The Completeness of Provable Realizability

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Abstract   Let A be a propositional formula and $r_A[x]$ express in the predicate logic the statement "x realizes A". We prove that the classical derivability of $r_A[t]$ for a lambda term t implies the intuitionistic derivability of A for the formulas A in the languges $(\supset, \wedge, \neg)$ and $(\supset, \&)$, where $\&$ is the so-called strong conjunction.

Intuitionistic logical connectives are often supposed to be determined by semantical constructions. For every such (binary) connective C it should be determined how the construction proving (or justifying) $C(A, B)$ is composed from the ones justifying A, B. This approach is traceable to Brouwer and was clearly formulated in [2] and [6]. Various notions of realizability beginning with [4] can be thought of as formalizations of these ideas. We follow [7] in formalizing further and show that the provability of the formula A in the intuitionistic propositional calculus with implication, negation, conjunction $\wedge$, and strong conjunction $\&$ (see below) coincides with the classical provability of the formula $r_A[t]$, expressing in the language of the predicate calculus the statement: "the λ-term t realizes the formula A". The main results of this paper were announced in [8].

Adding the disjunction connective changes the situation drastically: the familiar formula constructed by G. Rose is unprovable but realizable by a suitable if-then-else term.

Strong conjunction $\&$ is determined by the stipulation: x realizes $A \& B$ if x realizes both A and B.

The definition of the formula $r_A[t]$ and the statement of the problem are taken essentially from [7]. Nevertheless, the propositional calculus formulated in [7] for implication and strong conjunction is incomplete, contrary to the conjecture made in [7]: consider the formula $((A \supset B) \& D) \supset (((A \& C) \supset B) \& D)$. It is realizable by the term $\lambda x.x$ but unprovable in the calculus (see [7]).

If realizing terms are required to be typed (Howard's formulas-as-types) then completeness (for the &-free language) obviously follows from the familiar isomorphism between intuitionistic natural deductions and such terms. (The author learned this formulation from N. N. Nepeivoda.) We here investigate the situation where realizing terms are untyped and the derivation of realizability is classical.

The complete propositional calculus for the strong conjunction presented in this paper (cf. Section 2) uses formulas-as-types [3] to express the understanding of A & B as the existence of one and the same justification (construction) for A and B. (For the language without & the usual intuitionistic propositional calculus would be sufficient.) It turns out to be necessary to take η-conversion ($\lambda x.cx \vdash c$ and $\langle lt, rt \rangle \vdash t$) into account in the definition of the term assignment, although the definition of the equality of terms uses only β-conversion ($\lambda x.t)s \vdash t_x[s]$.

We present first the case of the propositional language without $\wedge$, $\neg$, then take account of $\wedge$ in Section 3, and then consider the system with negation.

Recall that λ-terms (here, simply terms) are formed from individual variables denoted by $x, y, z, \ldots$, by abstraction $\lambda x.t$, and application (ts). Left bracketing is assumed; i.e., $ts_1 \ldots s_n$ stands for $(\ldots (ts_1) \ldots s_n)$.

Predicate formulas will be formed by $\supset$, $\wedge$, $\forall$ from atomic formulas $P(t)$, where P is a monadic predicate symbol and t is a term.

To any propositional formula A (in the language $\supset$, &) we assign a predicate formula $r_A[x]$ in the following way. First we assign to any propositional variable p a predicate symbol P in a 1-1 correspondence.

Definition

$$r_p[x] \equiv P(x)$$

$$r_{A \supset B}[x] \equiv \forall z (r_A[z] \supset r_B[(xz)]) \tag{1}$$

$$r_{A \& B}[x] \equiv r_A[x] \wedge r_B[x]. \tag{2}$$

Let us recall some facts about the λ-calculus (see [1]). We do not distinguish terms differing only by renaming of bound variables. Conversion (or rather β-conversion) of the term u is rewriting according to equality

$$((\lambda x.t)s) = t_x[s] \tag{3}$$

of some subterm of u having the form $(\lambda x.t)s$ into $t_x[s]$. Reduction is a sequence of conversions. The notation

$$u \vdash v \tag{4}$$

means that there exists a reduction of the term u to v, i.e., that u reduces to v. Sometimes $u \vdash v$ stands for the reduction itself. The notation

$$u = v$$

means that u and v can be obtained from each other by applying the usual equality rules together with equation (3). In particular, $u \vdash v$, $u \vdash v_1$ implies that $u = v$, $v = v_1$.

The following statement is one of the basic propositions concerning the λ-calculus.

Church-Rosser Theorem $u = v$ *implies the existence of the term w satisfying $u \vdash w$, $v \vdash w$.*

The derivable objects of our version of the classical predicate calculus are sequents $\Gamma \to \Delta$ where Γ, Δ are lists of formulas.

Our axioms are $\Gamma, P(t) \to \Delta, P(u)$ for $t = u$. The analysis of such axioms should include a justification of the equality $t = u$.

The inference rules are the usual Gentzen-type cutfree rules for the classical predicate calculus.

The admissibility of the cut rule is proved in the standard way using the following obvious remark: If the justification of the equation $t = u$ is given, then we can transform the derivation of the sequent $S[t]$ into the derivation of $S[u]$ by simply replacing t by u.

1 Classical and intuitionistic derivability of formulas of the form $r_A[t]$ The main goal of this section is to prove that the use of the classical predicate calculus does not increase (compared with the intuitionistic predicate calculus) the number of provably realizable formulas if the realizing term is explicitly given. The last restriction is essential; if it is removed, all tautologies become realizable. Take for example the Pierce formula $A \equiv ((p \supset q) \supset p) \supset p$. Putting $B \equiv (p \supset q) \supset p$ we derive $\exists x r_A[x]$ as follows:

$$
\frac{\dfrac{\dfrac{P(b) \quad P((\lambda y.b)y)}{P(b) \to r_A[\lambda y.b]} \quad \dfrac{P(ca) \to P((\lambda y.ca)y)}{P(ca) \to r_A[\lambda y.ca]}}{\dfrac{P(b) \to Q(ab), \exists x r_A x \quad P(ca) \to r_A[\lambda y.ca]}{\dfrac{\to r_{p \supset q}[a], \exists x r_A[x] \quad P(ca) \to \exists x r_A[x]}{\dfrac{r_{p \supset q}[a] \supset P(ca) \to \exists x r_A[x]}{\dfrac{r_B[c] \to P(xc), \exists x r_A[x]}{\exists x \forall y (r_B[y] \supset r_p[xy])}}}}}{} \tag{1.1}
$$

We are interested in derivations of formulas of the form $r_A[t]$. By the subformula property of cutfree derivations they can contain only formulas of the form $r_B[u]$ and

$$
r_F[u] \supset r_G(tu). \tag{1.2}
$$

The latter arise as a result of splitting $r_{F \supset G}[t]$, i.e., $\forall z(r_F[z] \supset r_G[tz])$. Using the standard permutation of rules in Gentzen-type derivations [5] one can arrive at a situation where (1.2) is always split immediately above the formula $r_{F \supset G}[t]$ from which (1.2) was generated. Below we assume that this has been done and will not explicitly list (1.2) in the derivation. This stipulation is to be accounted for by the following proposition.

Lemma 1.1.   *Any sequent occurring in the derivation of the sequent*

$$
r_F[t] \tag{1.3}
$$

is of the form

$$
r_{F_1}[(x_1 t_1)], \ldots, r_{F_n}[(x_n t_n)] \to r_{G_1}[u_1], \ldots, r_{G_m}[u_m] \tag{1.4}
$$

where x_i are variables, t_i are sequences of terms, and u_i are terms, or in short

$$r_F[(xt)] \to r_G[u]. \tag{1.5}$$

Proof: The proof is by induction on the depth of the cutfree derivation, i.e., the distance from the lowermost sequent (1.3).

