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The Numerical Syllogism and Existential Presupposition

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Abstract The paper presents a numerical interpretation of the quantifiers of traditional categorical propositions and then offers a generalization to accommodate all other numerical values. Next, it considers the implications possible on the basis of both minimum and maximum existential presuppositions; and finally, it shows that every pair of categorical premises yields multiple conclusions when appropriate minimum and maximum presuppositions are made for the terms of the premises.

1 Introduction When its quantifiers are interpreted numerically, the assertions possible in categorical logic are increased *ad infinitum*; then, from each valid, traditional syllogistic form an endless number of numerically different inferences can be made. For example, one such expansion of Barbara is:

At least all but 5 M's are P
At least all but 4 S's are M
———
At least all but 9 S's are P.

Furthermore, the expanded quantifiers present the potential for countless additional inferences based on diverse *minimum* and *maximum* existential presupposition. For example, on the minimum presupposition that there exist at least eleven S's ($S \geq 11$), the numerical premises above also entail an additional conclusion:

At least 2 S's are P.

And on the maximum presupposition that there exist at most nine M's ($M \leq 9$), the following numerical instance of III, Fig. 1, is valid as well:

At least 7 M's are P
At least 5 S's are M
———
At least 3 S's are P.

This paper probes the limits of this numerically expanded logic, and in so doing it discloses a vast expanse of validity hitherto unclaimed for syllogistic reasoning. In

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fact, it shows that *every pair* of syllogistic premises yields *multiple conclusions* when appropriate minimum and maximum presuppositions are strategically combined.

In [2] and [3] I worked out the system of numerically expanded quantifiers and noted its possibility for further inferences based on minimum presupposition; however, in these I failed to consider maximum presupposition which, it turns out, is the key that unlocks the full potential of the logic.

In the first part of this paper I present the numerically expanded propositions. Here I summarize equivalences and distribution patterns before turning to the crucial topic of implications based on minimum, maximum, and combined existential presuppositions. Then in the second part I present the expanded syllogistic argument forms. I advance necessary criteria of validity for inferences based on numerically expanded propositions themselves, before commencing the principal project of deriving the various conclusions based on existential presupposition.

2 Basics of the numerical proposition Perhaps the most straightforward route into the numerically expanded syllogism begins by recasting the traditional propositions to make their implicit structure more pronounced. For the particulars this only amounts to adopting alternative expressions that already are in common usage, viz., "Some S's are P" becomes "At least one S is P," and "Some S's are not P" becomes "At least one S is not P."

The E proposition deviates just a bit more: "No S's are P" is rendered as, "At most zero S's are P." Here the substitution of "zero" for "no" should not be problematic, but it might be suggested that the qualifier should be "exactly," rather than "at most," if indeed a qualifier is needed at all. However, "Exactly zero S's are P" translates into the conjunction,

At least zero S's are P and at most zero S's are P

and since "At least zero S's are P" is vacuous (for it holds for every S and P whatever), it is "At most zero S's are P" which is the significant conjunct. In fact, it is precisely because the "at least…" alternative is vacuous that the "at most…" qualifier is not needed for ordinary expressions of this form. Accordingly, the addition of "at most" simply serves to make explicit a qualification that goes without saying in the economy of natural language; and with this having been made explicit, the E proposition exhibits a structure that it shares with other assertions whose qualifications do not go without saying.

Finally, the traditional "All S's are P" is rendered as "At least all but zero S's are P." Here again, the addition of "at least…" and "…but zero" only makes explicit what goes without saying in this case: that is, an indicated quantity might be "all but ten" or "all but five," and so on, but when "all" occurs without an exception specified, it is understood to carry the exception of zero, and the above rendition simply states this in the open. Moreover, "All (but zero) S's are P" is also understood to mean "at least all…" since the alternative, "At most all (but zero) S's are P," like "At least zero S's are P," is vacuous.

So again, the above rendition of "All S's are P" does not change its meaning. Rather, it makes explicit the qualification (at least) and exception (zero) that are understood in this case, and with these having been made explicit, the A proposition exhibits a structure that it shares with other claims whose qualifications and exceptions require articulation.

Now each proposition so rendered can be symbolized by prefixing the appropriate numeral to its traditional symbolization as follows:

Traditional Symbolization		Expanded Rendition		
SAP	=	0SAP	=	At least all but 0 S's are P
SEP	=	0SEP	=	At most 0 S's are P
SIP	=	1SIP	=	At least 1 S is P
SOP	=	1SOP	=	At least 1 S is not P

The numerical prefix indicates what will be called the "exception" of each proposition (even though it is only the A proposition that is clearly worded in an exceptive fashion here), and different propositions can be formed by altering the numerical value of the exceptions, such as:

7SAP	=	At least all but 7 S's are P
44SEP	=	At most 44 S's are P
567SIP	=	At least 567 S's are P
1000SOP	=	At least 1000 S's are not P

When values for the exceptions are not specified, the general types of propositions can be indicated as:

*x*SAP	=	At least all but *x* S's are P
*x*SEP	=	At most *x* S's are P
*x*SIP	=	At least *x* S's are P
*x*SOP	=	At least *x* S's are not P

(Alternatively, *x*SOP might be phrased as "At most all but *x* S's are P," as it is in [2] and [3].)

