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Truth Table Logic, with a Survey of Embeddability Results

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Abstract What logic is barely justified on the basis of the 'meanings' given to the connectives by the *left-right readings* of their truth tables?

The valid arguments involved in truth table computations are called *Kalmaric*. We set out a system T, consisting of normal proofs constructed by means of elegantly symmetrical introduction and elimination rules. In the system T there are two requirements, called (□) and (>), on applications of discharge rules. T is sound and complete for Kalmaric arguments. (□) requires nonvacuous discharge of assumptions; (>) requires that the assumption discharged be the sole one available of highest degree.

We then consider a 'Duhemian' extension T*, obtained simply by dropping the requirement (>). T* is a proper subsystem of intuitionistic relevant logic. Our main result is that T* is a double negation consistency companion to classical logic. Thus all one needs to add to T* to obtain classical logic is the (intuitionistic) absurdity rule, and the (classical) rule of double negation elimination. T* represents the inferential core that is justified by the left-right readings of the truth tables.

We survey all the embeddability results using various translation mappings "downwards" into subsystems of classical, intuitionistic, minimal, and intuitionistic relevant logic. This puts our main result into significant context.

1 How does one read off a logic from truth tables? It is often claimed that the standard two-valued truth tables for $\sim$, &, $\vee$, and $\supset$ capture the meanings

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of those connectives, and that those meanings are the classical ones. That is to say, the logic justified by the meanings in question is classical logic.

This claim bears interesting closer scrutiny. What do the truth tables for the connectives actually *say*? They say that *if* the truth values of the components of a compound sentence are such-and-such respectively, *then* the truth value of the compound itself is so-and-so. I shall call these the *left–right readings* of the truth tables; and I shall be interested in them, and them alone. There is a presupposition in these readings that truth values will be assigned uniquely, if at all. That is, truth value assignments are functions, or many-one mappings. But the truth tables do *not* also *say* that any sentence must be assigned either the value T (True) or the value F (False) by any given assignment. That is, truth value assignments need not be *total* functions (or, as I shall say, they need not be *classical*). If they are classical, this can only be *shown* by the statement of the truth tables, insofar as T and F are the only values appearing within them. But this aspect of what is shown is nowhere said explicitly in the left–right readings. If we confine ourselves to what the truth tables *say*, it is possible, *prima facie*, that some logic other than classical logic will emerge as the logic justified on the basis of the meanings conferred on the connectives by the truth tables.

This paper realizes that possibility. It investigates two systems, which I shall call T and T*, of *truth table logic*: a logic justified on the basis of what the truth tables *say*, rather than on what they might arguably also *show*. T is the smallest such system; T* is a very natural but modest extension of T. Their exact definitions will be given in due course.

2 Restricted and unrestricted transitivity Both T and T* are subsystems of the system of *intuitionistic relevant logic* (IR) developed in Tennant [17]. The deducibility relations of T and T*, like that of intuitionistic relevant logic, fail to be unrestrictedly transitive. That is, the following principle of unrestricted transitivity does not hold:

If P_1 is deducible from X_1,
. . .
P_n is deducible from X_n,
and Q is deducible from $X_0 \cup \{P_1, \ldots, P_n\}$,
then Q is deducible from $X_0 \cup X_1 \cup \ldots \cup X_n$.

But in the systems T* and IR the failures of this unrestricted transitivity principle are *virtuous*. Transitivity fails where, according to a relevantist, it *ought* to fail: when the deducibility would otherwise hold only by virtue of the fact that $X_0 \cup X_1 \cup \ldots \cup X_n$ is inconsistent, or Q is (intuitionistically) logically true. The deducibility relations of the systems T* and IR satisfy instead the following, epistemically superior, *restricted transitivity principle*:

If P_1 is deducible from X_1,
. . .
P_n is deducible from X_n,
and Q is deducible from $X_0 \cup \{P_1, \ldots, P_n\}$, then either Q or $\wedge$ (absurdity) is deducible from some subset of $X_0 \cup X_1 \cup \ldots \cup X_n$.

This major structural difference between, on the one hand, the systems T* and IR and, on the other hand, the unrestrictedly transitive systems of classical, intuitionistic, and minimal logic, is the reason for the interest attaching to results about the embeddability of any of the latter into any of the former. In particular, any result about the embeddability of classical logic (the largest unrestrictedly transitive system) into T* (the smallest restrictedly transitive system under consideration) would be of particular interest. To appreciate this, note the relative containments in Figure 1.

There are many well-known results about embeddability "downwards" in the chain C, I, M of unrestrictedly transitive logics (or theories closed under those logics); we will review these in a separate section. [17] gives a generalized Gödel–Glivenko theorem about the embeddability of C and of I into IR. In this paper the main result concerns the similar embeddability of C into T*:

Main Theorem *Every classical proof* $\begin{array}{c}X\\ P\\ A\end{array}$ *can be converted into a proof* $\begin{array}{c}\sim\sim Z\\ R\\ \sim\sim A\end{array}$ *or* $\begin{array}{c}\sim\sim Z\\ R\\ \wedge\end{array}$ *(for some subset Z of X) in* T*.

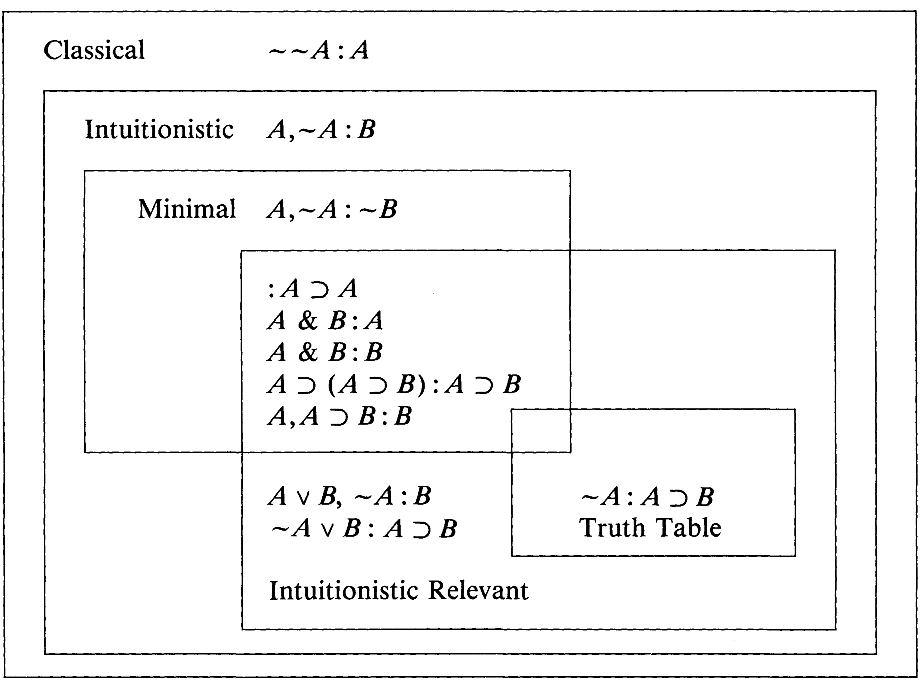

Figure 1.

This result tells us that the simplest translation in the book—prefixing with double negations—maps C into T* "where it counts". Let us call a classically valid argument *perfect* just in case its premises are satisfiable and no proper subset of them logically implies the conclusion. A corollary to our main result is this:

(i) *double negation maps perfect arguments into T**
(ii) *if a set of sentences is not satisfiable, T* will provide a reductio from (some subset of the set consisting of) their double negations, and*
(iii) *if a sentence is logically true, T* will provide a proof of its double negation.*

In order to appreciate the significance of how T* thus contains the restrictedly transitive fragment of classical logic under double negation, one has first to see just how "small" the system T* is.