The base is obvious. The induction step is split into cases according to the last inference rule used.

Case 1: $\to$ &

$$\frac{r_F[(xt)] \to r_G[u], r_K[v]; \quad r_F[(xt)] \to r_G[u], r_L[v]}{r_F[(xt)] \to r_G[u], r_{K\&L}[v]}.$$

Case 2: $\to \supset$

$$\frac{r_F[(xt)], r_A[y] \to r_G[u], r_B[(vy)]}{r_F[(xt)] \to r_G[u], \forall y(r_A[y] \supset r_B[(vy)])}.$$

The new antecedent member $r_A[y]$ is of the required form.

Case 3: & $\to$

$$\frac{r_F[(xt)], r_K[(yv)], r_L[(yv)] \to r_G[u]}{r_F[(xt)], r_{K\&L}[(yv)] \to r_G[u]}.$$

Case 4: $\supset \to$

$$\frac{r_F[(xt)] \to r_G[u], r_K[w] \qquad r_F[(xt)], r_L(zvw) \to r_G[u]}{r_F[(xt)], \underbrace{\forall y(r_K[y] \supset r_L[(zvy)])}_{r_{K\supset L}[(zv)]} \to r_G[u]}.$$

If t is a term and x is a variable, we write $x \in +t$ when x occurs free in any term t' such that $t \vdash t'$ (including t itself). $x \notin +t$ is the negation of $x \in +t$.

Lemma 1.2   *Let T be a sequence of terms, t, w be terms, and x, y, z be variables. Then*

(i) $x \in +(zT)$ *implies* $x \in +(zTt)$
(ii) $x \in +t$ *implies* $x \in +(yTt)$
(iii) $x \notin +t$, $t = w$ *imply* $x \notin +w$
(iv) *if x does not occur in t, then $x \notin +t$*
(v) *if $x \notin +w$, $x \notin +t$, then $x \notin +(wt)$.*

Proof: (i) Any reduction of the term zTt is of the form $zTt \vdash zT't'$, where $T \vdash T'$, $t \vdash t'$, so that x occurs free in the last term.

(ii) Any reduction of the term yTt is of the form

$$yTt \vdash yT't', \text{ where } T \vdash T', t \vdash t'. \tag{1.6}$$

By the latter relation x occurs free in t' and so in $yT't'$.

(iii) $x \in +t$ implies $t \vdash t'$ for some t' which does not contain x free. From $t \vdash t'$, $t = w$, and the Church-Rosser theorem it follows that $t' \vdash t''$, $w \vdash t''$ for some t''. Since no new free variable can appear during the reduction, x is not free in t'', so $x \notin +w$.

(iv) Obvious from $t \vdash t$.

(v) Assume that $w \vdash w'$, $t \vdash t'$ for some w', t' which do not contain x free. This implies $wt \vdash w't'$.

We could give a slightly shorter joint proof of the next two propositions, but we shall separate them in order to make the proofs more transparent.

Lemma 1.3   *Let $x \in +t$, $x \notin +u, yv$ in the sequent*

$$r_H[t], r_F[(yv)] \rightarrow r_G[u]. \tag{1.7}$$

Then the antecedent members of $r_H[t]$ are superfluous. More precisely, any derivation of this sequent can be pruned (i.e., transformed by deleting some formulas and whole sequents) into the derivation of the sequent

$$r_F[(yv)] \rightarrow r_G[u]. \tag{1.8}$$

Proof: By induction on the length of the derivation.

Induction base: Sequent (1.7) is the axiom

$$\Gamma, P(s) \rightarrow P(s'), \Delta, \text{ where } s = s'. \tag{1.9}$$

$P(s)$ cannot belong to the list $r_H[t]$ by Lemma 1.2(iii) since $x \notin +s'$. So $P(s)$ belongs to $r_F[(yv)]$, and hence the preceding part of the antecedent is superfluous and can be pruned.

Induction step: Consider the lowermost rule in the derivation.

1. $\rightarrow \&$ By the inductive assumption, the members of $r_H[t]$ can be pruned from both premises and so from the conclusion.

2. $\rightarrow \supset$

$$\frac{r_H[t], r_F[yv], r_K[z] \rightarrow r_G[u], r_L[wz]}{r_H[t], r_F[yv] \rightarrow r_G[u], r_{K \supset L}[w]}. \tag{1.10}$$

The premise of this rule satisfies the condition of the lemma, i.e., $x \notin +(yv), z, u, (wz)$. Indeed, the terms $(yv), u, w$ satisfy it since they occur in the conclusion, and $z, (wz)$ by Lemma 2(iv) and (v) since z is a new variable. Now we can apply the inductive assumption to the premise and prune $r_H[t]$.

3. $\& \rightarrow$ The inductive assumption is obviously applicable to the premise and the superfluous part can be pruned from it and from the conclusion.

4. $\supset \rightarrow$

$$\frac{r_H[t], r_F[yv] \rightarrow r_G[u], r_K[m]; \quad r_H[t], r_L[zpm], r_F[yv] \rightarrow r_G[u]}{r_H[t], r_{K \supset L}[zp], r_F[yv] \rightarrow r_G[u]}.$$

Consider the possible cases:

1. $x \in +(zp)$. Then $x \in zpm$ by Lemma 1.2(i), so the formulas $r_H[t]$, $r_L[zpm]$ can be pruned from the right premise by the inductive assumption. Therefore one can delete the whole inference (i.e., the branch ending in its left premise) as well as the formulas $r_H[t], r_{K \supset L}[zp]$ from the conclusion.

2. $x \notin +(zp)$.

    2.1. $x \notin +m$. Then we have $x \notin +(zpm)$ by Lemma 1.2(v), and the inductive assumption is applicable to both premises, so $r_H[t]$ can be pruned from them and consequently from the conclusion.

    2.2. $x \in +m$. Then we have $x \in +(zpm)$ by Lemma 1.2(ii) and so we can proceed as in case 1.

Theorem 1.1   *Any classical derivation of the sequent of the form (1.4) can be pruned into an intuitionistic derivation of the sequent*

$$r_F[xt] \to r_{G_i}[u_i] \tag{1.11}$$

for some i.

Proof: By induction on the length of the derivation. The formula G_i which is left (i.e., not pruned) in the succedent of (1.11) will be called a preserved formula.

The induction base is obvious.

In the induction step we consider cases according to the last applied rule:

Case 1: $\to\&$ If a side formula (i.e., one of the conjuncts) is not preserved in one of the premises, then prune the whole inference (i.e., the branch ending in the other premise) and preserve in the conclusion the same formula as in the considered premise. Otherwise preserve the whole inference and its main formula (i.e., the conjunction) in the conclusion.

Case 2: $\to\supset$

$$\frac{r_F[xt],\, r_K[z] \to r_G[u],\, r_L[wz]}{r_F[xt] \to r_G[u],\, \forall x(r_K[x] \supset r_L[wx])} \ .$$

If $r_L[wz]$ is preserved in the premise, then preserve the main formula in the conclusion. Otherwise, the formula $r_K[z]$ is superfluous by Lemma 1.3, since z is a new variable. So it can be pruned from the conclusion, and the same formula is preserved there as in the premise.

Case 3: $\&\to$ The formula preserved in the conclusion is the same one preserved in the premise.

Case 4: $\supset\to$

$$\frac{r_F[xt] \to r_G[u],\, r_K[w];\, r_F[xt],\, r_L[zvw] \to r_G[u]}{r_F[xt],\, \forall y(r_K[y] \supset r_L[zvy]) \to r_G[u]} \ .$$

If the formula $r_K[w]$ is not preserved in the left premise, then prune the whole inference, preserving in the conclusion the same formula as in the left premise. Otherwise preserve in the conclusion the same formula as in the right premise.

