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Uniform Short Proofs for Classical Theorems

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Abstract This note exploits back-and-forth characteristics to construct, using a single method, short proofs for ten classics of first-order and modal logic: interpolation theorems, preservation theorems, and Lindström's theorem.

1 Introduction

The classics alluded to in the title—among which are interpolation theorems, preservation theorems, and Lindström's theorem—all state the existence of a first-order formula satisfying a certain condition. By essentially the same simple argument for each separate case, it is shown that these conditions are satisfied by disjunctions of suitable characteristics. For Lindström's theorem, this was noticed already in Doets [3] (cf. exercise 165, p. 90). The wider applicability of this approach was suggested by Barwise and van Benthem [2] which has similar results for infinitary logic (and extensively discusses method).

2 Preliminaries

This section collects the few facts needed in what follows. These are applied verbatim in Sections 3 and 6 and, appropriately modified, in Sections 4 and 5. I assume some familiarity with this material and hence refrain from indicating the relationships with, for example, the Ehrenfeucht-Fraïssé game and infinitary logic (cf. [2], [3]). Vocabularies are finite (unless the contrary is evident) and consist of relation and constant symbols only.

Note 2.1 Some of the classical theorems considered here are valid without these restrictions. However, without them, the back-and-forth characteristics of the next definition do not exist.

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Definition 2.2 (Back-and-forth characteristics) For a model $\mathcal{A} = (A, \ldots)$, a finite sequence $\vec{a} = (a_1, \ldots, a_i)$ from A, and a nonnegative integer n, the *n-characteristic of $\vec{a}$ in $\mathcal{A}$* is the first-order formula $\varepsilon^n_{\mathcal{A},\vec{a}}$ with $x_1, \ldots, x_i$ free (and $x_{i+1}, \ldots, x_{i+n}$ bound) defined as follows:

1. $\varepsilon^0_{\mathcal{A},\vec{a}}$ is the conjunction of all literals in $x_1, \ldots, x_i$ satisfied by $\vec{a}$ in $\mathcal{A}$;
2. $\varepsilon^{n+1}_{\mathcal{A},\vec{a}}$ is $\bigwedge_{a\in A} \exists x_{i+1} \varepsilon^n_{\mathcal{A},(a_1,\ldots,a_i,a)} \wedge \forall x_{i+1} \bigvee_{a\in A} \varepsilon^n_{\mathcal{A},(a_1,\ldots,a_i,a)}$.

Finally, $\varepsilon^n_{\mathcal{A}} = \varepsilon^n_{\mathcal{A},\varnothing}$, where $\varnothing$ is the empty sequence. Note that for all i and n, there are only finitely many formulas of the form $\varepsilon^n_{\mathcal{A},\vec{a}}$ where $\mathcal{A}$ is any model and $\vec{a}$ any length-i sequence from it.

A *local isomorphism* between models $\mathcal{A} = (A, \ldots)$ and $\mathcal{B} = (B, \ldots)$ is a finite relation $\sim \subseteq A \times B$ such that, for atomic φ, we have that

$$\mathcal{A} \models \varphi[a_1, \ldots, a_i] \Leftrightarrow \mathcal{B} \models \varphi[b_1, \ldots, b_i]$$

when $a_j \sim b_j$ $(j = 1, \ldots, i)$.

Lemma 2.3

1. $\mathcal{A} \models \varepsilon^n_{\mathcal{A}}$.
2. *If $\mathcal{B} \models \varepsilon^n_{\mathcal{A}}$, then, defining $R_i \subseteq A^i \times B^i$ $(0 \leqslant i \leqslant n)$ by $R_i(\vec{a}, \vec{b}) \equiv \mathcal{B} \models \varepsilon^{n-i}_{\mathcal{A},\vec{a}}[\vec{b}]$, we have that*
   (a) *R_0 is 'true',*
   (b) *for $0 \leqslant i \leqslant n$: if $R_i(a_1, \ldots, a_i, b_1, \ldots, b_i)$, then $\{(a_j, b_j) \mid 1 \leqslant j \leqslant i\}$ is a local isomorphism between $\mathcal{A}$ and $\mathcal{B}$, and*
   (c) *for $0 \leqslant i < n$: if $R_i(a_1, \ldots, a_i, b_1, \ldots, b_i)$, then*
   ('forth') $\forall a \in A\, \exists b \in B\ R_{i+1}(a_1, \ldots, a_i, a, b_1, \ldots, b_i, b)$, and
   ('back') $\forall b \in B\, \exists a \in A\ R_{i+1}(a_1, \ldots, a_i, a, b_1, \ldots, b_i, b)$.