3 Rules of inference designed to mimic the truth tables: The systems T and T*
Any truth table logic has to be able, at the very least, to represent by means of deducibilities the left–right evaluative transitions corresponding to the rows of the truth tables. We need, that is, a sentential rendering of the input–output relations represented by the rows of each truth table. To that end, let us write in place of every occurrence of T in a truth table an occurrence of the sentence to which that T is assigned; and in place of every occurrence of F in a truth table an occurrence of the *negation* of the sentence to which that F is assigned. (There will never be any confusion over whether a given occurrence of T stands for the truth value or for the system of truth table logic soon to be defined.) Let us then read the components' entries in any row of a truth table as the premises of a simple argument whose conclusion is the compound's entry in that row. Each such argument mimics the action of the truth table as represented by that row; it captures exactly the left–right reading of the row. We shall use the letter ρ as a variable over rows, i.e., truth value assignments.

Each row in the tables below represents a truth value assignment, whose *truth set* (formed from the set of atoms in question) we shall define as the set consisting of those atoms (if any) to which the value T is assigned, along with the negations of those atoms (if any) to which the value F if assigned. For any sentence A, the truth set of ρ formed from A's atoms will be called $\rho[A]$. Truth sets are obviously consistent, in that no truth set contains any atom along with its negation.

In the diagram below, the truth tables are given in the left column. The middle column consists of the simple arguments set up, row by row, according to the method described above. The right column contains only those arguments that really need to be independently established after taking care of redundancies in the middle column. We are taking a 'fell swoop' view of what the truth tables say. Thus we read the truth table for disjunction as saying:

if the truth value of A is T then that of $(A \vee B)$ is T;
if the truth value of B is T then that of $(A \vee B)$ is T;
if the truth values of A and B are both F, then the truth value of $(A \vee B)$ is F.

Truth table inferences

A	$\sim A$		Sentential version	Nonredundant demonstranda
T	F		$A \therefore \sim\sim A$	$\boldsymbol{A \therefore \sim\sim A}$
F	T		$\sim A \therefore \sim A$	

A	B	$A \,\&\, B$	Sentential version	Nonredundant demonstranda
T	T	T	$A,B \therefore A \,\&\, B$	$\boldsymbol{A,B \therefore A \,\&\, B}$
T	F	F	$A,\sim B \therefore \sim(A \,\&\, B)$	$\boldsymbol{\sim B \therefore \sim(A \,\&\, B)}$
F	T	F	$\sim A,B \therefore \sim(A \,\&\, B)$	$\boldsymbol{\sim A \therefore \sim(A \,\&\, B)}$
F	F	F	$\sim A,\sim B \therefore \sim(A \,\&\, B)$	

A	B	$A \vee B$	Sentential version	Nonredundant demonstranda
T	T	T	$A,B \therefore A \vee B$	
T	F	T	$A,\sim B \therefore A \vee B$	$\boldsymbol{A \therefore A \vee B}$
F	T	T	$\sim A,B \therefore A \vee B$	$\boldsymbol{B \therefore A \vee B}$
F	F	F	$\sim A,\sim B \therefore \sim(A \vee B)$	$\boldsymbol{\sim A,\sim B \therefore \sim(A \vee B)}$

A	B	$A \supset B$	Sentential version	Nonredundant demonstranda
T	T	T	$A,B \therefore A \supset B$	
T	F	F	$A,\sim B \therefore \sim(A \supset B)$	$\boldsymbol{A,\sim B \therefore \sim(A \supset B)}$
F	T	T	$\sim A,B \therefore A \supset B$	$\boldsymbol{B \therefore A \supset B}$
F	F	T	$\sim A,\sim B \therefore A \supset B$	$\boldsymbol{\sim A \therefore A \supset B}$

Note two important casualties of our way of reading the truth tables. First, $\sim\sim A \therefore A$ (the law of double negation) fails. Nothing that the truth table says can guarantee that the falsity of $\sim A$ can arise only from the truth of A. Secondly, $A,A \supset B \therefore B$ (modus ponens) fails. The usual (classical) justification of modus ponens points to the second row of the truth table for $\supset$: one cannot have both A and $A \supset B$ true with B false. But from this it follows only that if both A and $A \supset B$ are true, then it is not the case that B is false. And this we shall recognize by the deducibility $A,A \supset B \therefore \sim\sim B$.

Kalmar's Theorem (sentential version) *For any truth value assignment ρ and any sentence A*
(i) *if A is true under ρ, then from $\rho[A]$ one can deduce A*
(ii) *if A is false under ρ, then from $\rho[A]$ one can deduce $\sim A$.*

Kalmar's Theorem (inferential version) *For any truth value assignment ρ and any sentence A*
(i) *if A is true under ρ, then from $\rho[A]$ one can deduce A*
(ii) *if A is false under ρ, then from $A,\rho[A]$ one can deduce $\wedge$.*

Let X be any subset of $\rho[A]$. Then arguments of the form $X:A$ or $X,A:\wedge$ will be called *simple.* A simple valid argument will be called *Kalmaric.*

The deducibilities in the rightmost column of the table above, which I shall call the list L, are just the ones we usually establish in order to prove (by induction on the length of sentences) the *sentential* version of Kalmar's Theorem for the logic in question. One can also, however, recast these deducibilities in a slightly more convenient form, in order to prove the *inferential* version of Kalmar's Theorem. The variant list L^ is

$A, \sim A \therefore \wedge$
$A, B \therefore A \,\&\, B$
$\sim B, A \,\&\, B \therefore \wedge$
$\sim A, A \,\&\, B \therefore \wedge$
$A \therefore A \vee B$
$B \therefore A \vee B$
$\sim A, \sim B, A \vee B \therefore \wedge$
$A, \sim B, A \supset B \therefore \wedge$
$B \therefore A \supset B$
$\sim A \therefore A \supset B.$

It is well-known that intuitionistic logic satisfies Kalmar's Theorem (in both its sentential and inferential versions) in that it serves up all the deducibilities in the union of the lists L and L^. Note that all of these are simple (and Kalmaric) in the senses defined above:

L ∪ L^

$A \therefore \sim\sim A$
$A, \sim A \therefore \wedge$
$A, B \therefore A \,\&\, B$
$\sim B, A \,\&\, B \therefore \wedge$
$\sim A, A \,\&\, B \therefore \wedge$
$\sim A \therefore \sim(A \,\&\, B)$
$\sim B \therefore \sim(A \,\&\, B)$
$A \therefore A \vee B$
$B \therefore A \vee B$
$\sim A, \sim B, A \vee B \therefore \wedge$
$\sim A, \sim B \therefore \sim(A \vee B)$
$A, \sim B, A \supset B \therefore \wedge$
$A, \sim B \therefore \sim(A \supset B)$
$B \therefore A \supset B$
$\sim A \therefore A \supset B.$

Thus, intuitionistic logic suffices to mimic the process of evaluating a compound sentence as true or false, given the truth values of the atoms occurring within it. The extra strength of *classical* logic enters into the picture when we go beyond Kalmar's Theorem to show that every sentence true in every row of its truth table is a theorem (that is, deducible from the empty set of assumptions). By Kalmar's Theorem any such sentence can be deduced from each of the possible truth sets. But in order to have a proof of that logically true sentence from the *empty* set of assumptions one must appeal to some classical rule such as dilemma, which allows one systematically to discharge those atoms and their negations that occur as assumptions within the proofs served up by Kalmar's Theorem.

What is less well-known, and indeed is rather obvious, is that *a much weaker sublogic of intuitionistic logic still suffices for Kalmar's Theorem.* The logic in question can simply be taken to be the closure (under cut) of the deducibility schemata in the list L ∪ L^.

But this way of specifying the weakest "Kalmar logic" is not very satisfying. Trivial though the deducibility schemata in L ∪ L^ may be, they nevertheless form a rather *ad hoc* collection from the proof-theoretic point of view. What would be preferable is a specification of introduction and elimination rules for the logical operators, taken one by one and in isolation.

