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Infinite Sets of Nonequivalent Modalities

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Abstract The set of irreducible nonequivalent modalities for a class of normal modal logics is determined. All the logics considered (KT, KD, K4, KB, among others) have this set infinite.

Introduction The study of irreducible nonequivalent modalities has been mostly devoted to those modal logics which have a finite number of such modalities. In all these cases the proof that a logic L has finitely many nonequivalent modalities is obtained by determining the set of irreducible modalities for L. Some attention has also been paid to the matter of showing that certain logics have infinitely many nonequivalent modalities; in fact it has been proved that the logics S2 [4], KT [5], KD4 and KTB [3], and $KBAlt_3$ [7] belong to this class (see also [1], §3.2, for a summary). But, in all these cases, the proof consists in finding an infinite set of modalities, usually obtained by reiterating one operator, which are nonequivalent for the logic under examination; therefore the set of the irreducible nonequivalent modalities for such logics remains undetermined (the only exception is constituted by the logic K, since Kit Fine has shown that no distinct modalities are K-equivalent).

In Section 2 we will show that if $L \subseteq KAlt_1$ or $L \subseteq KTAlt_3$ then no distinct modalities are L-equivalent; as a consequence we obtain, if we indicate by *Mod* the set of all modalities (in normal form) and by Y_L the set of nonequivalent irreducible modalities of L, that $Y_{KD} = Y_{KT} = Mod$. On the other hand, we will show in Section 3 that the set Y_{K4}, although infinite, is properly contained in *Mod* and is structurally simple; it is in fact obtained by reiterating the initial operator of the modalities of Y_{S4}, which is, as is well-known, a finite set. Moreover, in this context the addition of the axioms D and T is not useless, because it holds that $Y_{K4} \not\subseteq Y_{KD4} \not\subseteq Y_{KT4}$ (i.e., Y_{S4}).

1 Preliminaries Modal formulas are formed in the usual way from the language $\{P, \wedge, \vee, \neg, \rightarrow, \equiv, \Box, \Diamond\}$, where P is the set of propositional variables

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and $\Box$, $\Diamond$ are the so-called *modal operators.* A (*normal*) *modal logic* is a set of modal formulas containing all classical tautologies and the formula $\Box(p \to q) \to (\Box p \to \Box q)$, and which is closed under modus ponens, necessitation, and the rule of substitution. K denotes the smallest modal logic; it satisfies the following proposition.

Proposition 1.0 $\mathrm{K} \vdash \Phi \to \Psi$ *iff* $\mathrm{K} \vdash \Box\Phi \to \Box\Psi$ *iff* $\mathrm{K} \vdash \Diamond\Phi \to \Diamond\Psi$.

The names of the following formulas are standard:

$\mathrm{D} = \Box p \to \Diamond p, \quad \mathrm{T} = \Box p \to p, \quad \mathrm{B} = p \to \Box\Diamond p, \quad \text{and } 4 = \Box p \to \Box\Box p.$

We shall also consider the formula $\mathrm{Alt}_n = \Box p_1 \vee \Box(p_1 \to p_2) \vee \ldots \vee \Box(p_1 \wedge \ldots \wedge p_n \to p_{n+1})$. Semantic structures are *frames* (ordered pairs $F = \langle W,R \rangle$ where W is a nonempty set and $R \subseteq W \times W$) and *models* (ordered pairs $M = \langle F, \rho \rangle$ with F a frame and ρ a function, called a *valuation*, from P into $\mathcal{P}(W)$; we write $M = \langle W,R,\rho \rangle$ instead of $\langle\langle W,R \rangle, \rho \rangle$). The well-known Kripke truth-definition defines the notion "the formula Φ is true at the point w of M" (in symbols $M \models \Phi[w]$); as usual, $M \models \Phi$ means that $M \models \Phi[w]$ for each $w \in W$, and $F \models \Phi$ means that $\langle F, \rho \rangle \models \Phi$ for each ρ on F. The following result is standard: KD, KT (also called T), and K4 are complete with respect to the class of models whose relations are, respectively, serial, reflexive, transitive. Moreover (see [7]), $F \models \mathrm{Alt}_n$ iff $|\{v : wRv\}| \leq n$ for each $w \in W$ holds.

