \urcorner$: Then by the definition of $\leq_K$, $A \leftrightarrow \exists x (\mathrm{Proof}(x, \ulcorner B \urcorner) \wedge \forall y \leq x\ \neg (\mathrm{Proof}(y, \ulcorner C \urcorner) \vee (\exists i\, h(i) \Vdash \Box E \leq \Box D \wedge \mathrm{Proof}(y, \ulcorner \langle D \rangle_K \urcorner)))$.
- $A = \mathrm{Pr}(\ulcorner B \urcorner) \leq_K \mathrm{Pr}(\ulcorner C \urcorner)$: Similar.

For the reader's convenience, we give the informal considerations leading to the particular Σ_1^0-equivalents of A given in Cases 3 and 4 of the above proof. In Case 3 we know that proofs of B may have been pushed forward, but proofs of C not. Therefore we only have to check whether there is a proof y of C in the set of numbers z below the minimal proof x of B such that not $\mathrm{WP}_K(z,x)$. In Case 4 we know that proofs of C may have been pushed forward, but proofs of B not. So we now have to check whether there is a proof of C below the minimal proof x of B or whether the minimal proof of C is pushed before x, i.e., whether there is a proof of some $\langle D \rangle_K$ below x such that for some i, $h(i) \Vdash \Box E \leq \Box D$.

Now we can easily derive the following:

3.12 Theorem (Arithmetical completeness)
(i) $\mathrm{R} \vdash A$ *iff for all* * $\mathrm{PA} \vdash A^*$
(ii) $\mathrm{R}^\omega \vdash A$ *iff for all* * A^* *is true.*

Proof: By Theorem 3.2 we only have to prove the implications from the right to the left.

(i) Suppose $\mathrm{R} \nvdash A$. Then for some tail model K, A is not valid in K. Therefore $\mathrm{PA} + \neg [A]_K$ is consistent. By Theorem 3.9, $\mathrm{PA} + \neg\langle A\rangle_K$ is consistent, so we conclude that $\mathrm{PA} \nvdash \langle A\rangle_K$.

(ii) Suppose $\mathrm{R}^\omega \nvdash A$. Then $0 \not\Vdash A$, for some tail model K. Since $l = 0$ is true, $[A]_K$ is not true and therefore $\langle A\rangle_K$ is not true.

Notice that we may assume that an arithmetical interpretation * is based on a recursive relation $\leq^*$.

As remarked before, our definition of arithmetical interpretation is different from that of Guaspari and Solovay. Therefore, Theorem 5.6 of [2] and our Theorem 3.12(i) differ somewhat in meaning, in spite of their identical appearance.

It is easy to see that if K is a tail model, there is no interpretation * in the sense of Guaspari and Solovay such that $\mathrm{PA} \vdash [A]_K \leftrightarrow A^*$, since in every tail model there is a top node k such that $k \Vdash \bigwedge_{1\leq n<\omega} (\Box^n\bot > \Box^{n+1}\bot)$, so $\mathrm{PA} + \bigwedge_{1\leq n<\omega} [\Box^n\bot > \Box^{n+1}\bot]_K$ is consistent.

So $\mathrm{PA} + \bigwedge_{1\leq n<\omega} ((\Box^n\bot)^* >^* (\Box^{n+1}\bot)^*)$ would have to be consistent, which is clearly impossible if we only allow interpretations to be based on enumerations of the theorems of PA. A similar argument shows that the representation of any kind of Kripke model for R which is really embeddable into PA cannot be provably equivalent to an arithmetical interpretation in the sense of Guaspari and Solovay. The above formed our motivation for a definition of arithmetical interpretation which allows interpretations corresponding to more flexible ways of mingling witnesses of theorems.

However, even if one is not concerned with the problem of really embedding Kripke models into PA, one might be interested in interpretations where the box is interpreted as the usual provability predicate, rather than as any standard provability predicate. In [4] Smoryński argues that since the exact order used in the comparisons is irrelevant to results like Rosser's Theorem, "any decent completeness result ought to be with respect to a variety of orderings", and he mentions the proof of such a result as an open problem. Theorem 3.12 forms a solution to this problem, although we actually proved the theorem before we learned of the problem. Independently, this problem has also been solved by Tuttas [7], who extended a partial result already obtained in [4].