Question How can we frame rules of natural deduction for the smallest logic (call it T) for which both the sentential and inferential versions of Kalmar's Theorem hold?

(In what follows, I shall mean by "Kalmar's Theorem" both its sentential and inferential versions.)

Note that in each deducibility schema in the list L ∪ L^ the degree of its conclusion, if other than $\wedge$, exceeds the highest degree of its premises. It follows that any deducibility in the closure of these schemata has the same property. Thus the deducibility $A \mathbin{\&} B \therefore A$ will not be contained in the logic for which we are about to provide rules of natural deduction. Nor, as remarked earlier, will $\sim\sim A \therefore A$ or $A, A \supset B \therefore B$.

With an eye to the list L ∪ L^, and to the requirements of Kalmar's Theorem, we see that within the logic T we have to be able to prove

(i) at least some consequences of consistent sets

and

(ii) at least some inconsistencies.

A proof is said to be *simple* just in case its set of undischarged assumptions, along with its conclusion, form a simple argument. A proof is *in normal form* (or *normal*) just in case no sentence occurrence within it stands as the conclusion of an application of an introduction rule and as the major premise of an application of the corresponding elimination rule. A normal proof makes no unnecessary "detours" in leading one from its undischarged assumptions to its conclusion.

In order to generate the deducibility schemata in L ∪ L^, and the deducibilities required in general by Kalmar's Theorem, I now set up the system T of truth table logic.

Definition of T The system T of *truth table logic* consists of normal proofs built up by means of the following rules. Introduction rules appear on the left; the corresponding elimination rules appear on the right. Note that the absurdity rule is absent. In the statement of these rules, □> represents a dual requirement:

(□) There must be an undischarged assumption of the indicated form, available for discharge by the application of the rule in question

(>) That assumption must be the sole undischarged assumption of highest degree within the subproof in question.

Note further the pleasing symmetry between introduction and elimination rules. Introduction rules tell one how to reason towards a complex conclusion; elimination rules tell one how to reduce a complex assumption to absurdity.

Introduction rules | **Elimination rules**

$$\begin{array}{c} \Box\!>\!\overline{A}^{\,(i)} \\ \vdots \\ \dfrac{\wedge}{\sim A}\,(i) \end{array} \qquad\qquad \dfrac{A \quad \sim A}{\wedge}$$

$$\dfrac{A \quad B}{A \,\&\, B} \qquad\qquad \begin{array}{c} \qquad \Box\!>\!\underline{\;\;}\,(i) \\ \qquad A \\ \qquad \vdots \\ \dfrac{A \,\&\, B \quad \wedge}{\wedge}\,(i) \end{array} \qquad \begin{array}{c} \qquad \Box\!>\!\underline{\;\;}\,(i) \\ \qquad B \\ \qquad \vdots \\ \dfrac{A \,\&\, B \quad \wedge}{\wedge}\,(i) \end{array}$$

$$\dfrac{A}{A \vee B} \qquad \dfrac{B}{A \vee B} \qquad\qquad \begin{array}{c} \qquad \Box\!>\!\underline{\;\;}\,(i) \qquad \Box\!>\!\underline{\;\;}\,(i) \\ \qquad A \qquad\qquad\quad B \\ \qquad \vdots \qquad\qquad\quad \vdots \\ \dfrac{A \vee B \quad \wedge \qquad\qquad \wedge}{\wedge}\,(i) \end{array}$$

$$\begin{array}{c} \Box\!>\!\underline{\;\;}\,(i) \\ A \\ \vdots \\ \dfrac{\wedge}{A \supset B}\,(i) \end{array} \quad \dfrac{B}{A \supset B} \qquad\qquad \begin{array}{c} \qquad \Box\!>\!\dfrac{A}{\wedge}\,(i) \qquad \Box\!>\!\underline{\;\;}\,(i) \\ \qquad\qquad\qquad\qquad\quad B \\ \vdots \qquad \vdots \qquad\qquad\quad \vdots \\ \dfrac{A \supset B \quad \wedge \qquad\qquad \wedge}{\wedge}\,(i) \end{array}$$

Reminder: Applications of these rules in T are always subject to the overall requirement that the resulting proof be *normal.* So one cannot apply an introduction rule and then immediately afterwards apply the corresponding elimination rule. Therefore, given that all elimination rules have $\wedge$ as conclusion, we have:

Lemma 1 *In any proof in the system of truth table logic, every major premise for an elimination stands alone, with no sentence occurrences above it.*

By inspection of the rules, and induction on the complexity of proofs, we also have:

Lemma 2 *Every proof in the system* T *of truth table logic is simple; that is, the argument it proves is simple.*

This immediately yields what is in effect a *Kalmaric soundness theorem*: every argument provable in the system T of truth table logic is Kalmaric, that is, simple and valid. Kalmar's Theorem now in effect becomes a *Kalmaric completeness theorem* for the system T of truth table logic: every Kalmaric argument can be proved in T.

Let us now see how the proof of Kalmar's Theorem proceeds for the system T. Note that in the presence of the introduction and elimination rules for $\sim$ just given, it suffices to establish only the inferential version of Kalmar's Theorem.

Kalmar's Theorem (inferential version) *For any (total classical) truth value assignment* ρ *and any sentence* A

(i) *if* A *is true under* ρ, *then there is a* T*-proof of* A *from* $\rho[A]$

(ii) *if* A *is false under* ρ, *then there is a* T*-proof of* $\wedge$ *from* $A, \rho[A]$.

Proof: By induction on the length of sentences A.

Basis: If an atom A is true under ρ then $\rho[A] = \{A\}$ whence, trivially, there is a T-proof of A from $\rho[A]$. If on the other hand A is false under ρ then $\rho[A] = \{\sim A\}$, whence a single step of $\sim$-Elimination is a T-proof of $\wedge$ from $A, \rho[A]$.

Inductive Hypothesis: Assume that the result holds for all sentences less complex (that is, of lower degree) than A.

Inductive Step: Consider A by cases.

($\sim$) A is of the form $\sim B$:

Suppose that A is true under ρ. Then B is false. By the inductive hypothesis there is a T-proof of $\wedge$ from $B, \rho[B]$. By $\sim$-Introduction we extend this to a T-proof of $\sim B$ from $\rho[B]$; i.e., of A from $\rho[A]$.

Now suppose that A is false under ρ. Then B is true. By the inductive hypothesis there is a T-proof of B from $\rho[B]$. Now assume that $\sim B$ and apply $\sim$-Elimination to obtain a T-proof of $\wedge$ from $\sim B, \rho[B]$; i.e., from $A, \rho[A]$.

(&) A is of the form $(B \,\&\, C)$:

Suppose that A is true under ρ. Then B is true and C is true. By the inductive hypothesis there is a T-proof of B from $\rho[B]$ and a T-proof of C from $\rho[C]$. Now apply &-Introduction to obtain a T-proof of $B \,\&\, C$ from $\rho[B \,\&\, C]$.

Now suppose that A is false under ρ. Then either B is false or C is false. Without loss of generality suppose that B is false. Then by the inductive hypothesis there is a T-proof of $\wedge$ from $B, \rho[B]$. Now apply &-Elimination to obtain a T-proof of $\wedge$ from $B \,\&\, C, \rho[B]$:

$$\dfrac{B \,\&\, C \qquad \begin{matrix} \rho[B], \overline{B}^{\,(i)} \\ \vdots \\ \wedge \end{matrix}}{\wedge}\,(i)$$

The result is a T-proof of $\wedge$ from $B \,\&\, C, \rho[B]$.