Proposition 1.1 *Let M be a model whose relation is reflexive; then*
(i) $M \models \Phi$ *implies* $M \models \Box\Phi$ *and* $M \models \Diamond\Phi$
(ii) $M \models \neg\Phi$ *implies* $M \models \neg\Box\Phi$ *and* $M \models \neg\Diamond\Phi$.

Let $F = \langle W,R \rangle$ and $w, v \in W$; we write $wR^n v$ if there exist $u_0, \ldots, u_n$ of W such that $w = u_0 R u_1 R \ldots R u_n = v$. We denote by R^+ the *ancestral* of R, i.e. $R^+ = \{\langle w, v \rangle$: there exists an n such that $wR^n v\}$.

Following [7] we define a formula ϕ to be a *modality* if it is expressed in the language $\{\neg, \Box, \Diamond, p\}$.[1] A modality ϕ is in *normal form* if $\phi = \dagger_1\dagger_2, \ldots, \dagger_k p$ where $\dagger_i \in \{\neg, \Box, \Diamond\}$ and if $i \neq 1$ then $\dagger_i \neq \neg$. From $\mathrm{K} \vdash (\neg\Diamond\neg p \equiv \Box p) \wedge (\neg\Box\neg p \equiv \Diamond p)$ it follows that each modality ϕ is equivalent to a modality in normal form; throughout the paper we consider only modalities in normal form, denoted by ϕ, ψ, θ, ξ, etc. If $\dagger_1 \neq \neg$ then the modality is *positive*, otherwise it is *negative*. $\Box^m$ and $\Diamond^m$ denote strings of m $\Box$-operators and m $\Diamond$-operators respectively, and $\Box^0\phi = \Diamond^0\phi = \phi$. Thus we may indicate each positive modality by the form

$$\psi = \Box^{k_1}\Diamond^{k_2} \ldots \Box^{k_{r-1}}\Diamond^{k_r} p,$$

where $k_i = 0$ implies that $k_j = 0$ either for each $j < i$ or for each $j > i$. This condition, instead of the more usual "$k_i = 0$ only if $i = 1$ or $i = r$", will be convenient in our proofs; so, for instance, the modality $\Diamond\Box^2 p$ may be written in the forms $\Box^0\Diamond^1\Box^2\Diamond^0 p$ and $\Box^0\Diamond^0\Box^0\Diamond^1\Box^2\Diamond^0 p$, but not in the form $\Box^0\Diamond^1\Box^0\Diamond^0\Box^2\Diamond^0 p$. Moreover, when it will be useful to distinguish between the exponents of the $\Box$-operators and those of the $\Diamond$-operators, we will write ϕ in the form

$$\phi = \Box^{m_1}\Diamond^{n_1}\Box^{m_2}\Diamond^{n_2} \ldots \Box^{m_s}\Diamond^{n_s} p.$$

The *dual* of a modality ϕ is the modality which results from ϕ by interchanging $\Box$ and $\Diamond$ throughout ϕ. Obviously, if ϕ and ψ are equivalent in a logic L, so are their duals; we shall call this property "duality".

We set $F_0 = \langle\{w_0\},\varnothing\rangle$, $F_1 = \langle\{w_1\},\{\langle w_1,w_1\rangle\}\rangle$, $F_2 = \langle\{w_2,v_2\}, \{w_2,v_2\} \times \{w_2,v_2\}\rangle$, $\rho_{1+}(p) = \{w_1\}$, $\rho_{1-}(p) = \varnothing$, $\rho_2(p) = \{w_2\}$.

Lemma 1.2 *Let X be any finite string of modal operators (i.e., Xp is a positive modality). We have that* $F_0 \models \Box Xp$, $F_0 \not\models \Diamond Xp$, $\langle F_1,\rho_{1+}\rangle \models Xp$, $\langle F_1,\rho_{1-}\rangle \not\models Xp$, $\langle F_2,\rho_2\rangle \models X\Diamond p$, $\langle F_2,\rho_2\rangle \not\models X\Box p$.

The proof is trivial.

***2 Nonequivalent modalities in* KT** We define the frame $F_T = \langle W_T,R_T\rangle$ as follows: W_T is the set of finite sequences of 0 and 1 (as usual, $\langle a_1,\ldots,a_s\rangle * \langle b_1,\ldots,b_r\rangle = \langle a_1,\ldots,a_s,b_1,\ldots,b_r\rangle$), and $R_T = \{\langle w,w\rangle, \langle w, w*\langle 0\rangle\rangle, \langle w, w*\langle 1\rangle\rangle : w \in W_T\}$ (see Figure 1).