Perhaps it should be noted at this point that propositions of the same quantity and quality are stronger or weaker than one another according to their exception values. For example, "At least all but zero S's are P" (0SAP) is a stronger claim than "At least all but one S's are P" (1SAP), and as the exceptions are increased these claims become weaker. On the other hand, "At least one S is P" (1SIP) is a weaker claim than "At least two S's are P" (2SIP), and as the exceptions are increased these claims become stronger. Furthermore, the stronger claims always imply the weaker ones:

$\longrightarrow\longrightarrow\longrightarrow$ ——— progressively weaker claims ——— $\longrightarrow\longrightarrow\longrightarrow$

0SAP	implies	1SAP	implies	2SAP	implies	3SAP…etc.
0SEP	implies	1SEP	implies	2SEP	implies	3SEP…etc.

$\longrightarrow\longrightarrow\longrightarrow$ ——— progressively weaker claims ——— $\longrightarrow\longrightarrow\longrightarrow$

4SIP	implies	3SIP	implies	2SIP	implies	1SIP…etc.
4SOP	implies	3SOP	implies	2SOP	implies	1SOP…etc.

(It was mentioned earlier that 0SIP is vacuous and, of course, so is 0SOP; hence they are necessarily true, and are therefore implied by every proposition. And the same is case for −1SIP, −2SOP, and so on, if they are admissible at all.) Now, other than for the value of the exceptions, the numerically expanded forms are like the traditional ones.

2.1 Equivalences and distribution First of all, the immediate inferences are the same, as is recorded below where the expressions equivalent to the original ones are set in boldface. That is, for any exception x, the converse of the E and I, the contrapositive of the A and O, and the obverse of all four are equivalent to the original forms.

Original	Conversion(Cv)	Obversion(Ob)	Contraposition(Cp)
xSAP	xPAS	$\boldsymbol{x}\mathbf{SE\overline{P}}$	$\boldsymbol{x}\mathbf{\overline{P}A\overline{S}}$
xSEP	$\boldsymbol{x}$**PES**	$\boldsymbol{x}\mathbf{SA\overline{P}}$	$x\overline{\mathrm{P}}\mathrm{E}\overline{\mathrm{S}}$
xSIP	$\boldsymbol{x}$**PIS**	$\boldsymbol{x}\mathbf{SO\overline{P}}$	$x\overline{\mathrm{P}}\mathrm{I}\overline{\mathrm{S}}$
xSOP	xPOS	$\boldsymbol{x}\mathbf{SI\overline{P}}$	$\boldsymbol{x}\mathbf{\overline{P}O\overline{S}}$

Also, the distribution values of the terms are the same: that is, the subject of the universals (if they might still be so called, even with nonzero exceptions), and the predicates of the negatives remain distributed, while the others remain undistributed. (See [3] and [4] for more thorough treatments of distribution values of exceptive propositions.)

2.2 Existential import On the Boolean interpretation, particulars are considered inherently existential while universals are not, although the latter are considered capable of being existential by presupposition. It is this basic interpretation—generalized to cover the expanded propositions—that is required for the logic below. According to the generalized Boolean view, particulars with an exception of x (xSIP and xSOP) are inherently existential "to the magnitude of x"; that is, they entail that at least x S's exist, while universals (xSAP and xSEP) remain nonexistential except by presupposition. This generalization follows from the contention that

1. a statement is inherently existential to the magnitude of x if and only if the existence of at least x S's is a precondition for the statements truth,

together with the observation that

2. the existence of at least x S's is a precondition for the truth of xSIP and xSOP, but xSAP and xSEP are compatible with the existence of any number of S's whatever.

That is, a particular statement, such as "At least x S's are P" is true if and only if x or more S's are P in fact: that is, when and only when fewer than x S's are P is the statement false. Accordingly, to assert "At least x S's are P" is tacitly to assert also that there exist at least x S's, for it is only under this existential condition that it is possible for the asserted statement to be true. And the same holds for xSOP, since it is equivalent to $x\mathrm{SI\overline{P}}$.

On the other hand, a universal statement, such as "At most x S's are P" (xSEP) is true if and only if x *or fewer* S's are P in fact. Accordingly, to assert that at most x S's are P is not tacitly to assert that there exist at least x S's, for it can be—indeed, it will be—true if only $x-1$, $x-2$, and so on, or zero S's exist. The same holds for xSAP, since it is equivalent to $x\mathrm{SE\overline{P}}$.

Incidentally, Englebretsen [1] rejects the Boolean perspective at this point by arguing that SAP is inherently existential, while SEP is not, and since the presentation below proceeds from the Boolean perspective, the claims made of SAP will not be meaningful to any who read them through Englebretsen's eyes. However, since the Boolean view takes the A-form to be perfectly equivalent to and, hence, logically interchangeable with, its obverse E-form, the impasse can be averted: those who subscribe to Englebretsen's view can simply read the A-form by its obverse expression. Then, for example, the expansion of Barbara given in the introduction, viz.,

$$\begin{array}{l} \text{At least all but 5 M's are P} \\ \text{At least all but 4 S's are M} \\ \hline \text{At least all but 9 S's are P,} \end{array}$$

would be read as

$$\begin{array}{l} \text{At most 5 M's are nonP} \\ \text{At most 4 S's are nonM} \\ \hline \text{At most 9 M's are nonP;} \end{array}$$

and with the systematic application of this procedure, each point presented from the Boolean perspective will hold from Englebretsen's perspective as well. But clearly, there is no similar accommodation possible for those who reject the Boolean perspective in favor of the Aristotelian view, since the latter interprets both affirmative and negative universals as being inherently existential.