($\vee$) A is of the form $(B \vee C)$:

Suppose that A is true under ρ. Then B is true or C is true. Suppose without loss of generality that B is true. Then by the inductive hypothesis there is a T-proof of B from $\rho[B]$. Now apply $\vee$-Introduction to obtain a T-proof of $B \vee C$ from $\rho[B]$.

Now suppose that A is false under ρ. Then B is false and C is false. By the inductive hypothesis there is a T-proof of $\wedge$ from $B, \rho[B]$ and a T-proof of $\wedge$ from $C, \rho[C]$. Now apply $\vee$-Elimination to obtain a T-proof of $\wedge$ from $B \vee C, \rho[B], \rho[C]$ (i.e., from $B \vee C, \rho[B \vee C]$).

($\supset$) A is of the form $(B \supset C)$:

Suppose that A is true under ρ. Then B is false or C is true. Suppose first that B is false. Then by the inductive hypothesis there is a T-proof of $\wedge$ from $B, \rho[B]$. Now apply $\supset$-Introduction to obtain a T-proof of $B \supset C$ from $\rho[B]$. Suppose next that C is true. Then by the inductive hypothesis there is a T-proof of C from $\rho[C]$, which can be extended by $\supset$-Introduction to obtain a T-proof of $B \supset C$ from $\rho[C]$.

Now suppose that A is false under ρ. Then B is true and C is false. Thus by the inductive hypothesis there is a T-proof of B from $\rho[B]$ and a T-proof of $\wedge$ from $C, \rho[C]$. By $\supset$-Elimination there is a T-proof of $\wedge$ from $B \supset C$, $\rho[B]$, and $\rho[C]$ (i.e., from $B \supset C, \rho[B \supset C]$):

$$\dfrac{B \supset C \qquad \dfrac{\begin{array}{c}\rho[B]\\ \vdots\\ B\end{array}}{\wedge}\,(i) \qquad \begin{array}{c}\overline{C}^{\,(i)}, \rho[C]\\ \vdots\\ \wedge\end{array}}{\wedge}\,(i) \,.$$

This completes the proof of Kalmar's Theorem for the system T of truth table logic.

We shall now investigate how one might relax the requirement of simplicity in order to obtain more reasonable closure than we have in T. Recall the dual requirement on any application of a discharge rule in the system T:

($\Box$) there must be an undischarged assumption of the indicated form, available for discharge by the application of the rule in question

($>$) that assumption must be the sole undischarged assumption of highest degree within the subproof in question.

Note that although we have the T-proof

$$\dfrac{A \,\&\, B \qquad \dfrac{{}^{(1)}\overline{A} \qquad \sim A}{\wedge}}{\wedge}\,(1)$$

the requirement ($>$) on discharge rules allows us to extend it to

$$\dfrac{\dfrac{{}^{(2)}\overline{A \,\&\, B} \qquad \dfrac{{}^{(1)}\overline{A} \qquad \sim A}{\wedge}}{\wedge}\,(1)}{\sim (A \,\&\, B)}\,(2)$$

but *not* to

$$\dfrac{\dfrac{A \,\&\, B \qquad \dfrac{(1)\dfrac{}{A} \quad \dfrac{}{\sim A}(2)}{\wedge}}{\wedge}(1)}{\sim\sim A}(2)$$

But surely, insofar as truth-preservation is concerned, it does not matter *which* of the premises in a *reductio* one chooses for subsequent discharge and denial? Note that the requirement (>) concerns only subproofs having the form of a *reductio*. Therefore this Duhemian objection to the overly restrictive nature of (>) can be pressed quite generally.

The most obvious relaxation is therefore to drop the requirement (>) but still to retain the requirement (□) and the requirement that proofs be normal. We thereby arrive at what might be called the *Duhemian extension* of the system T, which I shall call T*.

Definition of T* Let T* be the system, based on the same rules as given above for T, that results from dropping the requirement (>) but still retaining the requirement (□) and the requirement that proofs be normal.

An immediate difference between T and T* is that in T* one can now prove $\sim A \therefore \sim(A \,\&\, B)$. It should also be obvious that T* contains T.

An interesting pathology of T* (hence also of T) is that it lacks $A \supset A$ as a theorem. One sees this by inspection of the (two halves of the) ⊃-Introduction rule. Yet there is the following proof in T (hence in T*) of $\sim\sim(A \supset A)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A}(1)}{A \supset A} \quad \dfrac{}{\sim(A \supset A)}(2)}{\wedge}}{A \supset A}(1) \qquad \dfrac{}{\sim(A \supset A)}(2)}{\dfrac{\wedge}{\sim\sim(A \supset A)}(2)}$$

Likewise, T* does not contain disjunctive syllogism: $A \vee B$, $\sim A \therefore B$, nor the usual forms of &-elimination: $A \,\&\, B \therefore A$ and $A \,\&\, B \therefore B$. But T (and hence T*) does contain $A \vee B$, $\sim A \therefore \sim\sim B$, and, as we saw above, $A \,\&\, B \therefore \sim\sim A$ and $A \,\&\, B \therefore \sim\sim B$.

Although our introduction and elimination rules for ⊃ in the systems T and T* have been framed so as to make $A \supset B$ look as though it ought to be equivalent to $\sim A \vee B$, these two formulas nevertheless fail to imply each other in T* (hence also in T). Note that in intuitionistic logic (and even in intuitionistic relevant logic) $\sim A \vee B$ implies $A \supset B$, though not conversely. To see that $\sim A \vee B$ nevertheless does *not* imply $A \supset B$ in T*, note that $\sim A \vee B$ does not imply B, and is consistent with A. Thus neither half of the rule of ⊃-Introduction in T* can be invoked to produce $A \supset B$ as the conclusion of a proof with $\sim A \vee B$ as its only undischarged assumption.

In T* the contraction inference $A \supset (A \supset B) : A \supset B$ also fails. Note that in any proof Π in T (T*) the subtree subtended by any sentence occurrence within Π is itself a proof in T (T*). That is to say, if one prunes away a subtree determined by a sentence occurrence within Π, what comes away is a proof. But what is left of Π (even after restoring the sentence occurrence in question) might not be. For example, in T* we have the following proof:

$$\dfrac{\dfrac{\dfrac{A \quad \overline{B}^{\,(1)}}{A \mathbin{\&} B} \qquad {\sim}(A \mathbin{\&} B)}{\wedge}}{{\sim}B}\,(1)$$

The subtree determined by the occurrence of $A \mathbin{\&} B$ is a T*-proof:

$$\dfrac{A \quad B}{A \mathbin{\&} B}\,.$$

But what is left of the original T*-proof after the latter has been pruned away, even after restoring the occurrence of $A \mathbin{\&} B$, is not a proof in T*:

$$\dfrac{\dfrac{A \mathbin{\&} B \qquad {\sim}(A \mathbin{\&} B)}{\wedge}}{{\sim}B}$$

because there is no occurrence within it of B as an assumption to be discharged by the final step of ~-Introduction.

This highlights the difference between two kinds of closure:

(I) Any pruned-away fragment of a proof is a proof.
(II) The residue of any pruning-away of a fragment from a proof is a proof.

Systems such as T and T*, and the relevant systems of classical and intuitionistic logic developed in [14] and [17], satisfy only (I). By contrast, the systems of classical, intuitionistic, and minimal logic satisfy both (I) and (II).

4 Proof theory for T and T* T* contains T. The main feature of proofs in T* is that they have to be in normal form, and must actually contain assumption occurrences as indicated by the various discharge rules. In this regard we shall call T* a *tight* system. T, obviously, is also tight. T, T*, and the relevant systems of intuitionistic and classical logic just mentioned are moreover what I shall call *trim*, in that they lack the absurdity rule (*ex falso quodlibet*):

$$\dfrac{\wedge}{A}\,.$$

These observations motivate the following definitions of various changes one can make to a given proof system framed in terms of introduction and elimination rules:

One *tightens* by requiring proofs to be in normal form and making assumption occurrences obligatory; one *trims* by banning the absurdity rule.