A *potential isomorphism* between models $\mathcal{A}$ and $\mathcal{B}$ is a nonempty set I of local isomorphisms such that for all $h \in I$, we have that $\forall a \in A\, \exists b \in B\ (h \cup \{(a, b)\} \in I)$ and $\forall b \in B\, \exists a \in A\ (h \cup \{(a, b)\} \in I)$.

Lemma 2.4 *If $R_0, R_1, R_2, \ldots$ is an infinite sequence of relations such that $R_i \subseteq A^i \times B^i$ and conditions 2.3.2(a–c) hold for all i, then*

$$\bigcup_i \{\{(a_1, b_1), \ldots, (a_i, b_i)\} \mid R_i(a_1, \ldots, a_i, b_1, \ldots, b_i)\}$$

is a potential isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Theorem 2.5 *If I is a potential isomorphism between two countable models, then some $h \subseteq \bigcup I$ is an isomorphism.*

Definition 2.6 (Model Pairs) Suppose that $\mathcal{A}_i = (A_i, \ldots)$ is a model for the vocabulary L_i $(i = 1, 2)$ and L is the disjoint union of L_1 and L_2 together with two new unary relation symbols $\mathrm{U}_1, \mathrm{U}_2$. The *model pair* $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ is the L-model with universe $A_1 \cup A_2$, with $\mathrm{U}_i^{\mathcal{A}} = A_i$ $(i = 1, 2)$, and where the L_i-symbols retain their old meanings.

3 Interpolation

Theorem 3.1 (Consistency Theorem) *Suppose that T_i is a set of L_i-sentences $(i = 1, 2)$ such that $T_1 \cup T_2$ has no model. Then there is an $L_1 \cap L_2$-sentence φ such that $T_1 \models \varphi$ and $T_2 \models \neg\varphi$.*

Proof Suppose that no such φ exists. The following constructs a model for $T_1 \cup T_2$. Put $L = L_1 \cap L_2$. $\square$

Claim 3.2 *For all $n \geqslant 0$, there exist $\mathcal{A} \models T_1$ and $\mathcal{B} \models T_2$ such that $\mathcal{B} \models \varepsilon^n_{\mathcal{A}|L}$.*

Proof Note that $\mathcal{A}|L$ denotes the L-reduct of the L_1-model $\mathcal{A}$. Suppose the claim fails for the integer n. Consider the finite set of L-sentences $\Sigma = \{\varepsilon^n_{\mathcal{A}|L} \mid \mathcal{A} \models T_1\}$. It suffices to show that (i) $T_1 \models \bigvee \Sigma$ and (ii) $T_2 \models \neg \bigvee \Sigma$.

(i) Assume that $\mathcal{A} \models T_1$. Then $\varepsilon^n_{\mathcal{A}|L} \in \Sigma$ and by Lemma 2.3.1, $\mathcal{A} \models \bigvee \Sigma$ follows.