So, in summary of the generalized Boolean perspective, *x*SIP and *x*SOP entail the existence of at least *x* S's since it is impossible for them to be true under any other condition; but *x*SAP and *x*SEP do not entail the existence of any S's, because it is possible for them to be true under any existential condition whatever. Nevertheless, the existence of any number of S's may be presupposed, or assumed, as a separate, or additional, consideration; and then inferences from the conjunction of the universals and the additional, existential presuppositions can be made.

2.3 Minimum presuppositions With the exceptions of the traditional logic limited to zero and one, the standard existential presupposition was understandably that of "at least one," even when many more may have been known or assumed to exist. However, with the introduction of alternative exceptions, alternative presuppositions become significant. For example, 0SAP implies 100SIP on the presupposition that there exist at least one hundred S's (S $\geq$ 100); and on the same presupposition,

1SAP	implies	99SIP	by presupposition,	
2SAP	implies	98SIP	by presupposition,	
3SAP	implies	97SIP	by presupposition,	
$\vert$			$\vert$	
97SAP	implies	3SIP	by presupposition,	
98SAP	implies	2SIP	by presupposition,	and
99SAP	implies	1SIP	by presupposition.	

Furthermore, since 99SIP implies 98SIP, 97SIP, and so on, because they are weaker claims, 1SAP likewise implies each of these by presupposition as well. Henceforth, only the strongest implication will be noted, although it is to be understood that any weaker proposition of the same form is also implied *a fortiori*. And the exception of the strongest particular implied by a universal by virtue of existential presupposition is: the value of exception of the particular is equal to the presupposition minus the exception of the implying universal. Accordingly, on the presupposition that there exist at least y S's (S $\geq y$), xSAP implies $y - x$SIP.

It should be noted that if x is equal to y, then the form implied is the vacuous one mentioned earlier, viz., 0SIP; and if x is greater than y then the form implied, for example, -1SIP, is also vacuous (if again, negative exceptions are admissible at all). Such implications can be avoided if it is stipulated that the minimum presupposition be greater than the exception.

Here such implications will be avoided by the convention that a, b, and c be variables whose instantiations may not be zero, although they may be any value greater than zero, while w, x, y, and z remain variables whose instantiations may either be zero or any greater value. Then the universal, xSAP, will always imply the nonvacuous particular, aSIP, if the presupposition is set for the sum of their exceptions: S $\geq a + x$. This will hold good for the traditional instantiations (where $a = 1$ and $x = 0$), as well as for each expanded value. Now analogous presuppositions can be made for E-forms: that is, on the presupposition that there exist at least $a + x$ S's (S $\geq a + x$), then xSEP implies aSOP.

The presuppositions so far have been made for the subject terms S of the two universals; however, a minimum presupposition can significantly be made for each term that is distributed relative to a universal, since each such term is the subject of an equivalent expression of that universal. Therefore, on the presupposition that there exist at least $a + x$ nonP's ($\overline{\mathrm{P}} \geq a + x$), xSAP implies $a\overline{\mathrm{P}}\mathrm{I}\overline{\mathrm{S}}$, since xSAP is equivalent to $x\overline{\mathrm{P}}\mathrm{A}\overline{\mathrm{S}}$ by contraposition,

$$x\mathrm{SAP} \iff \mathrm{C}p \implies x\overline{\mathrm{P}}\mathrm{A}\overline{\mathrm{S}},$$

and with the presupposition, ($\overline{\mathrm{P}} \geq a + x$), $x\overline{\mathrm{P}}\mathrm{A}\overline{\mathrm{S}}$ implies $a\overline{\mathrm{P}}\mathrm{I}\overline{\mathrm{S}}$. Likewise, on the presupposition that there exist at least $a + x$ P's, xSEP implies aPOS, since xSEP is equivalent to xPES by conversion,

$$x\mathrm{SEP} \iff \mathrm{C}v \implies x\mathrm{PES},$$

and with the presupposition, (P $\geq a + x$), xPES implies aPOS. Accordingly, both aSIP and $a\overline{\mathrm{P}}\mathrm{I}\overline{\mathrm{S}}$ follow from xSAP, and both aSOP and aPOS follow from xSEP, when the respective presuppositions are made.

2.4 Maximum presuppositions Although the traditional logic only employed the presupposition of "at least one," it might have made use of the presupposition of "at most one" also. Then, for example, from

Some God is omniscient (1SIP)

traditional logic might have inferred

Every God is omniscient (0SAP)

on the presupposition that there exists at most one God (S ≤ 1); and with the introduction of alternative exceptions, alternative maximum presuppositions, as well as alternative minimum presuppositions, now also become significant. For example,

1SIP implies 99SAP by presupposition,

on the presupposition that there exist at most one hundred S's (S $\leq$ 100), since "one of 100" is equal to "all but 99 of 100"; and on the same presupposition,

2SIP	implies	98SAP	by presupposition,	
3SIP	implies	97SAP	by presupposition,	
4SIP	implies	96SAP	by presupposition,	
|			|	
98SIP	implies	2SAP	by presupposition,	
99SIP	implies	1SAP	by presupposition,	and
100SIP	implies	0SAP	by presupposition.	