Conversely, respectively:

One *slackens* by dropping the requirements of normality and obligatory assumption occurrence; one *bloats* by adding the absurdity rule.

I want eventually to prove the result promised above about T*, that T* is a double negation consistency companion to classical logic in the sense of the

Main Theorem *Every classical proof* $\begin{matrix} X \\ P \\ A \end{matrix}$ *can be converted into a proof* $\begin{matrix} \sim\sim Z \\ R \\ \sim\sim A \end{matrix}$ *or* $\begin{matrix} \sim\sim Z \\ R \\ \wedge \end{matrix}$ *(for some subset Z of X) in* T*.

In this respect T* is like the system IR of intuitionistic relevant logic presented in [17]. But it is much weaker than IR, while yet being a double negation consistency companion to classical logic. It also has $\supset$ as a primitive connective, like the modified system of IR investigated in [18].

T* is thus a proper subsystem of IR. (It contains $\sim A \therefore A \supset B$, however, so it is not a subsystem of minimal logic.) Despite the fact that T* is a proper subsystem of IR, however, if we simply add the rule of double negation elimination to T* we produce a system of classical relevant logic that matches classical logic on all consistent sets of premises, and proves all inconsistencies. This is a striking corollary of our main theorem, which we shall prove shortly.

Even though intuitionistic logic is a double negation companion to classical logic, one cannot hope to achieve our main result—that the system T* of truth table logic is a double negation consistency companion to classical logic—by simply establishing that T* is a consistency companion to intuitionistic logic (thereby dividing the labor between two transitions, the first from C to I, the second from I to T*). For T* is *not* a consistency companion to intuitionistic logic, as the nontheoremhood in T* of $A \supset A$ dramatically shows.

We proceed now to the proof of the main theorem. First we recall two results, but framed in the new terminology just defined, due to Prawitz [11] and myself [14] respectively:

Normalization Theorem *Every proof of A from X in a given slack and bloated system can be converted into a proof in normal form of A from (some subset of) X in that slack and bloated system.*

Extraction Theorem *Every proof in normal form of A from X in the bloated slackening of a given trim and tight system S can be converted into a proof of A or of* $\wedge$ *from (some subset of) X in the system S.*

In order to build on these to obtain our main theorem we need two more theorems. Theorem 1 below draws on the normalization and extraction theorems. Theorem 2 is established independently.

Theorem 1 *Every proof* $\begin{array}{c}X\\ P\\ A\end{array}$ *in slackened bloated* T* *can be converted into a proof* $\begin{array}{c}Z\\ R\\ A\end{array}$ *or* $\begin{array}{c}Z\\ R\\ \wedge\end{array}$ *(for some subset Z of X) in* T*.

Proof: First we *normalize P* by means of the reduction procedures to obtain a proof $\begin{array}{c}Y\\ Q\\ A\end{array}$ (for some subset Y of X) in normal form within slackened bloated T*. Then we *extract* a proof $\begin{array}{c}Z\\ R\\ A\end{array}$ or $\begin{array}{c}Z\\ R\\ \wedge\end{array}$ (for some subset Z of Y) in T*.

Theorem 2 *In the language based on* $\sim$, &, $\vee$, *and* $\supset$ *every classical proof* $\begin{array}{c}X\\ P\\ A\end{array}$ *can be converted into a proof* $\begin{array}{c}\sim\sim X'\\ P'\\ \sim\sim A\end{array}$ *or* $\begin{array}{c}\sim\sim X'\\ P'\\ \wedge\end{array}$ *(for some subset X' of X) in slackened* T*.

Proof: By induction on the length of classical proofs. The method is exactly that of the proof of the generalized Gödel–Glivenko theorem for intuitionistic relevant logic given in Chapter 24 of [17]. One only has to take a little care to establish that the required transforms are indeed available in slackened T*.

Basis: Obvious.

Inductive Hypothesis: Assume that the result holds for all proofs less complex than *P*.

Inductive Step: Consider *P* by cases, according to the rule applied in the last step of *P*. Remember we are now dealing with proofs *P* constructed in accordance with the standard rules of inference (as in [11] or [13]) for classical logic. In particular, we do not require discharge rules actually to discharge assumptions of the indicated form. Moreover, the more familiar introduction and elimination rules for $\supset$ (conditional proof and modus ponens) are in use, as well as both the absurdity rule and the classical rule of reductio.

In each case below, the form of *P* is given in bold. Below it are given the forms of the transform P' in slackened T* as desired, depending on the form that might be taken by the transforms Π' in slackened T* that are guaranteed by the inductive hypothesis for the immediate subproofs Π of *P*.

Case 1: P ends with an application of classical reductio:

$$\begin{array}{c}\mathbf{Y},\ \overline{\mathbf{\sim A}}^{\,(i)}\\ \mathbf{\Pi}\\ \dfrac{\wedge}{\mathbf{A}}\,(i)\end{array}\ .$$

Then P' will have one of the following two forms, depending on the form of Π':

$$\begin{array}{c} \sim\sim Y', \quad \dfrac{\dfrac{(2)\,\dfrac{}{\sim A} \qquad \dfrac{}{\sim\sim A}\,(1)}{\wedge}}{\sim\sim\sim A}\,(1) \\ \Pi' \\ \dfrac{\wedge}{\sim\sim A}\,(2) \end{array}$$

or

$$\begin{array}{c} \sim\sim Y' \\ \Pi' \\ \wedge \end{array} .$$

Case 2: ***P*** ends with an application of the absurdity rule:

$$\begin{array}{c} \boldsymbol{X} \\ \boldsymbol{\Pi} \\ \dfrac{\wedge}{\boldsymbol{A}} \end{array} .$$

Then P' will have the form:

$$\begin{array}{c} \sim\sim X' \\ \Pi' \\ \wedge \end{array} .$$

Case 3: ***P*** ends with an application of $\sim$-Introduction:

$$\begin{array}{c} \boldsymbol{Y}, \overline{\boldsymbol{A}}\,(\boldsymbol{i}) \\ \boldsymbol{\Pi} \\ \dfrac{\wedge}{\sim\boldsymbol{A}}\,(\boldsymbol{i}) \end{array} .$$

Then P' will have one of the following two forms, depending on the form of Π':

$$\begin{array}{c} \sim\sim Y', \quad \dfrac{\dfrac{(2)\,\dfrac{}{A} \qquad \dfrac{}{\sim A}\,(1)}{\wedge}}{\sim\sim A}\,(1) \\ \Pi' \\ \dfrac{\dfrac{\dfrac{\wedge}{\sim A}\,(2) \qquad \dfrac{}{\sim\sim A}\,(3)}{\wedge}}{\sim\sim\sim A}\,(3) \end{array}$$

or

$$\begin{array}{c} \sim\sim Y' \\ \Pi' \\ \wedge \end{array} .$$

***Case 4:* P** ends with an application of ~-Elimination:

$$\dfrac{\begin{matrix} \boldsymbol{Y} \\ \boldsymbol{\Pi} \\ \boldsymbol{A} \end{matrix} \qquad \begin{matrix} \boldsymbol{Z} \\ \boldsymbol{\Sigma} \\ {\sim}\boldsymbol{A} \end{matrix}}{\Lambda} \; .$$