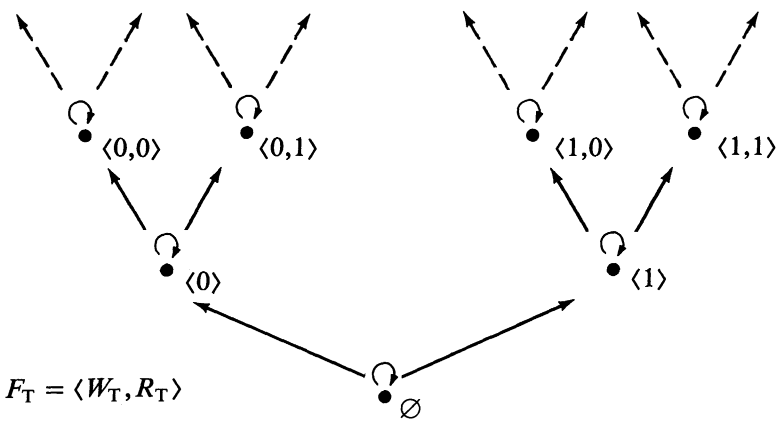

Figure 1

We indicate by ${}^0|w|$ and ${}^1|w|$ respectively the number of 0's and 1's in the sequence w. Let $X \subseteq W_T$; we set

$$X0^n = \{u \in W_T : u = v * \underbrace{\langle 0,\ldots,0\rangle}_{n\text{-times}} \text{ for } v \in X\}$$

$$X1^n = \{u \in W_T : u = v * \underbrace{\langle 1,\ldots,1\rangle}_{n\text{-times}} \text{ for } v \in X\}$$

$$X\Delta^n = \{u \in W_T : u = v * v', \text{ where } v \in X \text{ and the length of } v' \text{ is less than or equal to } n\}.$$

If $X = \{w\}$ we simply write $w0^n$, $w1^n$, and $w\Delta^n$ instead of $\{w\}0^n$, $\{w\}1^n$, and $\{w\}\Delta^n$. Observe that $w\Delta^n$ is the set $\{v \in W_T : wR_T^m v \text{ for } m \leq n\}$, and hence, since R_T is reflexive, $w\Delta^n = \{v \in W_T : wR_T^n v\}$.

Lemma 2.0 *Let* $\psi = \Box^{m_1}\Diamond^{n_1}\Box^{m_2}\Diamond^{n_2}\ldots\Box^{m_s}\Diamond^{n_s}p$; *there exists a valuation* ρ_ψ *on* F_T *such that* $\langle F_T,\rho_\psi\rangle \models \psi[w]$ *iff* $w = \emptyset$ *or* w *terminates with* 1.

The intuitive idea behind the proof is as follows: Suppose that $v = \emptyset$ or v ends in 1. Among the infinitely many sets $X \subseteq W_T$ having the property that if p is true at each point of X then ψ is true at v, we choose the set $f(v)$ (the set of "green" points relative to v) following this criterion: whenever the subformula $\Diamond\phi$ of ψ must be true at z in order that ψ be true at v, we impose that ϕ be true at $z * \langle 1\rangle$. In other words, looking at Figure 1, we choose the set $f(v)$ as far on the right as possible among the X's. If, on the contrary, the point u ends in 0, we choose $f(u)$ (the set of "red" points relative to u) among the infinitely many sets $Y \subseteq W_T$ having the property that if p is false at each point of Y then ψ is false at u, following this criterion: whenever a subformula $\Box\phi$ of ψ must be false at t in order that ψ be false at u, we impose that ϕ be false at $t * \langle 0\rangle$ (in this case $f(u)$ is as far on the left as possible). Since we shall show that no point of W_T must be simultaneously green and red, then $\rho(p) = \{w : w \text{ is green relative to some } v\}$ will be the required valuation.