Also, as with the case of minimum presupposition, since 0SAP implies the weaker claims of 1SAP, 2SAP, and so on, 100SIP implies each of these by presupposition as well.

Here, if the presupposition is greater then exception of the particular, the form implied is a universal with a positive exception (such as 1SAP), and if the presupposition is equal to the exception of the particular, the form implied is the full, traditional universal (0SAP). However, if the presupposition were less than the exception of the particular (such as would be the case for 101SIP on the presupposition above), then the form implied (-1SAP) would be necessarily false (if again, negative exceptions are admissible at all). Put another way, if the maximum presupposition is less than the exception of the particular proposition, then the presupposition is inconsistent with the proposition, as the presupposition that there exists at most 100 S's (S $\leq$ 100) is inconsistent with the claim that there exist at least 101 S's that are P (101SIP). To avoid such incongruousness it can be stipulated that the exception of the particular not be greater than the maximum presupposition, and the convention of setting the maximum presupposition at the sum of the exceptions of the particular and the implied universal will ensure compliance with the stipulation. Then the particular, aSIP, will imply the universal, xSAP, and also be consistent with the presupposition, (S $\leq a + x$).

Now analogous implications by presupposition hold for the O-forms, for on the presupposition that there exist at most $a+x$ S's, aSOP implies xSEP. Furthermore, on the presupposition that there exist at most $a+x$ P's, aSIP implies xPAS, since aSIP is equivalent to aPIS by conversion,

$$a\text{SIP} \Longleftarrow \text{C}v \Longrightarrow a\text{PIS},$$

and with the presupposition, (P $\leq a+x$), aPIS implies xPAS. Likewise, on the presupposition that there exist at most $a+x$ nonP's, aSOP implies $x\overline{\text{P}}\text{E}\overline{\text{S}}$, since aSOP is equivalent to $a\overline{\text{P}}\text{O}\overline{\text{S}}$,

$$a\text{SOP} \Longleftarrow \text{C}p \Longrightarrow a\overline{\text{P}}\text{O}\overline{\text{S}},$$

and with the presupposition, ($\overline{\text{P}} \leq a+x$), $a\overline{\text{P}}\text{O}\overline{\text{S}}$ implies $x\overline{\text{P}}\text{E}\overline{\text{S}}$.

There are several distinctive points of contrast between the two types of presupposition. Perhaps the most interesting is that while universals imply particulars by virtue of minimum presupposition, particulars imply universals by virtue of maximum presupposition, as the examples above illustrate.

A second point of contrast is that minimum presuppositions are to be made for distributed terms relative to universals, whereas maximum presuppositions are to be made for undistributed terms relative to particulars. Of course, any presupposition can be made for any term whatever; but it is only in these cases that the original proposition implies some other significant form by virtue of the presupposition. For example, nothing new follows from 1SIP on the presupposition that there exist at least one hundred S's; and nothing new follows from 0SAP on the assumption that there exist at most one hundred S's.

A third point of contrast is that while no proposition is true or false by virtue of minimum presupposition, universals can be true, and particulars can be false, by virtue of maximum presupposition. That is, on the presupposition that there exist at most x S's, xSAP and xSEP are both vacuous, and true by presupposition. For example, on the presupposition that there exist at most 0 S's, the traditional forms of 0SAP and 0SEP are no longer contraries, but are both true instead; and such is always the case when a maximum presupposition is equal to (or less than) the universal exception. On the presupposition that there exist at most b S's, $a+b$SIP and $a+b$SOP are both false by presupposition since, as was stated above, they are inconsistent with the presupposition. For example, on the presupposition that there exist at most 0 S's, the traditional forms of 1SIP and 1SOP are no longer subcontraries, but both are false by presupposition instead; and such is always the case when a maximum presupposition is less than the particular exception.

Finally, no categorical proposition follows from the mere presupposition that some minimum quantity of S's exist; however, every universal form having S as its subject follows from the mere presupposition that some maximum quantity of S's exists: that is, from the presupposition that there exist at most x S's, both xSEP and xSAP follow, for any P.

2.5 Combined presuppositions When appropriate minimum and maximum presuppositions are made in combination, each proposition implies three additional propositions, producing a total of four nonequivalent propositions in all. For example, a proposition of the SAP-form implies a proposition of the SIP-form, another of the $\overline{\text{S}}\text{A}\overline{\text{P}}$-form, and still another of the $\overline{\text{S}}\text{I}\overline{\text{P}}$-form. Of course, the two particulars follow from the SAP-form by minimum presupposition, as was shown earlier; but then the additional universal follows, in turn, as a secondary implication from one of the implied particulars by maximum presupposition.

The three implied forms follow when minimum presuppositions are made for the distributed terms relative to the original proposition and maximum presuppositions are made for the undistributed terms relative to it. This is shown below where the three implied forms are derived from the original form entered on line 5 along with the presuppositions listed on lines 1–4. The derivation proceeds by appealing to obversion (Ob) and valid applications of conversion (Cv) and contraposition (Cp), together with implication based on (maximum or minimum) existential presupposition (IP).