Appendix Let $\leq_K$, $<_K$ and $\langle\ \rangle_K$ be defined as in Definition 3.8 and let $\mathrm{M}(n,A)$ abbreviate $\mathrm{Minproof}(n, \ulcorner\langle A\rangle_K\urcorner)$.

Claim *$\leq_K$ is a weak pre-ordering and $<_K$ is its associated strict pre-ordering.*

Proof:

- $n \leq_K n$, since $n \leq n$ and not $\mathrm{SP}_K(n,n)$.
- $n <_K m$ implies $n \leq_K m$, since $\mathrm{SP}_K(m,n)$ implies $\mathrm{WP}_K(m,n)$, $n < m$ implies $n \leq m$, and $\neg\mathrm{WP}_K(n,m)$ implies $\neg\mathrm{SP}_K(n,m)$.

- $\neg(m \leq_K n) \Rightarrow n <_K m$:
$\neg(m \leq_K n) \Leftrightarrow \neg\mathrm{WP}_K(n,m) \wedge \neg(m \leq n \wedge \neg\mathrm{SP}_K(m,n))$
$\Leftrightarrow (\neg\mathrm{WP}_K(n,m) \wedge n < m) \vee (\neg WP_K(n,m) \wedge \mathrm{SP}_K(m,n))$
$\Rightarrow (\neg\mathrm{WP}_K(n,m) \wedge n < m) \vee \mathrm{SP}_K(m,n)$
$\Leftrightarrow n <_K m$.
- $n \leq_K m$ and $m \leq_K p \Rightarrow n \leq_K p$: Assume that $n \leq_K m$ and $m \leq_K p$.
Case 1: $\mathrm{WP}_K(m,n)$ and $\mathrm{WP}_K(p,m)$: Then $m < n$ and $p < m$, thus $p < n$. Further, we have $\mathrm{M}(n,A)$, $\mathrm{M}(m,B)$, for some i, $h(i) \Vdash \Box A \leq \Box B$, and for some j, $h(j) \Vdash \Box B \leq \Box C$, for some C such that $\mathrm{M}(p,C)$ or, if $p \notin$ MINPROOFS, $\mathrm{M}(q,C)$ for some $q < p$. Thus for $i_0 = \max(i,j)$, $h(i_0) \Vdash \Box A \leq \Box C$, for some C such that $\mathrm{M}(p,C)$ or, if $p \notin$ MINPROOFS, $\mathrm{M}(q,C)$ for some $q < p$. Hence $\mathrm{WP}_K(p,n)$ and thus $n \leq_K p$.
Case 2: $n \leq m \wedge \neg\mathrm{SP}_K(n,m) \wedge m \leq p \wedge \neg\mathrm{SP}_K(m,p)$: Then obviously $n \leq p$. Suppose $\mathrm{SP}_K(n,p)$, then $n < m < p$ and $\mathrm{M}(p,C)$ and for some i, $h(i) \Vdash \Box C < \Box A$ if $\mathrm{M}(n,A)$, or $h(i) \Vdash \Box C \leq \Box A$ if $n \notin$ MINPROOFS and for some $q < n$, $M(q,A)$. If $\mathrm{M}(m,B)$, then, by $\neg\mathrm{SP}_K(m,p)$, for some i, $h(i) \Vdash \Box B \leq \Box C$, which contradicts $\neg\mathrm{SP}_K(n,m)$. If $m \notin$ MINPROOFS, then for some i, $h(i) \Vdash \Box C \leq \Box B$, for some B such that for some $r < m$, $\mathrm{M}(r,B)$ ($r = n$ if $n \notin$ MINPROOFS, $r = q$ otherwise). This contradicts $\neg\mathrm{SP}_K(m,p)$. Hence $n \leq p$ and not $\mathrm{SP}_K(n,p)$, thus $n \leq_K p$.
Case 3: $\mathrm{WP}_K(m,n) \wedge m \leq p \wedge \neg\mathrm{SP}_K(m,p)$:
Case 3a: $n = p$: Then $n \leq p$ and not $\mathrm{SP}_K(n,p)$.
Case 3b: $n < p$: Suppose $\mathrm{SP}_K(n,p)$, then for some i, $h(i) \Vdash \Box C < \Box A$, with $\mathrm{M}(n,A)$ and $\mathrm{M}(p,C)$. By $\mathrm{WP}_K(m,n)$, we have for some i, $h(i) \Vdash \Box C \leq \Box B$ for some B such that $\mathrm{M}(m,B)$ or, if $m \notin$ MINPROOFS, $\mathrm{M}(q,B)$ for some $q < m$. This contradicts $\neg\mathrm{SP}_K(m,p)$. Hence not $\mathrm{SP}_K(n,p)$.
Case 3c: $n > p$: If $p \notin$ MINPROOFS, then we have, by $\mathrm{WP}_K(m,n)$ and $m \leq p$, for some i, $h(i) \Vdash \Box A \leq \Box B$, for some B such that for some $q < p$, $\mathrm{M}(q,B)$, and for A such that $\mathrm{M}(n,A)$. If $\mathrm{M}(p,C)$ and $\mathrm{M}(n,A)$, then we have, by $\neg\mathrm{SP}_K(m,p)$, for some i, $h(i) \Vdash \Box A \leq \Box C$. Hence $\mathrm{WP}_K(p,n)$.
Thus in all subcases, $n \leq_K p$.
Case 4: $n \leq m \wedge \neg\mathrm{SP}_K(n,m) \wedge \mathrm{WP}_K(p,m)$:
Case 4a: $n \leq p$: Suppose $\mathrm{SP}_K(n,p)$, i.e., $\mathrm{M}(p,C)$ and for some i, $h(i) \Vdash \Box C < \Box A$, for some A such that $\mathrm{M}(n,A)$ or, if $n \notin$ MINPROOFS and for some $q < n$, $\mathrm{M}(q,A)$, $h(i) \Vdash \Box C \leq \Box A$. By $\mathrm{WP}_K(p,m)$, for some i, $h(i) \Vdash \Box B \leq \Box C$, and $\mathrm{M}(m,B)$. Hence for some i, $h(i) \Vdash \Box B < \Box A$, for some A such that $\mathrm{M}(n,A)$ or, if $n \notin$ MINPROOFS and for some $q < n$, $M(q,A)$, $h(i) \Vdash \Box B \leq \Box A$. This contradicts $\neg\mathrm{SP}_K(n,m)$. Hence not $\mathrm{SP}_K(n,p)$.
Case 4b: $p < n$: By $\mathrm{WP}_K(p,m)$ we have $\mathrm{M}(m,B)$ and for some i, $h(i) \Vdash \Box B \leq \Box C$ for some C such that $\mathrm{M}(p,C)$ or, if $p \notin$