2 Completeness of the propositional provability for provable realizability
    The formulation of the main result of this section uses formulas-as-types

introduced by Howard [3]. H-terms are formed from variables $x^A, y^A, \ldots$ (for $\{\supset, \&\}$-propositional formulas A) by application (for suitable types) and λ-abstraction. Such terms are known to be in 1–1 correspondence to propositional natural deductions. t^- stands for the untyped λ-term obtained by the deleting of all types (i.e. superscripts) from the H-term t.

Let us describe a natural deduction system reminiscent of ones introduced in [7] and [9]. We will first prove its equivalence to provable realizability and then to the predicate calculus introduced in [7].

The formulas of this system are constructed from propositional variables denoted by $p, q, r, \ldots$ by the connectives $\&$ and $\supset$. Sequents are expressions of the form $\Gamma \to A$, where Γ is a list of formulas and A is a formula. An *enriched sequent* is an expression of the form $S(t)$ where S is a sequent and t is an H-term, such that to any free variable x^A of the term t some antecedent member A of the sequent S is assigned, and distinct antecedent members are assigned to different free variables.

Axioms $\qquad \Gamma, A \to A(x^A)$ $\hfill$ (2.1)

Inference rules

$(\&^+)$ $\quad \dfrac{\Gamma \to A(t); \; \Gamma \to B \;\; (t)}{\Gamma \to A \; \& \; B \qquad (t)}$

$(\&^-)$ $\quad \dfrac{\Gamma \to A_1 \; \& \; A_2 \;\; (t)}{\Gamma \to A_i \qquad (t)} \quad i = 1, 2$

$(\supset^+)$ $\quad \dfrac{\Gamma, A \to B \qquad (t)}{\Gamma \to (A \supset B) \;\; (\lambda x^A t)} \quad \text{or} \quad \dfrac{(cx^A)}{c}$

$(\supset^-)$ $\quad \dfrac{\Gamma \to A \supset B \;\, (t); \, \Gamma \to A \;\; (s)}{B \qquad\qquad\qquad (ts)}$

It is essential that the relations between enriching terms in the rules are to hold only up to β-conversions. For example, the $\&^+$-inference can have the form

$$\Gamma \to A \;(t_1); \quad \Gamma \to B \;(t_2)/\Gamma \to A \; \& \; B \;(t_3), \quad \text{where } t_1 = t_2 = t_3. \qquad (2.2)$$

The variable x^A in the axiom and in the rule $\supset^+$ is the one assigned to the explicitly shown antecedent occurrence of A.

We say that a formula A is derivable in our propositional system if a sequent $\to A \;(t)$ is derivable for some t.

The main result of this paper is the following proposition:

Theorem 2.1. *The propositional formula A is derivable iff $r_A[t]$ is derivable in the system L for some λ-term t.*

Proof: The easy half of the theorem is an instance of the following proposition.

Lemma 2.2 *Derivability of the propositional sequent*

$$B_1, \ldots, B_n \to A \;(t) \qquad (2.3)$$

implies derivability in L of the sequent

$$r_{B_1}[x_1], \ldots, r_{B_n}[x_n] \to r_A[t^-] \tag{2.4}$$

where $x_1, \ldots, x_n$ *are variables assigned in the term t from (2.3) to the formulas* $B_1, \ldots, B_n$.

Proof: By induction on the length of the derivation of the sequent (2.3).

Induction base: Here (2.3) is $\Gamma, A \to A$ and (2.4) is $\Gamma', r_A[x] \to r_A[x]$.

Induction step: Consider cases according to the last inference rule used:

Case 1: $\&^+$ By the inductive assumption we have L-derivations of the sequents $\Gamma' \to r_A[t_1^-]$ and $\Gamma' \to r_B[t_2^-]$. Taking into account the fact that $t_1^- = t_2^- = t$ we can replace the terms t_1^-, t_2^- by t. Applying $\&^+$ (this time in the natural deduction version of L) we obtain $\Gamma' \to r_{A\&B}[t^-]$ as required.

Case 2: $\&^-$ Similarly.

Case 3: $\supset^+$ By the inductive assumption we have $\Gamma', r_A[x] \to r_B[t^-]$. Using the equation $(\lambda x.u)x = u$ we obtain by the rules of equality the sequent $\Gamma', r_A[x] \to r_B[(\lambda xt^-)x]$, and using $\supset^+, \forall^+$ we have $\Gamma' \to r_{A\supset B}[\lambda xt^-]$ when t is not of the form cx^A. If $t \equiv cx^A$, then one need not even use β-conversion.

Case 4: $\supset^-$ By the inductive assumption we have

$$\Gamma' \to \forall x(r_A[x] \supset r_B[t^-x]) \text{ and } \Gamma' \to r_A[s^-]. \tag{2.5}$$

Using $\forall^-, \supset^-$ we obtain $\Gamma' \to r_B[t^-s^-]$.

The proof of the second half of Theorem 2.1 is based on the fact that the proof of Lemma 2.2 can be "reverted," although this is not so simple.

Lemma 2.3 *Let a sequent*

$$r_{B_1}[x_1], \ldots, r_{B_n}[x_n] \to r_A[t] \tag{2.6}$$

(*or shorter*

$$r_{\boldsymbol{B}}[\boldsymbol{x}] \to r_A[t]) \tag{2.7}$$

be provable in L. Then a propositional sequent

$$B_1, \ldots, B_n \to A \ (u) \tag{2.8}$$

(*or shorter,*

$$\boldsymbol{B} \to A \ (u)) \tag{2.9}$$

is provable for some u such that $u^- = t$.

Proof: If (2.7) is provable, we have by Theorem 1.1 its intuitionistic derivation from the axioms

$$\Gamma, P(s) \to P(s') \tag{2.10}$$

where $s' = s$, and $\Gamma, A \to A$.

Since natural deduction is normalizable, we normalize it and move equality rules into the axioms. The lemma will be proved by induction on the length of such a derivation. By the subformula property all sequents in such a derivation are of the form (2.7) (possibly with different B, A).

Induction base: If (2.10) is at the same time of the form (2.7), then $A[t] \equiv r_B[t]$ and $t = x$, so one can take $u \equiv x$.

Induction step: Consider the last inference rule used.

   1. $\&^+$

$$\frac{r_B[x] \to r_F[t]; \quad r_B[x] \to r_G[t]}{r_B[x] \to r_{F\&G}[t]} .$$

By the inductive assumption we have

$$B \to F \ (u_1); \ B \to G \ (u_2); \ u_1^- = u_2^- = t \tag{2.11}$$

which implies $B \to F \ \& \ G \ (u_1)$ and $u_1^- = t$.

   2. $\&^-$

$$\frac{r_B[x] \to r_{A_1\&A_2}[t]}{r_B[x] \to r_{A_i}[t]} .$$

We have $B \to A_1 \ \& \ A_2 \ (u)$, and obtain $B \to A_i \ (u)$ with $u^- = t$.

   3. $\forall^-, \supset^-$

$$\frac{r_B[x] \to \forall z(r_F[z] \supset r_G[tz]); \ r_B[x] \to r_F[s]}{r_B[x] \to r_G[ts]} . \tag{2.12}$$

It has to be verified that any inference according to the rules $\forall^-, \supset^-$ is indeed a part of some figure (2.12). In fact, since the deduction is normal, any elimination inference is situated in a branch which begins with the axiom $\Gamma, A \to A$ (where A occurs in the antecedent of sequents in (2.12)) and proceeds by elimination rules. So each succedent implication is of the form $r_F[s] \supset r_G[ts]$, where $r_{F\supset G}[t] \equiv \forall x(r_F[x] \supset r_G[tx])$ occurs in the antecedent of the branch in question immediately over the implication considered, so that $\forall^-, \supset^-$ are indeed included in blocks (2.12).