1.	$\text{S} \geq a+x$	Presup
2.	$\overline{\text{S}} \leq b+y$	Presup
3.	$\text{P} \leq a+y$	Presup
4.	$\overline{\text{P}} \geq c+x$	Presup
5.	***x*SAP**	original proposition

6.	***a*SIP**	IP 1, 5
7.	*a*PIS	Cv 6
8.	*y*PAS	IP 3, 7
9.	$\boldsymbol{y\overline{\mathrm{S}}\mathrm{A}\overline{\mathrm{P}}}$	Cp 8
10.	$\boldsymbol{b\overline{\mathrm{S}}\mathrm{I}\overline{\mathrm{P}}}$	IP 2, 9.

That is, given presuppositions 1–4, *x*SAP (5) implies *a*SIP (6), $y\overline{\mathrm{S}}A\overline{\mathrm{P}}$ (9), and $b\overline{\mathrm{S}}I\overline{\mathrm{P}}$ (10). In fact, these follow without any appeal to the presupposition for nonP (on line 4). However, the $\overline{\mathrm{S}}\mathrm{I}\overline{\mathrm{P}}$-form can also be derived by appealing to that presupposition, as was shown earlier:

11.	$x\overline{\mathrm{P}}\mathrm{A}\overline{\mathrm{S}}$	Cp 5
12.	$c\overline{\mathrm{P}}\mathrm{I}\overline{\mathrm{S}}$	IP 4, 11
13.	$\boldsymbol{c\overline{\mathrm{S}}\mathrm{I}\overline{\mathrm{P}}}$	Cv 12

Here $b\overline{\mathrm{S}}\mathrm{I}\overline{\mathrm{P}}$ (10) is equivalent to $c\overline{\mathrm{S}}\mathrm{I}\overline{\mathrm{P}}$ (13) if *b* equals *c*; otherwise, the one with the greater exception implies the other. This proof of the propositions implied by *x*SAP also holds for the propositions implied by *x*SEP, *mutatis mutandis*. Likewise, when maximum and minimum presuppositions are again made for its undistributed and distributed terms, respectively, a proposition of the SIP-form implies a proposition of the SAP-form, another of the $\overline{\mathrm{S}}\mathrm{I}\overline{\mathrm{P}}$-form, and still another of the $\overline{\mathrm{S}}\mathrm{A}\overline{\mathrm{P}}$-form. This is shown in the derivation below.

1.	$\mathrm{S} \leq a + x$	Presup
2.	$\overline{\mathrm{S}} \geq b + y$	Presup
3.	$\mathrm{P} \leq a + y$	Presup
4.	$\overline{\mathrm{P}} \geq c + x$	Presup
5.	***a*SIP**	original proposition
6.	***x*SAP**	IP 1, 5
7.	*a*PIS	Cv 5
8.	*y*PAS	IP 3, 7
9.	$\boldsymbol{y\overline{\mathrm{S}}\mathrm{A}\overline{\mathrm{P}}}$	Cp 8
10.	$\boldsymbol{b\overline{\mathrm{S}}\mathrm{I}\overline{\mathrm{P}}}$	IP 2, 9 (Compare with line 13.)

After line 9 the derivation might have continued:

11.	$x\overline{\mathrm{P}}\mathrm{A}\overline{\mathrm{S}}$	Cp 6
12.	$c\overline{\mathrm{P}}\mathrm{I}\overline{\mathrm{S}}$	IP 4, 11
13.	$\boldsymbol{c\overline{\mathrm{S}}\mathrm{I}\overline{\mathrm{P}}}$	Cv 12 (Compare with line 10.)

And this proof holds, *mutatis mutandis*, for the analogous implications of the O proposition.

3 Syllogistic inferences The fifteen valid forms of the traditional logic are also valid for the expanded syllogism so far as the common features are concerned. However, new possibilities for invalidity (as well as for validity) are introduced in the expanded logic by the unlimited number of exceptions such propositions can have; accordingly, supplemental rules are required to ensure that the exception value of the conclusion is warranted.

3.1 Inferences without presupposition The first supplementary rule for the numerically expanded logic is:

Rule 3.1 The exception for a universal conclusion must not be less than the sum of the exceptions in the premises.

If the exception is less than the totaled exceptions of the premises, then the argument is invalid; if the exception is equal to those exceptions (as in the cases below), then it is the strongest conclusion that is entailed; and if it is greater, then it is a valid conclusion that is weaker than the strongest one entailed.

The necessity of this rule should be obvious, I believe, at least after a moment's reflection. For example, the following instances of Barbara illustrate it.

Traditional Values	Some Expanded Values	Other Expanded Values	General Formula
0MAP	3MAP	487MAP	yMAP
0SAM	5SAM	1832SAM	xSAM
0SAP	8SAP	2319SAP	$x+y$SAP

The second supplementary rule is:

Rule 3.2 The exception for a particular conclusion must not be greater than the exception of the particular premise minus the exception of the universal premise.

Again, for the strongest conclusion entailed, the exception of the conclusion must equal the exception of the particular premise minus the exception of the universal, as they do in the instances of Darii below.

Traditional Values	Some Expanded Values	Other Expanded Values	General Formula
0MAP	3MAP	487MAP	xMAP
1SIM	9SIM	1832SIM	aSIM
1SIP	6SIP	1345SIP	$a-x$SIP

Here again, the conclusion, $a-x$SIP, will be vacuous unless the exception of the particular, a, is greater than the exception of the universal, x. So to ensure that the conclusion is not vacuous, the convention of including the exception of the universal in the exception of the particular will be adopted:

$$\begin{array}{l} x\text{MAP} \\ \underline{a+x\text{SIM}} \\ a\text{SIP} \end{array}$$

Perhaps it should be noted that these supplementary rules do not impose any new requirements on the syllogism. However, there was no need to articulate them when the exceptions were limited to the classical values, since then there was no way they could be violated.