(ii) Assume that $\mathcal{B} \models \bigvee \Sigma$. Then for some $\mathcal{A} \models T_1$ we have that $\mathcal{B} \models \varepsilon^n_{\mathcal{A}|L}$; by assumption on n, it follows that $\mathcal{B} \not\models T_2$. $\square$

Consider the theory of complexes $(\mathcal{A}, \mathcal{B}, R_0, R_1, R_2, \ldots)$ (i.e., model pairs $(\mathcal{A}, \mathcal{B})$ expanded with infinitely many relations $R_0, R_1, R_2, \ldots$) for which $\mathcal{A} \models T_1$, $\mathcal{B} \models T_2$, and such that $R_0, R_1, R_2, \ldots$ satisfy conditions 2.3.2(a–c) with respect to $\mathcal{A}|L$ and $\mathcal{B}|L$ for *all* i. From Claim 3.2 and Lemma 2.3.2, it is clear that this theory is finitely satisfiable. Thus, compactness and downward Löwenheim-Skolem provide a countable realization $(\mathcal{A}, \mathcal{B}, R_0, R_1, R_2, \ldots)$ for it. By Lemma 2.4, $\mathcal{A}|L$ and $\mathcal{B}|L$ are potentially isomorphic. By Theorem 2.5, it follows that $\mathcal{A}|L \cong \mathcal{B}|L$. Identification of $\mathcal{A}|L$ and $\mathcal{B}|L$ results in the required model for $T_1 \cup T_2$.

Theorem 3.3 (Interpolation Theorem) *Suppose the L_i-sentences φ_i $(i = 1, 2)$ are such that $\varphi_1 \models \varphi_2$. Then an $L_1 \cap L_2$-sentence φ exists such that both $\varphi_1 \models \varphi$ and $\varphi \models \varphi_2$.*

Proof Suppose there is no such φ. Let $L = L_1 \cap L_2$. As in the previous proof, for every $n \geqslant 0$ there exist $\mathcal{A} \models \varphi_1$ and $\mathcal{B} \models \neg\varphi_2$ such that $\mathcal{B} \models \varepsilon^n_{\mathcal{A}|L}$. As before, compactness, downward Löwenheim-Skolem, Lemma 2.4, and Theorem 2.5 yield a countermodel to $\varphi_1 \models \varphi_2$. $\square$

Theorem 3.4 (Lyndon's Refinement) *This is the same as Theorem 3.3 but φ satisfies additional polarity requirements: relation symbols (different from equality) occurring positively (respectively, negatively) in the interpolant should occur positively (respectively, negatively) in both φ_1 and φ_2.*

Proof Modify the argument for Theorem 3.3 as follows. Let P_i be the set of relation symbols from $L = L_1 \cap L_2$ that occur positively in φ_i and let N_i contain those that occur negatively in φ_i $(i = 1, 2)$. Modify the $\varepsilon^n_{\mathcal{A},\vec{a}}$ by letting $\varepsilon^0_{\mathcal{A},\vec{a}}$ be the conjunction of

1. all positive literals satisfied by $\vec{a}$ in $\mathcal{A}$ that carry a relation symbol in $P_1 \cap P_2$,
2. all negative literals satisfied by $\vec{a}$ in $\mathcal{A}$ that carry a relation symbol in $N_1 \cap N_2$,
3. all equality literals satisfied by $\vec{a}$ in $\mathcal{A}$.

These are the obvious modifications to make if one wants to conclude, as before, from the nonexistence of an interpolant, that for all n there are $\mathcal{A} \models \varphi_1$ and $\mathcal{B} \models \neg\varphi_2$ such that $\mathcal{B} \models \varepsilon^n_{\mathcal{A}|L}$.

After applying compactness and Löwenheim-Skolem, the relation $h \subseteq A \times B$, obtained from Theorem 2.5 by proviso (3), will be a bijection between $\mathcal{A}$ and $\mathcal{B}$. However, for $\mathrm{R} \in L$, we only get $\mathrm{R}^{\mathcal{A}}(a) \Rightarrow \mathrm{R}^{\mathcal{B}}(h(a))$ for $\mathrm{R} \in P_1 \cap P_2$, and $\mathrm{R}^{\mathcal{B}}(h(a)) \Rightarrow \mathrm{R}^{\mathcal{A}}(a)$ for $\mathrm{R} \in N_1 \cap N_2$. For symbols in $P_1 \cap P_2 \cap N_1 \cap N_2$, h

preserves in both directions (item 4 in the list below). For the remaining symbols, there is no preservation by h in any direction (item 3).