MINPROOFS, for some $q < p$, $\mathrm{M}(q, C)$. By $\neg\mathrm{SP}_K(n,m)$, we have $n \in$ MINPROOFS. Moreover, if $\mathrm{M}(n,A)$, then for some i, $h(i) \Vdash \Box A \leq \Box C$. Hence $\mathrm{WP}_K(p,n)$. Thus in both subcases $n \leq_K p$.

REFERENCES

[1] Boolos, G., *The Unprovability of Consistency*, Cambridge University Press, Cambridge, 1979.

[2] Guaspari, D. and R. M. Solovay, "Rosser sentences," *The Annals of Mathematical Logic*, vol. 16 (1979), pp. 81-99.

[3] de Jongh, D. H. J., "A simplification of a completeness proof of Guaspari and Solovay," *Studia Logica*, vol. 46 (1987), pp. 187-192.

[4] Smoryński, C., *Self-reference and Modal Logic*, Springer-Verlag, New York, 1985.

[5] Solovay, R. M., "Provability interpretations of modal logic," *Israel Journal of Mathematics*, vol. 25 (1976), pp. 287-304.

[6] Švejdar, V., "Modal analysis of generalized Rosser sentences," *The Journal of Symbolic Logic*, vol. 48 (1983), pp. 986-999.

[7] Tuttas, F., "A remark on a completeness theorem of Guaspari and Solovay," to appear.

[8] Visser, A., *Aspects of Diagonalization and Provability*, Dissertation, Utrecht, 1981.

[9] Visser, A., "The provability logic of recursively enumerable theories extending Peano Arithmetic at arbitrary theories extending Peano Arithmetic," *Journal of Philosophical Logic*, vol. 13 (1984), pp. 97-113.

[10] Voorbraak, F., "A simplification of the completeness proofs for Guaspari and Solovay's R," *Logic Group Preprint Series 8*, Department of Philosophy, University of Utrecht, May 1986.

Department of Philosophy
University of Utrecht
Heidelberglaan 2
3584 CS Utrecht
The Netherlands