With this numerical extension, the power and flexibility of the syllogism is immensely—indeed, *infinitely*—expanded, and it is expanded even *infinitely further* by the introduction of inferences based on existential presupposition.

3.2 Inferences based on presuppositions As was reported earlier, when appropriate presupposition is allowed, each set of premises yields multiple conclusions. Indeed, each yields a conclusion of the SAP[$\overline{\text{P}}$]-form, another of the SIP[$\overline{\text{P}}$]-form, still another of the $\overline{\text{S}}$A$\overline{\text{P}}$[P]-form, and also one of the $\overline{\text{S}}$I$\overline{\text{P}}$[P]-form; and, of course, each of these four nonequivalent conclusions has three other equivalent statements (by obversion, etc.), and each also represents the possibility of infinitely many specific instantiations.

Below I advance derivations to show that such conclusions do follow. Of course, alternative derivations are possible, and the conclusions follow from more than one pattern of presuppositions in each case; but I do not attempt to be exhaustive, instead I only make the presuppositions required for the derivations actually advanced.

Furthermore, presupposition and exception variables could be arbitrarily set, and general derivations could proceed on the basis of them. However, below I adjust the presupposition variables to those of the exceptions of the premises so as to preclude the possibility of any line's having an inconsistent or vacuous instantiation.

Now a set of propositions yields a conclusion if and only if (1) at least one of them is universal, and (2) they share a middle term whose occurrences have opposite distribution values. Accordingly, four groups of premise sets can be distinguished on the basis of whether they conform to both conditions: only the first, only the second, or neither, as follows:

	Fig. 1	Fig. 2	Fig. 3	Fig. 4	
	AA	AE	AI	AA	
	AI	AO	AO	AE	
First	EA	EA	EI	EI	Satisfies both
Group	EI	EI	EO	EO	criteria
	IE	IE	IA	IA	
	OE	OA	IE	IE	
			OA		
			OE		
	AE	AA		AI	
	AO	AI		AO	
Second	EE	EE	AA	EA	Satisfies first
Group	EO	EO	AE	EE	criterion only
	IA	IA	EA	OA	
	OA	OE	EE	OE	
Third	IO	IO		OI	Satisfies second
Group	OO	OI		OO	criterion only
			II		
Fourth			IO		Satisfies neither
Group	II	II	OI	II	criterion
	OI	OO	OO	IO	
	16	16	16	16	

Below I provide a derivation of the four conclusion forms for one or two sets of premises from each group; these derivations, then, will hold, *mutatis mutandis*, for the remaining sets of premises of each group.

Appeals will be made to Barbara (Ba) and Darii (Da), in addition to (Ob), (Cv), (Cp), and (IP), in the derivation of these conclusions from the premises and presuppositions. Each application of Barbara and Darii will conform to the relevant supplementary rule advanced above. The exception of a proposition inferred by Barbara will equal the combined exceptions of the universal propositions from which it is inferred, and the exception of a proposition inferred by Darii will equal the exception of the particular minus the exception of the universal proposition from which it is inferred.

Premises that conform to both conditions yield conclusions as they stand but again, they also imply three additional conclusions when the appropriate presuppositions are made. Perhaps the most obvious way to proceed with the sets of this group is to take the premises of Barbara and Darii, draw the presuppositionless conclusion from each, and then derive the remaining three from it.

AA_, Fig.1

1.	$\mathrm{S} \geq a+x+y$	Presup
2.	$\overline{\mathrm{S}} \leq b+z$	Presup
3.	$\overline{\mathrm{P}} \geq b+x+y$	Presup
4.	$x\mathrm{MAP}$	Premise
5.	$y\mathrm{SAM}$	Premise
6.	$\boldsymbol{x+y}\mathbf{SAP}$	Ba 4, 5
7.	$\boldsymbol{a}\mathbf{SIP}$	IP 1, 6
8.	$x+y\overline{\mathrm{P}}\mathrm{A}\overline{\mathrm{S}}$	Cp 6
9.	$b\overline{\mathrm{P}}\mathrm{I}\overline{\mathrm{S}}$	IP 3, 8
10.	$\boldsymbol{b}\overline{\mathbf{S}}\mathbf{I}\overline{\mathbf{P}}$	Cv 9
11.	$\boldsymbol{z}\overline{\mathbf{S}}\mathbf{A}\overline{\mathbf{P}}$	IP 2, 10

AI_, Fig 1

1.	$\mathrm{S} \leq a+x+y$	Presup
2.	$\overline{\mathrm{S}} \geq b+z$	Presup
3.	$\mathrm{P} \leq a+z$	Presup
4.	$x\mathrm{MAP}$	Premise
5.	$a+x\mathrm{SIM}$	Premise
6.	$\boldsymbol{a}\mathbf{SIP}$	Da 4, 5
7.	$\boldsymbol{x+y}\mathbf{SAP}$	IP 1, 5
8.	$a\mathrm{PIS}$	Cv 6
9.	$z\mathrm{PAS}$	IP 3, 8
10.	$\boldsymbol{z}\overline{\mathbf{S}}\mathbf{A}\overline{\mathbf{P}}$	Cp 9
11.	$\boldsymbol{b}\overline{\mathbf{S}}\mathbf{I}\overline{\mathbf{P}}$	IP 2, 10

(The first sample argument by presupposition given in the introduction instantiates lines 4–7 of this derivation from the premises of Barbara when $a = 2$, $x = 5$, and $y = 4$.)