Nevertheless, the rest of the argument not only needs $\mathcal{A} \models \varphi_1$ and $\mathcal{B} \models \neg\varphi_2$ but also that $\mathcal{A}|L \cong \mathcal{B}|L$. To get that, the interpretation $\mathrm{R}^{\mathcal{A}}$ or $\mathrm{R}^{\mathcal{B}}$ of some symbols $\mathrm{R} \in L$ is modified appropriately, forcing h to be an isomorphism but preserving φ_1 and $\neg\varphi_2$.

1. For R in $(P_1 - N_1) \cap P_2$, replace $\mathrm{R}^{\mathcal{A}}$ by $R(a) :\equiv \mathrm{R}^{\mathcal{B}}(h(a))$. Note that since $\mathrm{R}^{\mathcal{A}}(a) \Rightarrow \mathrm{R}^{\mathcal{B}}(h(a))$ holds, we have that $\mathrm{R}^{\mathcal{A}} \subseteq R$; and since $\mathrm{R} \in P_1 - N_1$, φ_1 will still hold in the so-modified $\mathcal{A}$. The same change is made for R in $(N_1 - P_1) \cap N_2$.
2. For R in $P_1 \cap N_1 \cap (N_2 - P_2)$ or in $P_1 \cap N_1 \cap (P_2 - N_2)$, replace $\mathrm{R}^{\mathcal{B}}$ by $R(b) :\equiv \mathrm{R}^{\mathcal{A}}(h^{-1}(b))$.
3. For R in $(N_1 - P_1) \cap (P_2 - N_2)$, replace both $\mathrm{R}^{\mathcal{A}}$ and $\mathrm{R}^{\mathcal{B}}$ by 'false'; for R in $(P_1 - N_1) \cap (N_2 - P_2)$, replace both $\mathrm{R}^{\mathcal{A}}$ and $\mathrm{R}^{\mathcal{B}}$ by 'true'.
4. In the remaining case, where R occurs in both $P_1 \cap N_1$ and $P_2 \cap N_2$, no relation needs to be changed as preservation by h is already guaranteed. □

Theorem 3.5 (Definability Theorem) *Suppose that $L^+ = L \cup \{\mathrm{R}\}$ and T is an L^+-theory such that for every two models $\mathcal{A}$ and $\mathcal{B}$ of T, if $\mathcal{A}|L = \mathcal{B}|L$, then $\mathrm{R}^{\mathcal{A}} = \mathrm{R}^{\mathcal{B}}$. Then an L-formula $\varphi = \varphi(x)$ exists for which $T \models \forall x(\mathrm{R}(x) \leftrightarrow \varphi)$.*

Proof Suppose no such "definition" φ for R exists. Then for all n there exist $\mathcal{A} \models T$ and $a \in A$ with $\mathrm{R}^{\mathcal{A}}(a)$, and $\mathcal{B} \models T$ and $b \in B$ with $\neg\mathrm{R}^{\mathcal{B}}(b)$ such that $\mathcal{B} \models \varepsilon^n_{\mathcal{A}|L,a}[b]$. (If this fails for n, $\bigvee\{\varepsilon^n_{\mathcal{A}|L,a} \mid \mathcal{A} \models T \wedge \mathrm{R}^{\mathcal{A}}(a)\}$ is a definition for R.)

The rest of the proof is as usual using (slight refinements of) Lemmas 2.3 and 2.4 and Theorem 2.5: we can take care that, for the resulting (countable) models $\mathcal{A}$ and $\mathcal{B}$ and the isomorphism h between $\mathcal{A}|L$ and $\mathcal{B}|L$, there is $a \in A$ such that $\mathrm{R}^{\mathcal{A}}(a)$ but $\neg\mathrm{R}^{\mathcal{B}}(h(a))$, contrary to hypothesis. □

4 Preservation

A sentence is *preserved under extensions* if it is true of every extension of one of its models.