Then P' will have one of the following three forms, depending on the forms of Π' and Σ':

$$\dfrac{\begin{matrix} {\sim}{\sim}Y' \\ \Pi' \\ {\sim}{\sim}A \end{matrix} \qquad \begin{matrix} {\sim}{\sim}Z' \\ \Sigma' \\ {\sim}{\sim}{\sim}A \end{matrix}}{\Lambda}$$

or

$$\begin{matrix} {\sim}{\sim}Y' \\ \Pi' \\ \Lambda \end{matrix}$$

or

$$\begin{matrix} {\sim}{\sim}Z' \\ \Sigma' \\ \Lambda \end{matrix} \; .$$

***Case 5:* P** ends with an application of &-Introduction:

$$\dfrac{\begin{matrix} \boldsymbol{Y} \\ \boldsymbol{\Pi} \\ \boldsymbol{A} \end{matrix} \qquad \begin{matrix} \boldsymbol{Z} \\ \boldsymbol{\Sigma} \\ \boldsymbol{B} \end{matrix}}{\boldsymbol{A} \;\&\; \boldsymbol{B}} \, .$$

Then P' will have one of the following three forms, depending on the forms of Π' and Σ':

$$\dfrac{\dfrac{\dfrac{\Lambda}{{\sim}B}\,(2) \qquad \begin{matrix} {\sim}{\sim}Z' \\ \Sigma' \\ {\sim}{\sim}B \end{matrix}}{\dfrac{\Lambda}{{\sim}{\sim}(A \,\&\, B)}\,(3)}}{} \quad\text{where}\quad \dfrac{\Lambda}{{\sim}B}\,(2) \text{ is obtained from } \dfrac{\dfrac{\Lambda}{{\sim}A}\,(1) \qquad \begin{matrix} {\sim}{\sim}Y' \\ \Pi' \\ {\sim}{\sim}A \end{matrix}}{\Lambda}, \quad \dfrac{\Lambda}{{\sim}A}\,(1) \text{ from } \dfrac{\dfrac{\overline{A}^{(1)} \quad \overline{B}^{(2)}}{A \,\&\, B} \qquad \overline{{\sim}(A \,\&\, B)}^{(3)}}{\Lambda}$$

or

$$\begin{matrix} {\sim}{\sim}Y \\ \Pi' \\ \Lambda \end{matrix}$$

or

$$\begin{array}{c} \sim\sim Z' \\ \Sigma' \\ \Lambda \end{array} .$$

Case 6: ***P*** ends with an application of &-Elimination:

$$\dfrac{\begin{array}{c} \boldsymbol{X} \\ \boldsymbol{\Pi} \\ \boldsymbol{A} \,\&\, \boldsymbol{B} \end{array}}{\boldsymbol{A}} .$$

Then P' will have one of the following two forms, depending on the form of Π':

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \,\&\, B}\,(2) \qquad \dfrac{\dfrac{}{A}\,(1) \quad \dfrac{}{\sim A}\,(3)}{\Lambda}}{\Lambda}\,(1)}{\dfrac{\Lambda}{\sim(A \,\&\, B)}\,(2)} \qquad \begin{array}{c} \sim\sim X' \\ \Pi' \\ \sim\sim(A \,\&\, B) \end{array}}{\dfrac{\Lambda}{\sim\sim A}\,(3)}$$

or

$$\begin{array}{c} \sim\sim X' \\ \Pi' \\ \Lambda \end{array} .$$

Case 7: ***P*** ends with an application of ∨-Introduction:

$$\dfrac{\begin{array}{c} \boldsymbol{X} \\ \boldsymbol{\Pi} \\ \boldsymbol{A} \end{array}}{\boldsymbol{A} \vee \boldsymbol{B}.}$$

Then P' will have one of the following two forms, depending on the form of Π':

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A}\,(1)}{A \vee B} \qquad \dfrac{}{\sim(A \vee B)}\,(2)}{\Lambda}}{\sim A}\,(1) \qquad \begin{array}{c} \sim\sim X' \\ \Pi' \\ \sim\sim A \end{array}}{\dfrac{\Lambda}{\sim\sim(A \vee B)}\,(2)}$$

or

$$\begin{array}{c} \sim\sim X' \\ \Pi' \\ \Lambda \end{array} .$$

***Case 8:* P** ends with an application of $\vee$-Elimination:

$$\dfrac{\begin{array}{c}\boldsymbol{Y}\\ \boldsymbol{\Pi}\\ \boldsymbol{A \vee B}\end{array} \qquad \begin{array}{c}\boldsymbol{Z}, \overline{\boldsymbol{A}}^{(i)}\\ \boldsymbol{\Sigma}\\ \boldsymbol{C}\end{array} \qquad \begin{array}{c}\boldsymbol{W}, \overline{\boldsymbol{B}}^{(i)}\\ \boldsymbol{\Xi}\\ \boldsymbol{C}\end{array}}{\boldsymbol{C}}\,(i).$$

Then P' will have one of the following nine forms, depending on the forms of Π', Σ', and Ξ':

$$\dfrac{\dfrac{\dfrac{\overline{A \vee B}^{(4)} \quad \dfrac{{}^{(5)}\overline{\sim C} \quad \begin{array}{c}\sim\sim Z', \dfrac{\dfrac{{}^{(3)}\overline{A} \quad \overline{\sim A}^{(1)}}{\Lambda}}{\sim\sim A}(1)\\ \Sigma'\\ \sim\sim C\end{array}}{\Lambda} \quad \dfrac{{}^{(5)}\overline{\sim C} \quad \begin{array}{c}\sim\sim W', \dfrac{\dfrac{{}^{(3)}\overline{B} \quad \overline{\sim B}^{(2)}}{\Lambda}}{\sim\sim B}(2)\\ \Xi'\\ \sim\sim C\end{array}}{\Lambda}}{\Lambda}(3)}{\sim(A \vee B)}(4) \quad \begin{array}{c}\sim\sim Y'\\ \Pi'\\ \sim\sim(A \vee B)\end{array}}{\dfrac{\Lambda}{\sim\sim C}(5)}$$

or

$$\dfrac{\dfrac{\dfrac{\overline{A \vee B}^{(4)} \quad \begin{array}{c}\sim\sim Z', \dfrac{\dfrac{{}^{(3)}\overline{A} \quad \overline{\sim A}^{(1)}}{\Lambda}}{\sim\sim A}(1)\\ \Sigma'\\ \Lambda\end{array} \quad \dfrac{{}^{(5)}\overline{\sim C} \quad \begin{array}{c}\sim\sim W', \dfrac{\dfrac{{}^{(3)}\overline{B} \quad \overline{\sim B}^{(2)}}{\Lambda}}{\sim\sim B}(2)\\ \Xi'\\ \sim\sim C\end{array}}{\Lambda}}{\Lambda}(3)}{\sim(A \vee B)}(4) \quad \begin{array}{c}\sim\sim Y'\\ \Pi'\\ \sim\sim(A \vee B)\end{array}}{\dfrac{\Lambda}{\sim\sim C}(5)}$$

or

$$\dfrac{\dfrac{\dfrac{\overline{A \vee B}^{(4)} \quad \dfrac{{}^{(5)}\overline{\sim C} \quad \begin{array}{c}\sim\sim Z', \dfrac{\dfrac{{}^{(3)}\overline{A} \quad \overline{\sim A}^{(1)}}{\Lambda}}{\sim\sim A}(1)\\ \Sigma'\\ \sim\sim C\end{array}}{\Lambda} \quad \begin{array}{c}\sim\sim W', \dfrac{\dfrac{{}^{(3)}\overline{B} \quad \overline{\sim B}^{(2)}}{\Lambda}}{\sim\sim B}(2)\\ \Xi'\\ \Lambda\end{array}}{\Lambda}(3)}{\sim(A \vee B)}(4) \quad \begin{array}{c}\sim\sim Y'\\ \Pi'\\ \sim\sim(A \vee B)\end{array}}{\dfrac{\Lambda}{\sim\sim C}(5)}$$

or

$$\dfrac{\dfrac{\dfrac{\overline{A \vee B}^{\,(4)} \qquad \begin{array}{c} \sim\sim Z',\ \dfrac{\dfrac{{}^{(3)}\overline{A} \quad \overline{\sim A}^{\,(1)}}{\Lambda}}{\sim\sim A}\,(1) \\ \Sigma' \\ \Lambda \end{array} \qquad \begin{array}{c} \sim\sim W',\ \dfrac{\dfrac{{}^{(3)}\overline{B} \quad \overline{\sim B}^{\,(2)}}{\Lambda}}{\sim\sim B}\,(2) \\ \Xi' \\ \Lambda \end{array}}{\Lambda}\,(3)}{\sim(A \vee B)}\,(4) \qquad \begin{array}{c} \sim\sim Y' \\ \Pi' \\ \sim\sim(A \vee B) \end{array}}{\Lambda}$$

or

$$\begin{array}{c} \sim\sim Y' \\ \Pi' \\ \Lambda \end{array}$$

$$\begin{array}{ccc} \text{or} & & \text{or} \\ \sim\sim Z' & & \sim\sim Z' \\ \Sigma' & & \Sigma' \\ \sim\sim C & & \Lambda \\ \text{or} & & \text{or} \\ \sim\sim W' & & \sim\sim W' \\ \Xi' & & \Xi' \\ \sim\sim C & & \Lambda\ . \end{array}$$