Here the maximum presupposition of y (in $\mathrm{S} \leq a + x + y$) is not necessary for the derivation of four different conclusions from the premises of Darii; however, it is included here to make the formula fully general. Otherwise, on the presupposition that ($\mathrm{S} \leq a + x$), the particular premise on line 5 ($a + x\mathrm{SIM}$) would imply the full universal, 0SAM, for every instantiation of $a + x$. So y is included in the presupposition to make alternative implications possible; but it also retains the original possibility, since y can be instantiated with zero. (The determination of the maximum presuppositions for the remaining derivations having a particular premise is based on this consideration as well.)

Now these same procedures hold, *mutatis mutandis*, for all other cases of the first group. That is, each other conclusion-yielding set of premises can be reduced either to AA_ or AI_, Fig. 1, and then the derivations above will produce the analogous conclusions from analogous presuppositions.

Premises that conform only to the first condition (viz., that at least one premise be universal) either have both or neither occurrence of the middle term distributed, as in PEM–SEM and MAP–SOM, or PAM–SAM and PAM–SIM. But the problem is the same: it merely manifests itself as having both middle terms distributed with one statement of the premises and as having neither distributed with an alternate statement of them. That is, since the distribution value of $\overline{\mathrm{M}}$ is always opposite to that of M, the equivalent expression of PEM–SEM, viz., PA$\overline{\mathrm{M}}$–SA$\overline{\mathrm{M}}$, suffers from having neither middle term distributed, whereas both are distributed in the original expression. The same is the case with each set of premises in this group.

The essential step in derivations from such premises is that of implication (IP) from a presupposition made for a middle term. This step fulfills the otherwise unmet second condition since the distribution value of the middle term in the implied proposition will be opposite that of the original one.

When both premises are universal, the obvious strategy is to make a minimum presupposition for the distributed middle, derive a particular by implication, draw a conclusion by Darii, and proceed from there, and when one premise is particular, the parallel strategy is to make a maximum presupposition for the undistributed middle, derive a second universal, draw a conclusion by Barbara, and proceed from there. These strategies are employed in the derivations below.

EE_, Fig.1

1.	$\mathrm{S} \geq a + w$	Presup
2.	$\overline{\mathrm{S}} \leq b + x + y$	Presup
3.	$\mathrm{M} \geq b + x + z$	Presup
4.	$\overline{\mathrm{P}} \leq b + w$	Presup
5.	$x\mathrm{MEP}$	Premise
6.	$z\mathrm{SEM}$	Premise
7.	$z\mathrm{MES}$	Cv 6
8.	$z\mathrm{MA}\overline{\mathrm{S}}$	Ob 7
9.	$b + x\mathrm{MI}\overline{\mathrm{S}}$	IP 3, 8
10.	$b + x\overline{\mathrm{S}}\mathrm{IM}$	Cv 9
11.	$x\mathrm{MA}\overline{\mathrm{P}}$	Ob 5
12.	$\boldsymbol{b}\overline{\mathbf{S}}\mathbf{I}\overline{\mathbf{P}}$	Da 10, 11
13.	$\boldsymbol{x + y}\overline{\mathbf{S}}\mathbf{A}\overline{\mathbf{P}}$	IP 2, 12
14.	$b\overline{\mathrm{P}}\mathrm{I}\overline{\mathrm{S}}$	Cv 12
15.	$w\overline{\mathrm{P}}\mathrm{A}\overline{\mathrm{S}}$	IP 4, 14
16.	$\boldsymbol{w}\mathbf{SAP}$	Cp 15
17.	$\boldsymbol{a}\mathbf{SIP}$	IP 1, 16

IA_, Fig 1

1.	$\mathrm{S} \geq a + x + y$	Presup
2.	$\overline{\mathrm{S}} \geq b + z$	Presup
3.	$\mathrm{M} \leq a + y$	Presup
4.	$P \geq a + z$	Presup
5.	$a\mathrm{MIP}$	Premise
6.	$x\mathrm{SAM}$	Premise
7.	$y\mathrm{MAP}$	IP 3, 5
8.	$\boldsymbol{x + y}\mathbf{SAP}$	Ba 6, 7
9.	$\boldsymbol{a}\mathbf{SIP}$	IP 1, 8
10.	$a\mathrm{PIS}$	Cv 9
11.	$z\mathrm{PAS}$	IP 4, 10
12.	$\boldsymbol{z}\overline{\mathbf{S}}\mathbf{A}\overline{\mathbf{P}}$	Cp 11
13.	$\boldsymbol{b}\overline{\mathbf{S}}\mathbf{I}\overline{\mathbf{P}}$	IP 2, 12

The same derivations work, *mutatis mutandis*, for the other sets of this group: that is, each other set of premises in this group can be reduced either to EE_ or IA_, Fig. 1, from which the same derivations yield analogous conclusions from analogous presuppositions.