Theorem 4.1 (Łoś-Tarski Theorem) *Every sentence preserved under extensions has an existential equivalent.*

Proof Modify the $\varepsilon^n_{\mathcal{A}}$ appropriately: let $\varepsilon^{n+1}_{\mathcal{A},(a_1,\dots,a_k)}$ be $\bigwedge_{a\in A} \exists x_{k+1} \varepsilon^n_{\mathcal{A},(a_1,\dots,a_k,a)}$. Note that this modification yields *existential* formulas. Now, if a sentence Φ doesn't have an existential equivalent, then, for every $n \geqslant 0$, there are $\mathcal{A} \models \Phi$ and $\mathcal{B} \models \neg\Phi$ such that $\mathcal{B} \models \varepsilon^n_{\mathcal{A}}$ (if this would be false for n, consider $\bigvee\{\varepsilon^n_{\mathcal{A}} \mid \mathcal{A} \models \Phi\}$). The rest of the proof follows the by now familiar pattern. The condition that $\mathcal{B} \models \varepsilon^n_{\mathcal{A}}$ entails the existence of a finite sequence of relations $R_0 =$ 'true', …, R_n that now code sets of local *embeddings* satisfying the 'forth'-property. By downward Löwenheim-Skolem and compactness, we obtain a countable complex $(\mathcal{A}, \mathcal{B}, R_0, R_1, R_2, \dots)$ with $\mathcal{A} \models \Phi$, $\mathcal{B} \models \neg\Phi$, and such that

$$\bigcup_i \{\{(a_1, b_1), \dots, (a_i, b_i)\} \mid R_i(a_1, \dots, a_i, b_1, \dots, b_i)\}$$

now (compare Lemma 2.4) is a potential *embedding*; and it follows (as in the proof for Theorem 2.5) that $\mathcal{A}$ embeds into $\mathcal{B}$. □

A sentence is *preserved under homomorphisms* if it is true of every homomorphic image of one of its models.

Theorem 4.2 (Lyndon's Theorem) *Every sentence that is preserved under homomorphisms has a positive logical equivalent.*

Proof Again, modify the characteristics: $\varepsilon^0_{\mathcal{A},\vec{a}}$ now is the conjunction of all *positive* literals satisfied by $\vec{a}$. Note that these modifications are positive.

Again, if a sentence Φ has no positive equivalent, then, for every $n \geqslant 0$, there exist $\mathcal{A} \models \Phi$ and $\mathcal{B} \models \neg\Phi$ such that $\mathcal{B} \models \varepsilon^n_{\mathcal{A}}$. (If this is false for n, consider $\bigvee\{\varepsilon^n_{\mathcal{A}} \mid \mathcal{A} \models \Phi\}$.)

The condition that $\mathcal{B} \models \varepsilon^n_{\mathcal{A}}$ entails the existence of a finite sequence of relations $R_0 =$ 'true', $\dots$, R_n that code sets of local *homomorphisms* satisfying the back-and-forth properties. By downward Löwenheim-Skolem and compactness, we obtain a countable complex $(\mathcal{A}, \mathcal{B}, R_0, R_1, R_2, \dots)$ with $\mathcal{A} \models \Phi$, $\mathcal{B} \models \neg\Phi$, and such that

$$\bigcup_i \{\{(a_1, b_1), \dots, (a_i, b_i)\} \mid R_i(a_1, \dots, a_i, b_1, \dots, b_i)\}$$

now is a potential *homomorphism*; and it follows that $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$. □

An $L \cup \{\mathrm{R}\}$-sentence Φ is *preserved under* R-*extensions* if for every two models $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A} \models \Phi$, $\mathcal{A}|L = \mathcal{B}|L$ and $\mathrm{R}^{\mathcal{A}} \subseteq \mathrm{R}^{\mathcal{B}}$, then $\mathcal{B} \models \Phi$. We have, similarly, the following theorem.