Proof of Lemma 2.0: If ϕ is a positive modality then, for each $w \in W_T$ and each ρ on F_T, there hold:

(1) $\langle F_T,\rho\rangle \models \Box^s\phi[w]$ iff $\langle F_T,\rho\rangle \models \phi[w']$ for each $w' \in w\Delta^s$
(2) $\langle F_T,\rho\rangle \not\models \Diamond^s\phi[w]$ iff $\langle F_T,\rho\rangle \not\models \phi[w']$ for each $w' \in w\Delta^s$
(3) $\langle F_T,\rho\rangle \models \Diamond^s\phi[w]$ if $\langle F_T,\rho\rangle \models \phi[w1^s]$
(4) $\langle F_T,\rho\rangle \not\models \Box^s\phi[w]$ if $\langle F_T,\rho\rangle \not\models \phi[w0^s]$.

Let $V = \{v \in W_T : v = \emptyset \text{ or } v \text{ terminates with } 1\}$ and $U = W_T - V$. We define a function f from W_T into $\mathcal{P}(W_T)$ as follows: if $v \in V$ then $f(v) = v\Delta^{m_1}1^{n_1}\Delta^{m_2}1^{n_2}\ldots\Delta^{m_s}1^{n_s}$ and if $u \in U$ then $f(u) = u0^{m_1}\Delta^{n_1}0^{m_2}\Delta^{n_2}\ldots0^{m_s}\Delta^{n_s}$. From (1) and (3) it follows that for each ρ on F_T and each $v \in V$

(5) if $f(v) \subseteq \rho(p)$ then $\langle F,\rho\rangle \models \psi[v]$,

while, from (2) and (4) we obtain that for each ρ on F_T and each $u \in U$

(6) if $f(u) \cap \rho(p) = \emptyset$ then $\langle F,\rho\rangle \not\models \psi[u]$.

Next we will show that

(7) $\bigcup_{v\in V} f(v) \cap \bigcup_{u\in U} f(u) = \emptyset.$

Let $v \in V$ and $u \in U$: we show that $f(v) \cap f(u) = \emptyset$. Let $z \in f(v)$ and $t \in f(u)$. If vR^+u (i.e., $u = v * x$ for a sequence x) then

$$^0|z| \le {}^0|v| + \sum_{i=1}^{s} m_i \text{ and } {}^0|t| \ge {}^0|u| + \sum_{i=1}^{s} m_i.$$

But vR^+u, $v \in V$, and $u \in U$ imply that $^0|u| > {}^0|v|$ and, therefore, $^0|z| < {}^0|t|$ and $z \neq t$. On the other hand, if uR^+v then $^1|z| \ge {}^1|v| + \sum_{i=1}^{s} n_i > {}^1|u| + \sum_{i=1}^{s} n_i \ge {}^1|t|$ and hence $z \neq t$. Finally, suppose that $u\not R^+v$ and $v\not R^+u$ and let $v = \langle a_1,\ldots,a_r\rangle$ and $u = \langle b_1,\ldots,b_{r'}\rangle$. Since $u \neq v$ there exists an $i \le \inf\{r,r'\}$

such that $a_i \neq b_i$ and hence, since t and z start respectively with u and v, the ith component of z differs from the ith component of t, i.e. $t \neq z$, and this concludes the proof of (7). Now let $\rho_\psi(p) = \bigcup_{v \in V} f(v)$. By (7) we have that $\rho_\psi(p) \cap \bigcup_{u \in U} f(u) = \varnothing$ and hence from (5) and (6) it follows that ρ_ψ satisfies the Lemma.

Theorem 2.1 *If ξ and θ are distinct modalities then* KT $\nvDash \xi \equiv \theta$.

Proof: If ξ is positive and θ negative, then from $F_1 \vDash$ KT and Lemma 1.2 it follows that KT $\nvDash \xi \equiv \theta$. If ξ and θ are both positive we distinguish two cases:

Case A. θ is a subformula of ξ. Then, since $\xi \neq \theta$, either $\Diamond\theta$ or $\Box\theta$ are subformulas of ξ. Let us consider $\langle F_{\mathrm{T}},\rho_\theta\rangle$; since for each $w \in W_{\mathrm{T}}$ there exist a $v \in V$ and a $u \in U$ such that $wR_{\mathrm{T}}v$ and $wR_{\mathrm{T}}u$, from Lemma 2.0 we obtain that $\Diamond\theta$ and $\Box\theta$ are respectively true and false at every point of W_{T}; i.e., $\langle F_{\mathrm{T}},\rho_\theta\rangle \vDash \Diamond\theta$ and $\langle F_{\mathrm{T}},\rho_\theta\rangle \vDash \neg\Box\theta$. Now, if $\Diamond\theta$ is a subformula of ξ, then by Proposition 1.1(i) we obtain that $\langle F_{\mathrm{T}},\rho_\theta\rangle \vDash \xi$ which, together with $\langle F_{\mathrm{T}},\rho_\theta\rangle \nvDash \theta[\langle 0\rangle]$ (see Lemma 2.0), implies that KT $\nvDash \xi \equiv \theta$; on the other hand, if $\Box\theta$ is a subformula of ξ, then from Proposition 1.1(ii) it follows that $\langle F_{\mathrm{T}},\rho_\theta\rangle \vDash \neg\xi$ which, together with $\langle F_{\mathrm{T}},\rho_\theta\rangle \vDash \theta[\varnothing]$, again implies that KT $\nvDash \xi \equiv \theta$.

Case B. Neither θ is a subformula of ξ nor ξ is a subformula of θ. Let ψ be the largest common subformula of ξ and θ, and suppose that $\Diamond\psi$ is a subformula of θ and $\Box\psi$ of ξ. Following the proof of Case A we obtain that $\langle F_{\mathrm{T}},\rho_\psi\rangle \vDash \theta$ and $\langle F_{\mathrm{T}},\rho_\psi\rangle \vDash \neg\xi$.

Remark 2.2 In order to show that two modalities ξ and θ are not equivalent in a logic L, it is enough to find a point w of an L-model M such that $M \vDash \xi[w]$ and $M \nvDash \theta[w]$, or vice versa. By the proof of Theorem 2.1 we realize that KT "separates" the modalities in a stronger sense; in fact, for each ξ and θ we have obtained a model M such that $M \vDash \xi$ and $M \nvDash \theta$ or vice versa. Furthermore, the proof of Case B is still stronger: If neither ξ is a subformula of θ nor θ of ξ, there is an M such that $M \vDash \xi$ and $M \vDash \neg\theta$ or vice versa. But this fact cannot be extended to Case A: If $\xi = \Diamond p$ and $\theta = p$ it is impossible to find a KT-model M such that $M \vDash \xi$ and $M \vDash \neg\theta$ or $M \vDash \theta$ and $M \vDash \neg\xi$. This limitation that KT-models have in separating modalities persists even for the class of all Kripke models (i.e., the class of models for K): we show that it is impossible to find a model $M = \langle W,R,\rho\rangle$ such that $M \vDash \Box p$ and $M \vDash \neg\Box\Diamond\Box p$, or $M \vDash \Box\Diamond\Box p$ and $M \vDash \neg\Box p$. In fact, if W contains a terminal point (a point w such that $w \not R v$ for each $v \in W$) then $M \nvDash \neg\Box p$ and $M \nvDash \neg\Box\Diamond\Box p$. Hence suppose that W is without terminal points; then $M \vDash \Box p$ implies that $M \vDash \Diamond\Box p$ and $M \vDash \Box\Diamond\Box p$, and thus $M \nvDash \neg\Box\Diamond\Box p$. On the other hand, $M \vDash \Box\Diamond\Box p$ implies that $M \vDash \Diamond\Box p$, and hence $M \vDash \Box p[w]$ for a $w \in W$; therefore $M \nvDash \neg\Box p$.

From Theorem 2.1 it also follows that in KD all distinct modalities are nonequivalent. Moreover, this theorem can be strengthened as follows:

Corollary 2.3 *If ξ and θ are distinct modalities and* L $\subseteq$ KTAlt_3 *then* L $\nvDash \xi \equiv \theta$.

Proof: $F_T \models \mathrm{KTAlt}_3$.

At this point the following question arises: Does Corollary 2.3 hold if we substitute KTAlt_3 by KTAlt_2? The answer is negative, as shown in the following theorem.

Theorem 2.4 $\mathrm{KTAlt}_2 \vdash \Box\Diamond\Box^2 p \equiv \Box^2\Diamond\Box p$.

Proof: First we show that

(1) $\mathrm{KTAlt}_2 = Th(F^*)$, where $F^* = \langle \omega, \{\langle m,m\rangle, \langle m+1,m\rangle : m \in \omega\}\rangle$.