Case 9: ***P*** ends with an application of ⊃-Introduction:

$$\begin{array}{c} \boldsymbol{Y}, \overline{\boldsymbol{A}}^{\,(\boldsymbol{i})} \\ \boldsymbol{\Pi} \\ \dfrac{\boldsymbol{B}}{\boldsymbol{A} \supset \boldsymbol{B}}\,(\boldsymbol{i}). \end{array}$$

Then P' will have one of the following four forms, depending on the form of Π':

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{B}^{\,(2)}}{A \supset B} \qquad \overline{\sim(A \supset B)}^{\,(4)}}{\Lambda}}{\sim B}\,(2) \qquad \begin{array}{c} \sim\sim Y',\ \dfrac{\dfrac{{}^{(3)}\overline{A} \quad \overline{\sim A}^{\,(1)}}{\Lambda}}{\sim\sim A}\,(1) \\ \Pi' \\ \sim\sim B \end{array}}{\Lambda}}{A \supset B}\,(3) \qquad \overline{\sim(A \supset B)}^{\,(4)}}{\Lambda}}{\sim\sim(A \supset B)}\,(4)$$

or

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\sim\sim Y',\ \dfrac{\dfrac{{}^{(3)}\overline{A}\qquad\overline{\sim A}^{(1)}}{\wedge}}{\sim\sim A}\,(1)\\ \Pi'\\ \wedge\end{array}}{A\supset B}\,(3)\qquad\overline{\sim(A\supset B)}\,(4)}{\wedge}}{\sim\sim(A\supset B)}\,(4)$$

or

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{B}^{(1)}}{A\supset B}\qquad\overline{\sim(A\supset B)}^{(2)}}{\wedge}}{\sim B}\,(1)\qquad\begin{array}{c}\sim\sim Y'\\ \Pi'\\ \sim\sim B\end{array}}{\wedge}}{\sim\sim(A\supset B)}\,(2)$$

or

$$\begin{array}{c}\sim\sim Y'\\ \Pi'\\ \wedge\end{array}\ .$$

Case 10: ***P*** ends with an application of $\supset$-Elimination:

$$\dfrac{\begin{array}{c}\boldsymbol{Y}\\ \boldsymbol{\Pi}\\ \boldsymbol{A}\end{array}\qquad\begin{array}{c}\boldsymbol{Z}\\ \boldsymbol{\Sigma}\\ \boldsymbol{A\supset B}\end{array}}{\boldsymbol{B}}\ .$$

Then P' will have one of the following three forms, depending on the forms of Π' and Σ':

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{}^{(1)}\overline{A}\qquad\overline{A\supset B}^{(2)}}{B}\qquad\overline{\sim B}^{(3)}}{\wedge}}{\sim A}\,(1)\qquad\begin{array}{c}\sim\sim Y'\\ \Pi'\\ \sim\sim A\end{array}}{\wedge}}{\sim(A\supset B)}\,(2)\qquad\begin{array}{c}\sim\sim Z'\\ \Sigma'\\ \sim\sim(A\supset B)\end{array}}{\wedge}}{\sim\sim B}\,(3)$$

or

$$\begin{array}{c}\sim\sim Y'\\ \Pi'\\ \wedge\end{array}$$

or

$$\begin{array}{c} \sim\sim Z' \\ \Sigma' \\ \wedge \end{array} .$$

This completes the proof of Theorem 2.

Finally, we have our:

Main Theorem *Every classical proof* $\begin{array}{c} X \\ P \\ A \end{array}$ *can be converted into a proof* $\begin{array}{c} \sim\sim Z \\ R \\ \sim\sim A \end{array}$ *or* $\begin{array}{c} \sim\sim Z \\ R \\ \wedge \end{array}$ *(for some subset Z of X) in* T*.

Proof: Apply the method of Theorem 2 and then that of Theorem 1.

In the light of our main result, we see that T* satisfies two rather nice new closure conditions, generalizing the conditions, noted earlier, that in T we had to be able to prove:

(i) at least some consequences of consistent sets
(ii) at least some inconsistencies.

The two new closure conditions are, respectively, that the logic deliver:

(i′) transitivity of deducibility under consistent accumulation of premises (which I shall call the *consistent transitive closure condition*) and
(ii′) deducibility of *all* inconsistencies (which I shall call the *inconsistency closure condition*).

The satisfaction of each of these conditions follows immediately from the main theorem.

5 Comparison with other embeddability results The literature contains many examples of embeddings of one system into another by means of schematically definable translations. In this section we survey all of these, and summarize the results so far obtained. This gives the context that imparts the special interest to our main result above.

Abbreviations of systems

C	is classical logic
I	is intuitionistic logic
M	is minimal logic
MR	is minimal relevant logic
IR	is intuitionistic relevant logic
T*	is the system of truth table logic of this paper
CA (IA)	is classical (intuitionistic) arithmetic
CZF	is classical Zermelo-Fraenkel set theory
IZF\E	is intuitionistic Zermelo-Fraenkel set theory minus the axiom of extensionality.

The translation mappings

	atom A	**$\sim A$**	**$A \,\&\, B$**	**$A \vee B$**	**$A \supset B$**	**$\exists xA$**	**$\forall xA$**
id	A	#	#	#	#	#	#
=	the double negation translation, involving only prefixing with $\sim\sim$						
+	#	#	#	#	#	#	$\forall x \sim\sim A$
t	the composition + followed by =						
D	$\sim\sim A$	#	#	#	#	#	#
*	replaces atomic A within S by $\sim\sim A$ iff A is S or is immediately within the scope of &, $\vee$, or $\supset$ (as consequent)						
ˆ	replaces atomic A within S by $\sim\sim A$ iff A is S or is immediately within the scope of & or $\vee$						
P	$A \vee \bot$ $(A \neq \bot)$	#	#	#	#	#	#
K	$\sim\sim A$	$\sim\sim\sim KA$	$\sim\sim(KA \,\&\, KB)$	$\sim\sim(KA \vee KB)$	$\sim\sim(KA \supset KB)$	$\sim\sim\exists xKA$	$\sim\sim\forall xKA$
c	$\sim\sim A$	$\sim cA$	$\sim\sim(cA \,\&\, cB)$	$\sim\sim(cA \vee cB)$	$\sim\sim(cA \supset cB)$	$\sim\sim\exists xcA$	$\sim\sim\forall xcA$
o	#	#	#	$\sim(\sim oA \,\&\, \sim oB)$	#	$\sim\forall x \sim oA$	#
G	$\sim\sim A$	#	#	$\sim(\sim GA \,\&\, \sim GB)$	#	$\sim\forall x \sim GA$	#
′	#	#	#	$\sim(\sim{}'A \,\&\, \sim{}'B)$	$\sim({}'A \,\&\, \sim{}'B)$	$\sim\forall x \sim{}'A$	#
×	#	#	#	#	${}^{\times}A \supset ({}^{\times}B \vee \bot)$	#	$\forall x({}^{\times}A \vee \bot)$
F	$\sim\sim A$	#	#	$\sim\sim(FA \vee FB)$	#	$\sim\sim\exists xFA$	#