Theorem 4.3 *Every sentence preserved under* R-*extensions has an* R-*positive equivalent.*

5 Modal Logic

Theorem 5.1 (van Benthem's Theorem) *If a first-order formula in one free variable is preserved under bisimulation, then it has a modal equivalent.*

Proof The modal vocabulary has a binary "accessibility" relation symbol R plus a set $\mathcal{U}$ of unary relation symbols. For a model $\mathcal{A} = (A, \mathrm{R}^{\mathcal{A}}, \mathrm{U}^{\mathcal{A}})_{\mathrm{U}\in\mathcal{U}}$ and an element $a \in A$, define the modal characteristics $\sigma^n_{\mathcal{A},a}(x)$ in one free variable x as follows:

1. $\sigma^0_{\mathcal{A},a}(x)$ is the conjunction of all literals in x using some $\mathrm{U} \in \mathcal{U}$ that are satisfied by a in $\mathcal{A}$.
2. $\sigma^{n+1}_{\mathcal{A},a}(x) = \sigma^0_{\mathcal{A},a} \wedge \bigwedge_{\mathrm{R}^{\mathcal{A}}(a,b)} \exists y[\mathrm{R}(x, y) \wedge \sigma^n_{\mathcal{A},b}(y)]$
$\wedge \forall y[\mathrm{R}(x, y) \rightarrow \bigvee_{\mathrm{R}^{\mathcal{A}}(a,b)} \sigma^n_{\mathcal{A},b}(y)]$.

As in Lemma 2.3.1, we have that $\mathcal{A} \models \sigma^n_{\mathcal{A},a}[a]$.

Suppose that the first-order formula $\Phi(x)$ is preserved under bisimulation but has no modal equivalent. Then for all n there are $\mathcal{A}, a$ and $\mathcal{B}, b$ such that $\mathcal{A} \models \Phi[a]$, $\mathcal{B} \models \neg\Phi[b]$ and such that $\mathcal{B} \models \sigma^n_{\mathcal{A},a}[b]$. (If this is false for n, then $\bigvee\{\sigma^n_{\mathcal{A},a}(x) \mid \mathcal{A} \models \Phi[a]\}$ is a modal equivalent for Φ.)

As in Lemma 2.3.2, if $\mathcal{B} \models \sigma^n_{\mathcal{A},a}[b]$ holds, then, defining $u \sim_i v$ as $\mathcal{B} \models \sigma^{n-i}_{\mathcal{A},u}[v]$, we have that

(a) $a \sim_0 b$,

(b) if $u \sim_i v$, then $\mathrm{U}^{\mathcal{A}}(u) \Leftrightarrow \mathrm{U}^{\mathcal{B}}(v)$ ($\mathrm{U} \in \mathcal{U}$),

(c) ('forth') $i < n$, $u \sim_i v$ and $\mathrm{R}^{\mathcal{A}}(u, u')$ imply $\exists v' \in B[\mathrm{R}^{\mathcal{B}}(v, v') \wedge u' \sim_{i+1} v']$, ('back') similar.

By compactness we find $\mathcal{A}, a, \mathcal{B}, b$, and $\sim_0, \sim_1, \sim_2, \ldots \subseteq A \times B$ such that $\mathcal{A} \models \Phi[a]$, $\mathcal{B} \models \neg\Phi[b]$, and such that conditions (a – c) are satisfied for all i. As in Lemma 2.4, it follows that $\bigcup_i \sim_i$ is a bisimulation, contradicting the assumption on Φ. $\square$

Theorem 5.2 (Modal Interpolation Theorem) *Suppose that the modal formulas φ_i ($i = 1, 2$) are such that $\varphi_1 \models \varphi_2$. Then a modal φ exists such that $\varphi_1 \models \varphi$, $\varphi \models \varphi_2$, and every unary relation symbol in φ occurs in both φ_1 and φ_2.*