In fact, using methods analogous to those used in the Tree Lemma of Sahlquist (see [6]), it is possible to show that KTAlt_2 is complete with respect to the class X of finite reflexive linear intransitive trees; i.e. it is complete with respect to $\{\langle\{0,\ldots,n\}, \{\langle m,m\rangle : m \le n\} \cup \{\langle m+1,m\rangle : m < n\}\rangle, n \in \omega\}$. Since each frame of X is a generated subframe of F^*, we obtain (1). Hence, setting $\psi_1 = \Box\Diamond\Box^2 p$ and $\psi_2 = \Box^2\Diamond\Box p$, we must show that $F^* \models \psi_1 \equiv \psi_2$, i.e. that for each ρ on F^* and each $m \in \omega$, $\langle F^*, \rho\rangle \models \psi_1 \equiv \psi_2[m]$ (we simply write $m \models \psi_1 \equiv \psi_2$). If $m \ge 3$ then the following (a)–(e) are equivalent, and so are (a′)–(e′):

(a) $m \models \psi_1$
(b) $m \models \Diamond\Box^2 p$ and $m-1 \models \Diamond\Box^2 p$
(c) $m-1 \models \Box^2 p$ or ($m \models \Box^2 p$ and $m-2 \models \Box^2 p$)
(d) $m-i \models p$ for $1 \le i \le 3$ or $m-j \models p$ for $0 \le j \le 3$
(e) $m-i \models p$ for $1 \le i \le 3$
(a′) $m \models \psi_2$
(b′) $m-i \models \Diamond\Box p$ for $0 \le i \le 2$
(c′) ($m \models \Box p$ and $m-2 \models \Box p$) or ($m-1 \models \Box p$ and $m-2 \models \Box p$) or ($m-1 \models \Box p$ and $m-3 \models \Box p$)
(d′) $m-i \models p$ for $0 \le i \le 3$ or $m-j \models p$ for $1 \le j \le 3$ or, if $m > 3$, $m-r \models p$ for $1 \le r \le 4$
(e′) $m-j \models p$ for $1 \le j \le 3$.

Since (e) = (e′) we obtain that $m \models \psi_1 \equiv \psi_2$. In the same way we can show that

$2 \models \psi_1 \Leftrightarrow (0 \models p \text{ and } 1 \models p) \Leftrightarrow 2 \models \psi_2$
$1 \models \psi_1 \Leftrightarrow 0 \models p \Leftrightarrow 1 \models \psi_2$
$0 \models \psi_1 \Leftrightarrow 0 \models p \Leftrightarrow 0 \models \psi_2$,

thus concluding the proof.

Theorem 2.5 *The set Γ of all logics without equivalent distinct modalities, ordered by inclusion, is without maximum.*

Proof: Let us consider the logic KAlt_1 ($=\mathrm{K} \cup \{\Diamond p \to \Box p\}$). Since F_1 is a frame of KAlt_1, if ξ is positive and θ negative then $\mathrm{KAlt}_1 \nvdash \xi \equiv \theta$. Hence suppose that ξ and θ are both positive and let X be the largest initial part common to ξ and θ; the possible cases are the following:

(i) $\xi = X\Box\psi$ and $\theta = X\Diamond\phi$ (or vice versa)
(ii) $\xi = X\Box\psi$ and $\theta = Xp$ (or vice versa)
(iii) $\xi = Xp$ and $\theta = X\Diamond\phi$ (or vice versa).

Let s be the number of operators in X and consider the frame $N = \langle \omega, \{\langle n+1, n\rangle : n \in \omega\}\rangle$. We have that $N \models X\Box\psi[s]$, $N \not\models X\Diamond\phi[s]$, and $\langle N, \rho\rangle \models Xp[s]$ iff $0 \in \rho(p)$. Hence, for all of the above cases, $N \not\models \xi \equiv \theta$. But, on the other hand, $N \models \Diamond p \rightarrow \Box p$, and therefore no distinct modalities are equivalent in KAlt_1, and $\mathrm{KAlt}_1 \in \Gamma$. Moreover, from Theorem 2.1 it follows that $\mathrm{KT} \in \Gamma$. Now, let us consider KTAlt_1 ($=\mathrm{KT} \cup \mathrm{KAlt}_1$). From $\mathrm{KT} \vdash \Box p \rightarrow \Diamond p$ and $\mathrm{KAlt}_1 \vdash \Diamond p \rightarrow \Box p$ it follows that $\mathrm{KTAlt}_1 \vdash \Box p \equiv \Diamond p$, and hence $\mathrm{KTAlt}_1 \notin \Gamma$.