Now using the inductive assumption we obtain $B \to F \supset G \ (u)$, $B \to F \ (v)$, $u^- = t$, $v^- = s$, which implies $B \to G \ (uv)$ with $(uv)^- = ts$.

   4. $\forall^+, \supset^+$

$$\frac{r_B[x], r_F[z] \to r_G[tz]}{r_B[x] \to \forall z(r_F[z] \supset r_G[tz])} .$$

By the inductive assumption we have $B, F \to G \ (u)$, where $u^- = (tz)$. By the Church-Rosser theorem there exists a v such that $u^- \vdash v$, $(tz) \vdash v$. If $v \equiv (t'z)$ with $t \vdash t'$, then in view of $u^- \vdash v$ there exists a term c such that $u \vdash cz$ and $c^- = t'$. In this case we have $B \to F \supset G \ (c)$ and $c^- = t$. Otherwise $(tz) \vdash (\lambda z t')z \vdash t' \vdash v$, where $t \vdash \lambda z t'$ and $u^- = t'$. Then $(\lambda z u)^- = \lambda z t' = t$ and $B \to F \supset G \ (\lambda z u)$, $(\lambda z u)^- = t$.

5.

$$\forall^- \quad \frac{r_B[x] \to \forall y(r_F[y] \supset r_G[ty])}{}$$
$$\forall^+ \quad \frac{r_B[x] \to r_F[b] \supset r_G[tb]}{r_B[x] \to \forall z(r_F[z] \supset r_G[tz]).}$$

To obtain $B \to F \supset G$ (u) with $u^- = t$ it is sufficient to apply the inductive assumption to the upper sequent.

We now show that no other case is possible.

The rules $\supset^-$, $\&^-$ cannot occur immediately over $\forall^+$, since the corresponding main formulas are not subformulas of the formulas having the form $r_H[w]$. Finally, the premise of $\forall^+$ cannot be an axiom since the corresponding antecedent formula would not have the form $r_B[x]$.

3 Completeness of provable realizability for $(\&,\supset,\wedge)$ Now we will extend some of the results and proofs obtained in previous sections to the case where the usual intuitionistic conjunction is added to the propositional language. Here the definition of a (realizing) term is to be extended by adding the binary function constant **pair** (for pairing function) and unary function constants l,r (for left and right projections extracting components of a pair). We use the notation

$$\langle u,v \rangle \equiv \textbf{pair } uv.$$

β-conversion is defined for the new language by the familiar equations

$$(\lambda x.t)u = t_x[u]$$
$$l\langle t,u \rangle = t$$
$$r\langle t,u \rangle = u.$$

We put

$$r_{A \wedge B}[t] \equiv r_A[lt] \wedge r_B[rt].$$

We denote by $\Pi,\Pi_1,\Pi_2,\ldots$ arbitrary (possibly empty) finite sequences of projections l,r, for example l,r,lr,rllr etc. For the language $\supset,\&$ an important part was played by terms of the form xt (i.e., head normal forms beginning with a variable). Now this part will be played by terms of the form

$$\Pi_n(\ldots \Pi_1((\Pi xT)T_1 \ldots T_n)) \tag{3.1}$$

which we call h-terms or xh-terms, to distinguish the head variable x (and we write $t \in h$ or $t \in xh$). The letters $T,T_1,\ldots$ will stand for finite sequences of terms.

Now, in the formulation of Lemma 1.1 one has to replace the terms $(x_1t_1),\ldots,(x_nt_n)$ by $t_1,\ldots,t_n$, that is, to replace the sequent (1.5) by

$$r_F[t] \to r_G[u]$$

with $t \in h$.

To the proof of this lemma given in Section 1 one has to add two cases, corresponding to the rules $\to\wedge$ and $\wedge\to$. It is the latter that introduces the constants l, r into realizing terms.

Lemma 1.2 is restated as follows:

Lemma 1.2*

(i) $x \in +\Pi_n(\ldots(\Pi x T_1)\ldots T_n)$, *i.e.,* $t \in xh$ *implies* $x \in +t$

(iia) *if* $w \in h$, $x \in +t$, *then* $x \in +wt$

(iib) *if* $w \in h$, $x \in +w$, *then* $x \in +\mathrm{l}w, \mathrm{r}w$

(iii) *if* $x \notin +t$, $t = w$, *then* $x \notin +w$

(iv) *if* x *does not occur in* t, *then* $x \notin +t$

(v) *if* $x \notin +w$, $x \notin +t$, *then* $x \notin +(wt), \mathrm{l}t, \mathrm{r}t$.

Proof: (i) Any term t' occurring in any reduction of the term t of the form (3.1) has the same form $\Pi_n(\ldots\Pi_1((\Pi x T')T_1')\ldots T_n')$ with $T \vdash T'$, $T_i \vdash T_i'$. So x is free in t'; that is, $x \in +t$.

(iia) if w is of the form (3.1), then $wt \vdash u$ implies $u \equiv w't'$, with $w \vdash w'$, $t \vdash t'$, so x is free in t' and consequently in u. This implies $x \in +wt$.

(iib) $\Pi w \vdash u$ for a term w of the form (3.1) implies $u \equiv \Pi w'$ with $w \vdash w'$.

(iii), (iv), (v) are proved as in Section 1.

Lemma 1.3 now takes the following form:

Lemma 1.3*   *If we have* $q, t \in h$, $x \in +q$, $x \notin +t, u$ *in the sequent*

$$r_H[q], r_F[t] \to r_G[u]$$

then the antecedent members $r_H[q]$ *can be pruned.*

One has only to add $\wedge$-cases to the proof in Section 1. These are treated even more simply than $\supset$-cases. The proof of Theorem 1.1 is modified with the same ease.

The definition of an *H*-term at the beginning of Section 2 should be supplemented by the constants **pair**, l, r and corresponding term construction rules. The natural deduction calculus is supplemented with the following familiar rules for $\wedge$:

$$(\wedge^-) \quad \frac{\Gamma \to A \wedge B \quad (t)}{\Gamma \to A \quad\quad (\mathrm{l}t)}$$

$$(\wedge^-) \quad \frac{\Gamma \to A \wedge B \quad (t)}{\Gamma \to B \quad\quad (\mathrm{r}t)}$$

$$(\wedge^+) \quad \frac{\Gamma \to A; \Gamma \to B \quad t \quad u}{\Gamma \to A \wedge B \quad\quad \langle t, u \rangle} \ \text{or} \ \frac{\mathrm{l}t \quad \mathrm{r}t}{t}.$$

Let us now show how to supplement the proof of Theorem 2.1. In Lemma 2.2 the rules for $\wedge$ have to be considered. The $\wedge^-$ case is obvious, and that of $\wedge^+$ is treated similarly to $\supset^+$.

Finally, Lemma 2.3 is to be supplemented by treatment of $\wedge$-rules corre-

sponding to formulas $r_{A \wedge B}[t]$. The case of $\wedge^-$ is treated similarly to that of $\supset^-$. Let us consider the case of $\wedge^+$:

$$\frac{r_B[x] \to r_F[1t]; \quad r_B[x] \to r_G[1t]}{r_B[x] \to r_{F\&G}[t]}.$$

By the inductive assumption we have

$$B \to F\ (u);\ B \to G\ (v),\ \text{where}\ u^- = 1t,\ v^- = rt.$$

By the Church-Rosser theorem there are terms u_1, v_1 such that $u^- \vdash u_1$, $1t \vdash u_1$, $v^- \vdash v_1$, $rt \vdash v_1$.