Proof Let L_i consist of R and the unary relation symbols that occur in φ_i ($i = 1, 2$); $L = L_1 \cap L_2$. Suppose no such φ exists. Then for all n, there are an L_1-model $\mathcal{A}, a$ and an L_2-model $\mathcal{B}, b$ such that $\mathcal{A} \models \varphi_1[a]$, $\mathcal{B} \models \neg\varphi_2[b]$, and $\mathcal{B} \models \sigma^n_{\mathcal{A}|L,a}[b]$. (If this is false for n, then $\bigvee\{\sigma^n_{\mathcal{A}|L,a} \mid \mathcal{A} \models \varphi_1[a]\}$ would be an interpolant.) By compactness, we find $\mathcal{A}, a$ and $\mathcal{B}, b$ and a bisimulation $\sim \subseteq A \times B$ such that $\mathcal{A} \models \varphi_1[a]$, $\mathcal{B} \models \neg\varphi_2[b]$, and $a \sim b$.

Define the $(L_1 \cup L_2)$-model $\mathcal{C}$ as follows (cf. Andréka, Németi, and van Benthem [1]). $C = \{(u, v) \in A \times B \mid u \sim v\}$; $(u, v)\mathrm{R}^{\mathcal{C}}(u', v') :\equiv u\mathrm{R}^{\mathcal{A}}u' \wedge v\mathrm{R}^{\mathcal{B}}v'$; and, for $\mathrm{U} \in \mathcal{U}$, $(u, v) \in \mathrm{U}^{\mathcal{C}}$ if and only if

$$\mathrm{U} \in L_1 \text{ and } u \in \mathrm{U}^{\mathcal{A}},$$

or

$$\mathrm{U} \in L_2 \text{ and } v \in \mathrm{U}^{\mathcal{B}}.$$

(Note that if $\mathrm{U} \in L$, then, since $u \sim v$, $u \in \mathrm{U}^{\mathcal{A}} \Leftrightarrow v \in \mathrm{U}^{\mathcal{B}}$.)

It is straightforward to check that the projection relations $\sim_1$ and $\sim_2$ defined by $(u, v) \sim_1 u$ and $(u, v) \sim_2 v$ are bisimulations between $\mathcal{C}$ and $\mathcal{A}$, and respectively, $\mathcal{C}$ and $\mathcal{B}$. Thus, φ_1 is true at (a, b) in $\mathcal{C}$, but φ_2 is false. $\square$

6 Lindström's Theorem

A *logic* is a schema $\mathcal{Z}$ that associates to any vocabulary L a set $\mathcal{Z}(L)$ of *sentences* together with a truth-relation $\models$ between L-models and sentences from $\mathcal{Z}(L)$ such that the following hold.

(L1) Isomorphism preserves truth.

(L2) If L^+ extends the vocabulary that is appropriate for model pairs built from two L-models, then for every $\Phi \in \mathcal{Z}(L)$ there exist $\Phi^i \in \mathcal{Z}(L^+)$ ($i = 1, 2$) such that, for every L^+-model $(\mathcal{A}_1, \mathcal{A}_2, \ldots)$, we have $(\mathcal{A}_1, \mathcal{A}_2, \ldots) \models \Phi^i \Leftrightarrow \mathcal{A}_i \models \Phi$.

For a logic to *extend* first-order logic means closure under negation and inclusion of all first-order sentences in the given vocabulary (with their usual meaning).

Theorem 6.1 (Lindström's Theorem) *If the logic $\mathcal{Z}$ extends first-order logic and satisfies the downward Löwenheim-Skolem and compactness theorems, then every $\mathcal{Z}$-sentence has a first-order equivalent.*

Proof Let Φ be a $\mathcal{Z}$-sentence. If it doesn't have a first-order equivalent, we have, once more, for every $n \geqslant 0$, models $\mathcal{A} \models \Phi$ and $\mathcal{B} \models \neg\Phi$ such that $\mathcal{B} \models \varepsilon^n_{\mathcal{A}}$.

The proof is again finished using Lemma 2.3, compactness, and downward Löwenheim-Skolem (employing condition L2), Lemma 2.4, and Theorem 2.5, by constructing (countable) $\mathcal{A} \models \Phi$ and $\mathcal{B} \models \neg\Phi$ such that $\mathcal{A} \cong \mathcal{B}$, contradicting condition L1. $\square$

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