3 Nonequivalent modalities of K4

Lemma 3.0 *Let X be a finite, possibly empty, string of modal operators. The following are theorems of* K4:

$\Phi_0 : \Box\Diamond X\Diamond p \equiv \Box\Diamond p$
$\Phi_1 : \Box\Diamond X\Box p \equiv \Box\Diamond\Box p$
$\Phi_2 : \Diamond\Box X\Box p \equiv \Diamond\Box p$
$\Phi_3 : \Diamond\Box X\Diamond p \equiv \Diamond\Box\Diamond p.$

Proof: First note that the following may be easily established as theorems of K4:

(1) $\Box\Diamond\Diamond p \equiv \Box\Diamond p$
(2) $\Box\Diamond\Box\Box p \equiv \Box\Diamond\Box p$
(3) $\Box\Diamond\Box\Diamond p \equiv \Box\Diamond\Diamond p.$

We now prove Φ_0 inductively. The case where X is empty is (1). Otherwise X is $\Diamond Y$ or $\Box Y$. We show that in each case the following is a theorem:

(4) $\Box\Diamond X\Diamond p \equiv \Box\Diamond Y\Diamond p.$

If X is $\Diamond Y$ then (4) is (1) with $Y\Diamond p/p$. If X is $\Box Y$ and Y is empty then (4) is (3). If X is $\Box Y$ and Y is $\Diamond Z$ then (4) is (3) with $Z\Diamond p/p$. If X is $\Box Y$ and Y is $\Box Z$ then (4) is (2) with $Z\Diamond p/p$.

As regards Φ_1, if X is empty then Φ_1 is a tautology. Otherwise X is $\Diamond Y$ or $\Box Y$. We show that in either case

(5) $\Box\Diamond X\Box p \equiv \Box\Diamond Y\Box p.$

If X is $\Diamond Y$ then (5) is (1) with $Y\Box p/p$. If X is $\Box Y$ then: if Y is empty then (5) is (2); if Y is $\Diamond Z$ then (5) is (3) with $Z\Box p/p$; if Y is $\Box Z$ then (5) is (2) with $Z\Box p/p$.

Since Φ_2 and Φ_3 are duals respectively of Φ_0 and Φ_1, the proof is concluded.

Theorem 3.1 *Let $Y_4' = \{\Box^{m_1}\Diamond^{m_2}\Box^{m_3}p, \Diamond^{m_1}\Box^{m_2}\Diamond^{m_3}p : m_2, m_3 \in \{0,1\}\}$ and $Y_4 = Y_4' \cup \{\neg\phi : \phi \in Y_4'\}$. We have:*
(i) *for each modality ψ there exists a $\xi \in Y_4$ such that* $\mathrm{K4} \vdash \psi \equiv \xi$
(ii) *if $\xi, \theta \in Y_4$ and $\xi \neq \theta$ then* $\mathrm{K4} \not\vdash \xi \equiv \theta$.

Proof: The proof of (i) trivially follows from Lemma 3.0. Regarding (ii), since F_0, F_1, and F_2 are K4-frames, we need only, via Lemma 1.2 and duality, to consider the case in which ξ and θ are positive modalities whose first and whose last operators are $\Box$. Then we set $\xi = \Box^{m_1}\Diamond^{m_2}\Box^{m_3}p$ and $\theta = \Box^{n_1}\Diamond^{n_2}\Box^{n_3}p$ and consider a model $N = \langle \omega, >, \rho\rangle$ where ρ is such that $0 \notin \rho(p)$. If $m_1 > n_1$ then (since $N \not\models p[0]$, $N \not\models \Diamond\psi[0]$, and $N \models \Box\psi[0]$ for each ψ) we obtain that $N \models \xi[n_1]$ and $N \not\models \theta[n_1]$. Hence suppose that $m_1 = n_1$; since ξ and θ terminate