Case 1: $u_1 \equiv 1t'$, $v_1 \equiv rt''$, $t \vdash t', t''$. By the Church-Rosser theorem there is a t_1 such that $t', t'' \vdash t_1$, whence $u^- \vdash 1t_1$, $v^- \vdash rt_1$. So there exists a w such that $u \vdash 1w$, $v \vdash rw$, $w = t$. So $B \to F \& G\ (w)$.

Case 2: Case 1 does not hold. Then at least one of the terms u_1, v_1 is not of the form $\Pi t'$ with $t \vdash t'$. Let it be u_1. Then the reduction $1t \vdash u_1$ is of the form

$$1t \vdash 1\langle t_1, t_2 \rangle \vdash t_1 \vdash u_1$$

with $t \vdash \langle t_1, t_2 \rangle$. So we have $u^- = t_1$, $v^- = t_2$, $t = \langle t_1, t_2 \rangle$, whence $\langle u, v \rangle^- = t^-$. Therefore $B \to F \wedge G\ (\langle u, v \rangle)$, which concludes the proof of the theorem.

4 The system with negation   Let us now extend the preceding results to the language $\supset, \wedge, \neg$. Put

$$r_{\neg A}[x] =_{def} \forall z \neg r_A[z].$$

The basic ideas of the proof are as before, but the details become more complicated. The proof will probably go through for the language with &, but some additional work is needed here, since familiar reduction to formulas of depth 3 does not go through for &. This is because the equivalent replacement theorem is invalid for &. If

$$E = (a \equiv a') \supset (a \& b \equiv a' \& b),\ \text{where}\ (A \equiv B) = ((A \supset B) \& (B \supset A)),$$

then $r_E(t)$ is underivable for any t.

In the remainder of this section we treat only of the language $\supset, \wedge, \neg$.

Recall that familiar depth-reducing transformations are based on the replacement of the subformula P of the given formula F by a new variable p, and adding the equivalence $p \leftrightarrow P$.

Lemma 4.1
(i) *If a formula D is derivable in the intuitionistic propositional calculus, then $r_D(t)$ is derivable for some term t.*
(ii) *For any propositional formula $F[A]$ with a distinguished occurrence of the formula A there is a term t such that*

$$r_{A \leftrightarrow B}[x] \wedge r_{F[A]}[y] \supset r_{F[B]}[(txy)] \tag{4.1}$$

is derivable, where $(A \leftrightarrow B) =_{def} (A \supset B) \wedge (B \supset A)$.

Proof: (i) is established as before by induction on the derivation. The term t encodes the essential content of that derivation.

(ii) follows from (i) and intuitionistic derivability of the implication

$$(A \leftrightarrow B) \supset (F[A] \supset F[B]). \tag{4.2}$$

Definition   A *standard formula* is any propositional formula of the form

$$E_1 \supset (\ldots \supset (E_n \supset p) \ldots) \tag{4.3}$$

where p is a propositional variable and $E_1, \ldots, E_n$ are formulas of one of the forms

$$f, \neg f, f \supset g, f \supset (g \supset h), (f \supset g) \supset h, \neg f \supset g, g \supset \neg f \tag{4.4}$$

where f, g, h are propositional variables.

The following result and its proof are standard.

Lemma 4.2   *Any propositional formula F can be transformed into a standard formula S such that*
(i) $F \supset S$ is provable intuitionistically
(ii) $S^* \supset F$ is provable intuitionistically, where S^* is a substitution instance of S.

Proof: Let us write F in the form

$$E_1 \supset (\ldots \supset (E_k \supset G) \ldots) \tag{4.5}$$

where $E_1, \ldots, E_k$ are of any one of the forms (4.4) and use an induction on the length of G. If G is a variable (induction base) we are done. Otherwise write G as $G[A]$ where A has one of the forms

$$f \,\&\, g, f \supset g, \neg f. \tag{4.6}$$

So F is written as $F[A]$. Note that the following formulas are intuitionistically derivable:

$$
\begin{aligned}
(h \leftrightarrow f \,\&\, g) &\leftrightarrow [(h \supset f) \wedge (h \supset g) \wedge (f \rightarrow (g \rightarrow h))], \\
(h \leftrightarrow (f \supset g)) &\leftrightarrow [((f \supset g) \supset h) \wedge (h \supset (f \supset g))], \\
(h \leftrightarrow \neg f) &\leftrightarrow [(h \supset \neg f) \wedge (\neg f \supset h)], \\
(B \wedge C \supset D) &\leftrightarrow [B \rightarrow (C \rightarrow D)].
\end{aligned} \tag{4.7}
$$

For any A of one of the forms (4.6) denote by E^1, E^2, E^3 (respectively E^1, E^2) conjuncts in the right-hand sides of the equivalences (4.7). We apply the inductive assumption to the formula

$$E^1 \supset (E^2 \supset (E^3 \supset G[h])) \tag{4.8}$$

or, respectively,

$$E^1 \supset (E^2 \supset G[h]), \tag{4.8'}$$

where h is a new propositional variable, and see that the standard formula S constructed for (4.8) or (4.8') would also do for A. Indeed, by the last relation in (4.7) the formula (4.8) or (4.8') is equivalent to

$$(h \leftrightarrow A) \supset F[h].$$

It remains only to apply (4.2).

Now we see that it is sufficient to prove our theorem for standard formulas. In fact we shall transform formulas into sequents.

Definition   A *standard propositional sequent* is an expression

$$E_1, \ldots, E_n \to p \tag{4.9}$$

where p is a propositional variable and $E_1, \ldots, E_n$ are formulas of any of the forms (4.4).

A *standard predicate sequent* is an expression

$$r_{E_1}[x_1], \ldots, r_{E_n}[x_n] \to r_p[t] \tag{4.10}$$

where $E_1, \ldots, E_n$ are formulas of any of the forms (4.4), p is a propositional variable, t is a λ-term, and $x_1, \ldots, x_n$ are distinct variables.

In other words, (4.10) says that (4.9) is realizable by the term t.

The formulation of the classical predicate calculus with the thinning rule presented below is designed for pruning derivations, i.e., deleting superfluous formulas and sequents.

Axioms are of the form:

$$P(t) \to P(s), \quad \text{where } t = s \tag{4.11}$$

and the thinning rule and rules for negation $\neg$ are added:

$$\text{(Th)} \quad \frac{\Gamma \to \Delta}{\Pi, \Gamma \to \Delta, \Sigma} \qquad (\to\neg) \quad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A} \qquad (\neg\to) \quad \frac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta}$$

This formulation with the thinning rule and axioms (4.11) is well known to be equivalent to a formulation without thinning but with the axioms:

$$\Gamma, P(t) \to P(s), \Delta, \quad \text{where } t = s.$$

The admissibility of cut for (and completeness of) our formulation is also established in a standard way. The following proposition will be often used below without special reference.

Note 1   If $t = t'$ then any derivation of the sequent $S[t]$ can be transformed in the derivation of $S[t']$ by replacing some occurrences of t by t'. Indeed, such a replacement preserves all rules (if eigenvariables of $\to\forall$ are suitably renamed), and the axioms are preserved in view of the fact that $t = t'$.

Below we shall always assume that all derivations possess the strong pure variable property: No variable occurs both free and bound, and eigenvariables of different $(\to\forall)$-inferences are distinct and occur only over the premises of these inferences.

Definition   A derivation in the predicate calculus is *pruned* (cf. [5]) if all thinnings are moved maximally downward, i.e., (Th)-inferences occur only in the following three situations: (a) immediately before the lowermost sequent; (b) thinning introduces a side formula of a one-premise rule having two side formulas, and the other side formula is not introduced by a thinning; (c) thinning introduces a parametric formula into one of the premises of a two-premise rule,

while the corresponding formula in the remaining premise is not introduced by thinning.