α replaces, in the language of set theory, each atomic formula $a \in b$ by the formula

$$(\exists v \in b)(\exists u)(\forall x \forall y(\langle x,y\rangle \in u \supset ((\forall z \in x)(\exists w \in y)\langle z,w\rangle \in u \,\&\, (\forall w \in y)(\exists z \in x)\langle z,w\rangle \in u)) \,\&\, \langle v,a\rangle \in u).$$

If f is a mapping from the language L into itself, we say that *f distributes over @* iff

$$f(A \ @ \ B) = (f(A)) \ @ \ (f(B)) \text{ and } f(QxA) = Qxf(A).$$

We list in tabular form on p. 480 the various translation mappings that have appeared in the literature. Whenever the entry # appears the translation mapping is understood to be distributive for the case in question. The sentence S to be translated can have one of the forms shown, with respect to which the translation mappings can be specified as given.

Forms of claims

$f: U, V$ means "f maps deducibility in U to deducibility in V"

$f: (U, V)$ means "f maps theoremhood in U to theoremhood in V"

$f: [U, V]$ means "if A is deducible in U from X, then either fA or $\bot$ is deducible in V from some subset of fX".

(In the terminology used earlier, we could also say that *V is an f consistency companion to U.* Note that in terms of sequents, we could also say "if the sequent $X : Y$ is provable in U, then for some subsequent $X' : Y'$, $fX' : fY'$ is provable in V"; or, thinking of *concentration* as the converse of *dilution*, we could say that f maps every deducibility in U to some (possibly more concentrated) deducibility in V; or, *f concentrates deducibility in U to deducibility in V.*)

$f: U, V - \{\ldots\}$ means "in the language without the operators . . ., f maps deducibility in U to deducibility in V"

$f: [U, V] - \{\ldots\}$ means "in the language without the operators . . ., f concentrates deducibility in U to deducibility in V".

Note that when the target system V is relevant, in the sense that it does not contain the first Lewis paradox $A, \sim A \therefore B$, and the source system is not, the best results we can hope for in general are of the form $f: [U, V]$ rather than $f: U, V$.

Results	**Sources**
K : C,I	Kolmogorov 1925 [8] (propositional logic only)
= : (C,I)	Glivenko 1929 [5] (propositional logic only)
id : (C,I) − {∨,⊃}	Gödel 1932-3 [6] (propositional logic only)
′(C,I)	Gödel 1932-3 [6]
′ : CA,IA	Gödel 1932-3 [6]
G : C,I	Gentzen 1934 [3]
o : CA,IA	Gentzen 1936 [4]
o^ : CA,IA	Kleene 1952 [7]
*′ : CA,IA	Kleene 1952 [7]

if S is free of $\vee,\exists$ then S is deducible in IA from $\sim\sim S$	Kleene 1952 [7]
$=: C,I - \{\forall\}$	Kleene 1952 [7]
$=: CA,IA - \{\forall\}$	Kleene 1952 [7]
$id: CA,IA - \{\vee,\exists\}$	Kleene 1952 [7]
if all atomic constituents other than those that are antecedents of implications are negated, then id : C,I	Kleene 1952 [7]
c : (C,M)	Church 1956 [1]
if S has no negative occurrence of $\forall$ and $\sim S$ is a classical theorem then $\sim S$ is an intuitionistic theorem	Minc and Orevkov 1963 [10]
K : C,M	Prawitz-Malmnäs 1968 [12]
G : C,I	Prawitz-Malmnäs 1968 [12]
$^{\times}$: (I,M)	Prawitz-Malmnäs 1968 [12]
F : (C,I)	Friedman 1973 [2]; via Kleene 1952 [7]
$F\alpha$: CZF, IZF\E	Friedman 1973 [2]
$=: C,M - \{\forall,\supset\}$	Tennant 1978 [13]
t : C,I	Tennant 1978 [13]
$t: C,M - \{\supset\}$	Tennant 1978 [13]
P : I,M	Leivant 1985 [9]
D : I,M	Leivant 1985 [9]
$id: [I,IR] - \{\supset\}$	Tennant 1987 [17]
$=: [C,MR] - \{\vee,\supset,\forall\}$	Tennant 1987 [17]
$=: [C,IR] - \{\supset,\forall\}$	Tennant 1987 [17]
$t: [C,MR] - \{\vee,\supset\}$	Tennant 1987 [17]
$t: [C,IR] - \{\supset\}$	Tennant 1987 [17]
id : [I,IR]	Tennant 1988 [18]
$=: [C,IR] - \{\forall\}$	Tennant 1988 [18]
t : [C,IR]	Tennant 1988 [18]

The main result (for propositional logic) of this paper is:

$$= : [C,T^*].$$

It says that the simplest translation concentrates the largest system into the smallest (relevant) one.

6 The question of a semantics Our inquiry had a "semantical" starting point—the left-right readings of the truth tables—but ended with purely proof-theoretical results. We make no apology in reply to any objection that a semantics is somehow "missing" for the system T*. For the whole burden of the analysis has been to discover just what the supposedly semantic nature of the logical constants, as "given" by the truth tables, really consists in. The general philosophical conviction behind the enterprise is that the intuitions of semanticists really only arise from the manipulations they would be disposed to make in syntax, that is, from the rules of inference and of proof according to which they would make their deductive transitions. (For a more extended defense of this standpoint, see [16].) Our inquiry has focused on the question of what exactly these transitions ought to be taken to be, if one is simply given the truth tables as a supposedly "semantical" account of the workings of the logical connectives.

If the reader were to insist on an orthodox "semantics", in terms of which soundness and completeness results could be given, then let it be the one given earlier. Focus on the *simple* sequents, namely those of the form $\rho[A]:A$ or $\rho[A],A:\bot$. Now consider those among them that are valid, that is, the *Kalmaric* sequents. These are the valid arguments used in calculating the truth value of A under the various assignments ρ. The proof system T given above captures exactly the Kalmaric sequents.

Dropping the discharge requirement (>) on proofs in the system T, thereby getting the system T*, corresponds to widening one's interest in a Duhemian spirit so as to include all premises of *reductio* proofs as candidates for discharge. This widening takes place without changing the fundamental "semantical" character of the logical connectives, as given by the left–right readings of their truth tables, and as now captured by the rules of proof for the system T*. The thought was that if T* were the right syntactic story to be told about the connectives thus given, we should be able to show that we had got hold of the uncorrupted core of the slack and bloated "understanding" of the connectives claimed by the classicist. We have seen that double negation concentrates classical deducibility in the system T*, thereby showing that the classicist is only two illicit steps beyond the pale. First, he or she uses *ex falso quodlibet*, which, in the terminology developed here, shows a gross lapse of concentration; and secondly, he or she uses double negation elimination (or some equivalent): a metaphysical article of faith which is all Wittgensteinian *show* but which enjoys no truth-tabular *go*.

7 A further application of truth table logic Quantified versions of T and T* can be defined in the obvious way. The adoption of potentially infinitary versions of ∀-Introduction and ∃-Elimination for T yields a system that enables one, in a perfectly precise way, to account for the way in which basic facts about the world can render a complex sentence true, or render it false. This enables one to reformulate Ayer's celebrated and ill-fated criterion of verifiability in a most satisfactory way. The explication is consonant with the logical empiricist's original motivating intuitions, and is immune to the Church-type collapses that have bedeviled various attempts to refine Ayer's criterion. This application of the system T, however, is material for a future paper.

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