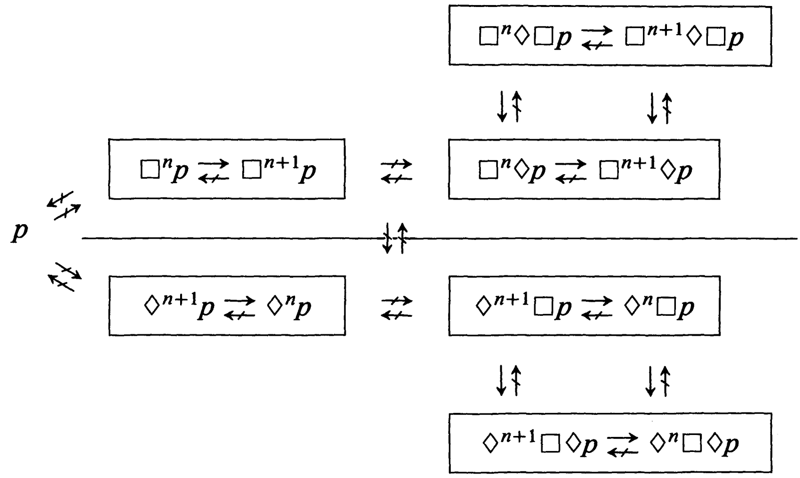

Figure 2.

with $\Box$, from the fact that $\xi \neq \theta$ it follows that $m_2 = m_3 = 1$ and $n_2 = n_3 = 0$ (or vice versa). But $N \nvDash \Box^{m_1}\Diamond\Box p[m_1]$ and $N \vDash \Box^{m_1} p[m_1]$, and so the proof is concluded.

Using Theorem 3.1 we can also draw the pattern of implications among irreducible modalities of K4. Figure 2 pictures the implications among positive modalities; for their negations reverse the arrows. The arrows not proved in Theorem 3.1 are straightforward.

Corollary 3.2 *Let* $Y'_{\mathrm{D4}} = \{\Box\Diamond\Box p, \Box\Diamond p, \Box^n p : n \in \omega\}$ *and* $Y_{\mathrm{D4}} = Y'_{\mathrm{D4}} \cup \{\neg\phi : \phi \in Y'_{\mathrm{D4}}\}$. *For each modality* ψ *there exists a* $\xi \in Y_{\mathrm{D4}}$ *such that* KD4 $\vdash \psi \equiv \xi$ *and if* $\xi,\theta \in Y_{\mathrm{D4}}$ *and* $\xi \neq \theta$ *then* KD4 $\nvdash \xi \equiv \theta$.

Proof: By Theorem 3.1 we need only show that: (i) KD4 $\nvdash \Box^{n+1}p \to \Box^n p$, (ii) KD4 $\vdash \Box^{n+1}\Diamond p \to \Box^n\Diamond p$, and (iii) KD4 $\vdash \Box^{n+1}\Diamond\Box p \to \Box^n\Diamond\Box p$. Let us consider the model $N' = \langle\omega,<,\rho\rangle$ where $\rho(p) = \omega - \{n\}$; we have that $N' \nvDash \Box^n p[0]$ and $N' \vDash \Box^{n+1}p[0]$, and hence (i) is proved. Regarding (ii) and (iii), we show that for each ϕ KD4 $\vdash \Box^{n+1}\Diamond\phi \to \Box^n\Diamond\phi$. Suppose that $F \nvDash \Box^n\Diamond\phi[w]$, i.e. $F \vDash \Diamond^n\Box\neg\phi[w]$; hence there exists a v such that $wR^n v$ and $F \vDash \Box\neg\phi[v]$. By transitivity and seriality it follows that $F \vDash \Box^2\neg\phi[v]$ and $F \vDash \Diamond\Box\neg\phi[v]$; thus we obtain that $F \vDash \Diamond^{n+1}\Box\neg\phi[w]$ and $F \nvDash \Box^{n+1}\Diamond\phi[w]$.

Finally, we observe that for the systems KB (i.e., K + $p \to \Box\Diamond p$), KTB, and KDB it is possible to show that any positive modality is equivalent to a modality of the form $\Box^{m_1}\Diamond^{m_2}\ldots\Box^{m_{s-1}}\Diamond^{m_s}p$ where for each i, $1 < i < s$, if $m_i \neq 0$ and $m_i \leq m_{i+1}$ then $m_{i-1} < m_i$. No distinct modalities of this form are equivalent.

NOTE

1. Under an alternative definition (see for instance [1] or [3]), a modality is a sequence X of symbols from the set $\{\neg,\Box,\Diamond\}$. In such a case X and X' are said to be equivalent in L if L $\vdash Xp \equiv X'p$.

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