Let us call a derivation in the predicate calculus *nonsuperfluous* if it does not contain in a single branch identical inferences, i.e. inferences with identical main formulas and (in the case of $\forall\rightarrow$) identical substituted terms.

The following pruning lemma comes essentially from [5].

Lemma 4.3   *Any derivation can be pruned into a pruned derivation with the same last sequent. Moreover, this transformation preserves nonsuperfluousness.*

Proof: Move thinnings downward, deleting if necessary formulas (for example, parametric formulas in one-premise rules) as well as whole inferences when all side formulas of a one-premise rule, or at least one side formula of a two-premise rule, were pruned. The remaining rules are in the same mutual position as before, so the derivation remains nonsuperfluous if it was so before.

The next proposition is obvious and well-known in similar situations.

Lemma 4.4   *Any derivation can be transformed into a nonsuperfluous one by deleting some formulas and inferences.*

Note 2   The derivations of the sequents (4.10) can contain sequents which do not have the form

$$r_F[t] \rightarrow r_G[u]. \tag{4.12}$$

This is because the side formula of the rule introducing the main formula $r_{A\supset B}[s] \equiv \forall z(r_A[z] \supset r_B[(sz)])$ is itself of the form $r_A[u] \supset r_B[(su)]$, and similarly for $r_{\neg p}[s] \equiv \forall z \neg P(z)$. However, by using the invertibility of $\neg, \supset$-rules it is easy to ensure that this side formula would be split immediately, i.e. $\neg, \supset$-rules would have the form:

$$\frac{\Gamma, r_A[z] \rightarrow \Delta, r_B[(uz)]}{\Gamma \rightarrow \Delta, r_{A\supset B}[u]} \qquad \frac{\Gamma \rightarrow \Delta, r_A[u]; \; r_B[(tu)], \Gamma \rightarrow \Delta}{r_{A\supset B}[t], \Gamma \rightarrow \Delta}$$

$$\frac{\Gamma, P(z) \rightarrow \Delta}{\Gamma \rightarrow \Delta, r_{\neg p}[u]} \qquad \frac{\Gamma \rightarrow \Delta, P[(tu)]}{r_{\neg p}[t], \Gamma \rightarrow \Delta} \tag{4.13}$$

We shall always assume that this has been done.

The restriction on the depth of formulas in (4.10) will allow us to restrict the complexity of terms occurring in a derivation.

The notation $t \in h$, $t \in xh$ has almost the same meaning as before: t is of the form $xt_1 \ldots t_n$ where x is a variable but $0 \le n \le 2$.

Lemma 4.5   *Any sequent in the derivation of the sequent (4.10) has the form*

$$r_{E'}[x'], \; r_F[(x''t'')], \; r_H[(x'''t_1''' t_2''')] \rightarrow r_G[u]. \tag{4.12'}$$

Here $r_{E'}[x']$ is a subsequence of the sequence $r_{E_1}[x_1], \ldots, r_{E_n}[x_n]$. $r_F[(x''t'')]$ consists of formulas of the form $r_{F_i}[(x_i t)]$, where E_i is of the form $A \supset F_i$ and below the sequent in question there is a $(\forall\rightarrow)$-inference applied to the formula $r_{E_i}[x_i]$ with the substituted term t. The sequence $r_H[(x'''t_1''' t_2''')]$ consists of formulas of the form $r_H[(x_i t_1 t_2)]$, where E_i is of the form $A \supset$

$(B \supset H)$ and below the sequent in question there is a $(\forall\rightarrow)$-inference with substituted term t_1 introducing $r_{E_i}[x_i]$ and another $(\forall\rightarrow)$-inference with substituted term t_2 introducing the side formula $r_{B\supset H}[(x_i t_1)]$ of the previous inference. The formula G is of the form $f \supset g$, $\neg f, f$ where f and g are propositional variables.

Proof: The proof is by an easy bottom-up induction over the derivation.

In the presence of negation the simple pruning used in Section 2 is insufficient to turn a classical derivation into an intuitionistic one, as the following example suggested by T. Uustalu shows:

$$r_{\neg p \supset g}[x], \ r_{\neg\neg p\supset g}[y] \rightarrow r_g[(xx)], \ r_g(yx). \tag{4.14}$$

The derivability of this sequent is immediate, but deleting any of its succedent members turns it into an underivable one. So a deeper transformation is needed.

Definition   A derivation of the sequent

$$\Gamma \rightarrow \Delta, P(u) \tag{4.15}$$

is *directed at* the formula $P(u)$, if it is an axiom (i.e., consists only of the sequent (4.15)) or has (up to the thinning-inferences) one of the following forms:

$$\tag{4.16}$$

where the upper rightmost sequent is an axiom with succedent $P(u)$, and the explicitly listed $\supset,\forall$-inferences lead immediately to this axiom; i.e., the main formula of the upper rule is the side formula of the lower rule.

Note that if (4.15) is not the axiom on $P(u)$ then in the case of our definition the main formula of the lowermost inference in (4.16) has the form:

$$r_{A\supset p}[t] \text{ or } r_{f\supset(g\supset p)}[t]. \tag{4.17}$$

Lemma 4.6   *Let some derivation of the sequent (4.10) contain an occurrence of the sequent (4.15)*

$$\Gamma \rightarrow \Delta, \ P(u)$$

and let the derivation of the latter be directed at the formula $P(u)$. Then u has up to equality one of the forms

$$x_i, \ (x_i t_1), \ (x_i t_1 t_2). \tag{4.18}$$

Here if $u = x_i$ then $E_i \equiv p$. If $u = x_i t_1$ then the right branch of the derivation (shown explicitly in (4.16)) contains an $(\forall\rightarrow)$-inference with substituted term t_1 and main formula $r_{E_i}[x_i]$. If $u = (x_i t_1 t_2)$ then this branch also contains an $(\forall\rightarrow)$-inference with substituted term t_2 and main formula $r_{B\supset p}[(x_i t_1)]$, where $E_i \equiv A \supset (B \supset p)$, and the main formula of the lower mentioned $\forall\rightarrow$ is the successor of the main formula of the upper $\forall\rightarrow$.

Proof: Use induction on the given derivation and Lemma 4.5.

Lemma 4.7   *Let a sequent*

$$\Gamma \rightarrow \Delta, P(u) \tag{4.19}$$

occur in a pruned nonsuperfluous derivation of the sequent (4.10), and let the derivation of (4.19) be directed at a $P(u)$ which is not an axiom. Then the succedent formula $P(u)$ (or more precisely, predecessors of the explicitly shown occurrence in (4.19)) occurs only in the rightmost branch of (4.16), or, more precisely, is introduced into other branches by thinnings.

Proof: The derivation is directed at $P(u)$, so by Lemma 4.6 the antecedent formula of the axiom on $P(u)$ is of the form $P(s)$ where $s \equiv (x_i t_1 \ldots t_n)$, $n \leq 2$, and under this axiom there is a pair $(\forall\rightarrow, \supset\rightarrow)$ with main formula E_i and substituted term t_1, and in the case $n = 2$ also the pair $(\forall\rightarrow, \supset\rightarrow)$ with main formula $r_{B\supset p}[(x_i t_1)]$ and substituted term t_2. In the case $n = 2$ the second of these main formulas is the side formula of the rule introducing the first one (since the derivation is nonsuperfluous). So $u = s \equiv (x_i t_1 \ldots t_n)$ and any axiom with the succedent $P(u)$ should have the antecedent $P(v)$ where v contains x_i. In the pruned derivation an atomic formula is not introduced into a sequent by thinning if above it there is an axiom on this atomic formula (or rather its predecessor). Let us investigate the successors of antecedent formulas $P(v)$ in axioms

$$P(v) \rightarrow P(u) \tag{4.20}$$

having succedent $P(u)$. These successors in the sequent (4.19) are positive occurrences of the formulas $P(v)$. These are occurrences into succedent formulas $r_{p\supset q}[w]$ and into antecedent formulas of the forms $r_{(p\supset q)\supset f}[w]$, $r_{\neg p\supset f}[w]$, $r_{f\supset p}[w]$, $r_{f\supset(g\supset p)}[w]$, $r_{\neg f\supset p}[w]$. The first three types of occurrences (corresponding to the succedent $p \supset q$ and antecedent $(p \supset q) \supset f$, $\neg p \supset f$) are not suitable: in the path leading from (4.19) upwards to the antecedent $P(v)$ there occurs an $(\rightarrow\forall)$-inference introducing the new variable v which is not free in u. So the equation $v = u$ is impossible and $P(v) \rightarrow P(u)$ is not an axiom.

We are left with the antecedent formulas corresponding to $f \supset (g \supset p)$, $f \supset p$, $\neg f \supset p$. By Lemma 4.5 they have the form $\forall z_1 (Fz_1 \supset \forall z_2 (Fz_2 \supset P(x_j z_1 z_2)))$ in the case of $f \supset (g \supset p)$ or simpler in the other cases. Then the term v from (4.20) has the form $(x_i w_1 \ldots w_m)$, $m \leq 2$, and the equation $u = v$ implies $i = j$, $m = n$, $t_1 = w_1$, $t_2 = w_2$ (if $n = 2$). So $P(v)$ from (4.20) is the result of splitting the very same formula which produced the antecedent of the axiom in the rightmost branch of (4.16), and the substituted terms in $\forall\rightarrow$ are the same up to equality.

Now we see that the nonsuperfluousness of the derivation prevents axioms

of the form (4.20) outside of the rightmost branch of (4.16), which was to be proved.

Lemma 4.8   *Let sequent (4.19) occur in a pruned, nonsuperfluous derivation of a sequent (4.10), where the explicitly shown occurrence of $P(u)$ in (4.19) is not introduced by thinning. Then, by permuting inferences in the (sub)derivation of (4.19) one can make it directed at $P(u)$, and the whole derivation of (4.10) will remain pruned and nonsuperfluous.*

Proof: The proof is by induction on the given derivation. The induction base is obvious.

To prove the induction step we consider cases depending on the rule L introducing (4.19). Thinnings are understood to be included in the rule which immediately follows them:

1. One-premise rule, that is, a pair $(\rightarrow\forall,\rightarrow\supset)$ corresponding to the succedent formula $r_{f\supset g}[s]$:

$$\frac{\Gamma' \rightarrow \Delta', P(u)}{\Gamma \rightarrow \Delta, P(u)} \text{ (L)}.$$

By the inductive assumption we have a derivation of the premise $\Gamma' \rightarrow \Delta'$, $P(u)$ directed at $P(u)$. Let us show that this premise is not an axiom. Indeed, since $P(u)$ is not introduced into the succedent by thinning, it would be the main succedent formula of the axiom in question, and the corresponding antecedent formula would be a side formula of the rule L. But $r_{f\supset g}[s] \equiv \forall z(F(z) \supset G(sz))$ and that succedent formula contains a new variable b which is not free in u, so the equation $b = u$ is impossible. We see that the derivation of the premise directed at $P(u)$ should have one of the forms (4.16). By Lemma 4.7 the predecessors of $P(u)$ occur (up to thinnings) only in the rightmost branch of (4.16). Let us permute L with (4.16). In the case of the longest formula the transformation is as follows:

(4.21)
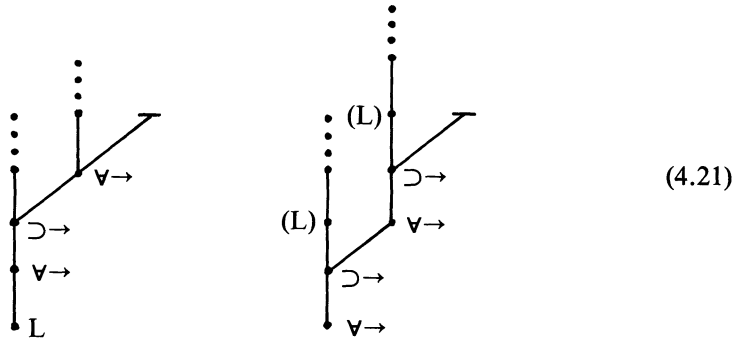
The inference L is written over a node of the new derivation only in the case when the preimage of that node in the original derivation contained at least one of the side formulas of L, i.e., if not all of them were introduced by thinnings.

Note that the right figure in (4.21) can fail to be a derivation only in the case if the proviso for variables is violated in the new L's. This means that in

the original figure (depicted here modulo thinnings using E for $\forall z_1(F(z_1) \supset \forall z_2(G(z_2) \supset P(xz_1z_2)))$ and E' for $\forall z_2(G(z_2) \supset P(xs_1z_2)))$),

$$\frac{\dfrac{(u=xs_1s_2) \qquad \Pi,E,E',A[z] \to B[z], G(s_2); \qquad P(xs_1s_2) \to P(u)}{\Pi,E,A[z] \to B[z], F(s_1); \qquad \Pi,E,E',A[z] \to B[z], P(u)}}{\dfrac{\Pi,E,A[z] \to B[z], P(u)}{\Pi,E \to \forall z(A \supset B), P(u)}}$$

one of the terms s_1, s_2 contains the variable z. The equation $xs_1 s_2 = u$ together with the fact that z is not free in u by the proviso for variables, allows us by the Church-Rosser theorem to conclude that $s_i = s_i'$ ($i = 1,2$) for some terms s_i' which do not contain z. Replacing s_i by s_i' we can perform the required permutation which concludes the treatment of Case 1.

2. L is a two-premise rule. (More precisely, L is a combination of the $(\forall\to)$-inference which will be inessential and the two-premise rule $\supset\to$ which will be denoted by L.)

Using the induction assumption we assume that the derivation is directed to $P(u)$ in the premises where $P(u)$ is not introduced by thinning. If the whole derivation of (4.19) is already directed at $P(u)$ we are done. Otherwise, if $P(u)$ is introduced by thinning in one of the premises, we proceed as in Case 1 (and here one need not even bother about the proviso for variables).

Assume that $P(u)$ is not introduced by thinning in any of the premises. We shall now prove that the derivations of these premises directed at $P(u)$ end in one and the same rightmost branch of Figure (4.16) applied to one and the same formula.

Indeed, the term u from the directed derivation is equal to a term $x_i t_1 \ldots t_n$ ($n \le 2$), and rules explicitly shown in Figure (4.16) are applied either to the formula E_i itself or to its predecessors, and terms t_j are substituted in the corresponding rules $\forall\to$. So relevant parts of Figure (4.16) situated over different premises of L can be different only if in one of them (say the left one) the formula E_i itself is split, and in another one its predecessor is split. But then this predecessor was itself introduced by splitting of E_i (with the same substituted term in $\forall\to$), so its splitting over the left premise contradicts the derivation being nonsuperfluous. Now we can perform the permutation:

$$\frac{\dfrac{\to G(s_2), A; \quad P(xs_1s_2) \to P(u)}{E \to F(s_1), A; \quad E' \to A, P(u)} \qquad \dfrac{B \to G(s_2); \quad P(xs_1s_2) \to P(u)}{B \to F(s_1); \quad E', B \to P(u)}}{\dfrac{\dfrac{E \to A, P(u)}{}\qquad\qquad \dfrac{E, B \to P(u)}{}}{A \supset B, \forall z_1(F(z_1) \supset \forall z_2(G(z_2) \supset P(xz_1z_2))) \to P(u)}}$$

Here L is written over a node of the new derivation (copied from (4.16)) only if in both preimages of that node (over both the left and right premise of L in (4.16)) the corresponding side formula of L is not introduced by thinning. Otherwise, take the derivation where it was introduced by thinning.

Theorem 4.1   *Any derivation of sequent (4.10) can be transformed into an intuitionistic derivation of that sequent.*

Proof: The transformation will be done in the unusual direction — bottom up — and not from axioms. We apply Lemma 4.8 beginning with the lowermost sequent (4.10). It seems that the termination of such a process is not guaranteed: the part of the derivation which remains above and has to be transformed can even increase. But since our transformation steps are only deletions, permutations, and replacements of terms by equal terms, the total supply of terms to be used in the $(\forall\rightarrow)$-inferences does not increase (up to equality) and the transformed part of the derivation remaining below is increasing. So the process will terminate in a finite number of steps. Let us see how such a strategy works.

Call a set $\mathfrak{B}$ of nodes in a derivation a *bar* if no two nodes from $\mathfrak{B}$ are in the same branch and every branch of the derivation contains a node from $\mathfrak{B}$.
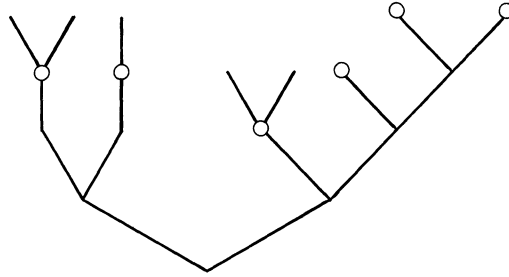
Here a bar will form a boundary between the already transformed and the remaining part of the derivation. By $\mathfrak{B}^-$ we denote the part of the derivation from $\mathfrak{B}$ down. Inversion of the rule $\rightarrow\forall$ is by definition the passage from the figure

$$\frac{\Pi, A[a_1] \rightarrow B[a_1], \Sigma}{\Pi \rightarrow \forall x(A \supset B), \Sigma} \qquad \frac{\Theta, A[a_n] \rightarrow B[a_n], \Lambda}{\Theta \rightarrow \forall x(A \supset B), \Lambda}$$

$$\Gamma \rightarrow \forall x(A \supset B)$$

to the figure

$$\Pi, A[a_1] \rightarrow B[a_1], \Sigma \qquad \Theta, A[a_1] \rightarrow B[a_1], \Lambda$$

$$\frac{\Gamma, A[a_1] \rightarrow B[a_1]}{\Gamma \rightarrow \forall x(A \supset B)}$$

and similarly in the case of $\forall x \neg A$.

Note that if the upper figure is a nonsuperfluous, pruned derivation (as always, with the strong pure variable property) then the lower figure is also such. Let us prove a Lemma which allows us to move a bar upwards.

Lemma 4.9   *Let D be a nonsuperfluous, pruned derivation of sequent (4.10) and $\mathfrak{B}$ be a bar in this derivation, such that all inferences in $\mathfrak{B}^-$ (but not in D) satisfy intuitionistic restrictions. Then applying the transformations from Lemma 4.8 and inversion of the rules one can construct a nonsuperfluous, pruned deri-*

vation D_1 of sequent (4.10) and a bar $\mathcal{B}_1$ in D_1 such that $\mathcal{B}_1^-$ properly contains $\mathcal{B}^-$ and all inferences in $\mathcal{B}_1^-$ are intuitionistic.

Proof: By the assumption $\mathcal{B}^-$ is properly contained in D, i.e., one of the upper nodes of $\mathcal{B}$ is not an axiom. Consider the rule L introducing that node. If L is intuitionistic, it is sufficient to include its premise(s) into $\mathcal{B}_1$. If the succedent is not an atomic formula, use inversion and again extend $\mathcal{B}$. If the succedent is atomic, apply Lemma 4.8 and note that by Lemma 4.7 the resulting Figure (4.16) consists of intuitionistic rules. Now one can again extend $\mathcal{B}$.

Now we can finish the proof of Theorem 4.1. Consider a pruned, non-superfluous derivation D of sequent (4.10). If one identifies the eigenvariables of ($\rightarrow\forall$)-inferences, then one can obtain only a finite number of new terms from terms occurring in D. Only such terms (up to equality) can occur in the derivations obtained from the given one by transformations used in Lemma 4.9. So these transformations can produce from D only a finite number of nonsuperfluous derivations, and the process of iterative application of Lemma 4.9 will be finished in a finite number of steps. When this happens one has the required intuitionistic derivation.

Theorem 4.2   *If (4.10) is derivable, then the sequent (4.9)*

$$E_1, \ldots, E_n \rightarrow p$$

is also intuitionistically derivable.

Proof: Add $\neg$-cases to the proof of Theorem 2.1.

Theorem 4.3   *The propositional formula F in the language $(\supset, \wedge, \neg)$ is intuitionistically derivable iff $r_F[u]$ is derivable in L for some term u.*

Proof: In one direction this is Lemma 4.1(i). Assume now that $r_F[u]$ is derivable. Construct by Lemma 4.2 a standard formula S such that

$$F \supset S \quad \text{and} \quad S^* \supset F$$

are intuitionistically derivable for some substitution instance S^* of S. The derivability of $F \supset S$ implies, by Lemma 4.1(ii), the derivability of $\forall x(r_F[x] \supset r_S[(vx)])$ for some term v, which together with the derivability of $r_F[u]$ implies the derivability of $r_S[(vu)]$. Since S is of the form (4.3), the derivability of $r_S[w]$ is equivalent to the derivability of sequent (4.10) for $t = (wx_1 \ldots x_n)$. By Theorem 4.2, sequent (4.9) is intuitionistically derivable, and so is the formula S. Making substitutions and applying $S^* \supset F$ we obtain F as required.

REFERENCES

[1] Barendregt, H., *The Lambda Calculus*, North Holland, Amsterdam, 1981.

[2] Heyting, A., "Die intuitionistische Grundlegung der Mathematik," *Erkenntnis*, vol. 2 (1931-1932), pp. 106–115.

[3] Howard, W., "The formulae-as-types notion of construction," pp. 479–490 in *Essays on Combinatory Logic, Lambda Calculus and Formalism*, eds. J. Hindley and J. Seldin, Academic Press, New York and London, 1980.

[4] Kleene, S. C., *Introduction to Metamathematics*, van Nostrand, New York, 1952.

[5] Kleene, S. C., "Permutability of inferences in Gentzen's calculi *LK* and *LJ*," *Memoirs of the American Mathematical Society*, vol. 10 (1952) pp. 1–26.

[6] Kolmogorov, A. N., "Zur Deutung der intuitionistischen Logik," *Mathematische Zeitschrift*, vol. 35 (1932), pp. 56–85.

[7] Lopez-Escobar, E. G. K., "On intuitionistic sentential connectives, I," *Revista Colombiana de Mathematicas*, vol. 19 (1985), pp. 117–130.

[8] Mints, G. E., "Provability and provable realizability" (in Russian), pp. 37–39 in *Logic and Systemic Methods of Analysis. Abstracts of Talks at 9th Soviet Conference in Logic, Methodology and Philosophy of Science*, Moscow, 1986.

[9] Pottinger, G., "A type assignment for the strongly normalizable lambda terms," pp. 559–577 in *Essays on Combinatory Logic, Lambda Calculus and Formalism*, eds. J. Hindley and J. Seldin, Academic Press, New York and London, 1980.

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