Premises that conform only to the second condition have the middle terms distributed properly but lack a universal premise. Here a maximum presupposition for an extreme term, either S or P, will allow the derivation of a universal premise, whereas it will leave the distribution value of the middle terms the same. It might seem that the inference of the four conclusions could then proceed as with the earlier cases. However, this group is different in that the derivations require maximum presuppositions for both complements of a set of terms in order to reach all four conclusions. In the derivation below these presuppositions are made for P and $\overline{\text{P}}$, although they might have been for either of the other sets of complementary terms.

IO__, Fig. 1

1.	$\text{S} \leq a + x + y$	Presup	9.	$x\overline{\text{M}}\text{A}\overline{\text{P}}$	Cp 8
2.	$\overline{\text{S}} \geq b + z$	Presup	10.	$a + x\text{SI}\overline{\text{M}}$	Ob 6
3.	$\text{P} \leq b + x$	Presup	11.	$\boldsymbol{a}\mathbf{SI}\overline{\mathbf{P}}$	Da 9, 10
4.	$\overline{\text{P}} \leq a + z$	Presup	12.	$\boldsymbol{x} + \boldsymbol{y}\mathbf{SA}\overline{\mathbf{P}}$	IP 1, 11
5.	$b\text{MIP}$	Premise	13.	$a\overline{\text{P}}\text{IS}$	Cv 11
6.	$a + x\text{SOM}$	Premise	14.	$z\overline{\text{P}}\text{AS}$	IP 4, 13
7.	$b\text{PIM}$	Cv 5	15.	$\boldsymbol{z}\overline{\mathbf{S}}\mathbf{AP}$	Cp 14
8.	$x\text{PAM}$	IP 3, 7	16.	$\boldsymbol{b}\overline{\mathbf{S}}\mathbf{IP}$	IP 2, 15

The noteworthy consequence of having maximum presuppositions for both complements of a term is that it restricts the domain of discourse to the totality defined by the sum of those presuppositions. Of course, there is no limitation on how great this finite domain may be, but for the pattern of presuppositions above it must be some instantiation of $(b + x) + (a + z)$, and this holds, *mutatis mutandis*, for the other sets of this group.

Premises that conform to neither condition are the particular premises whose middle terms have like distribution value, such as POM–SOM and MIP–SIM. However, both of these conditions can be corrected at once with a maximum presupposition for the undistributed middle, for then a universal proposition whose subject is the distributed middle is implied. The implied universal, together with the remaining original premise, yields a conclusion by Darii, and the three additional conclusions follow from it. This is illustrated below where the introductory example is generalized and completed.

II__, Fig. 1

1.	$\text{S} \leq a + w + x$	Presup	8.	$\boldsymbol{a}\mathbf{SIP}$	Da 6, 7
2.	$\overline{\text{S}} \geq b + x + y$	Presup	9.	$\boldsymbol{w} + \boldsymbol{x}\mathbf{SAP}$	IP 1, 8
3.	$\text{M} \leq a + w + y$	Presup	10.	$a\text{PIS}$	Cv 8
4.	$\text{P} \leq a + x + y$	Presup	11.	$x + y\text{PAS}$	IP 4, 10
5.	$a + y\text{MIP}$	Premise	12.	$\boldsymbol{x} + \boldsymbol{y}\overline{\mathbf{S}}\mathbf{A}\overline{\mathbf{P}}$	Cp 11
6.	$a + w\text{SIM}$	Premise	13.	$\boldsymbol{b}\overline{\mathbf{S}}\mathbf{I}\overline{\mathbf{P}}$	IP 2, 12
7.	$w\text{MAP}$	IP 3, 5			

Here the crucial consideration is to ensure that (1) the sum of the members asserted to belong to M by both premises combined is greater than the maximum amount presupposed for M, but that (2) neither individual premise has an exception greater than the maximum presupposed for M. For if condition 1 is unmet as in,

3.	$\text{M} \leq 10$	Presup
5.	$4MIP$	Premise
6.	$6SIM$	Premise

the application of Darii produces a vacuous conclusion on line 8:

7.	6MAP	IP 3, 5
8.	0SIP	Da 6, 7;

and if condition 2 is unmet as in,

3.	$M \leq 10$	Presup
5.	11MIP	Premise;

the premise is inconsistent with the presupposition.

The choice of the variables above ensures that both conditions will be met for every instantiation. The other sets of particular premises that assign the same value to both occurrences of the middle term can be handled in like fashion. That is, they can all be reduced to II__, Fig. 1, and their analogous conclusions derived from analogous presuppositions.

4 Summary The system of numerically expanded quantifiers increases the scope of each of the standard syllogisms infinitely, and this increase, furthermore, introduces the possibility of endless further inferences by presupposition for each different quantificational value. As mentioned at the outset, this quantificational extension, together with the further inferences possible on the basis of minimum presuppositions, was worked out earlier. In the above analysis the field is expanded once again to include inferences that are valid on the basis of maximum and combined presuppositions, and the magnitude of this final expansion is extraordinary: it discloses an infinite quantificational range for each of four nonequivalent conclusion forms for every pair of syllogistic premises.

The logic resulting with this final expansion "fills a gap" in that it delineates a set of problems which raw, critical thinking would likely struggle with and whose solutions by derivation in the first order predicate calculus with identity would be impractically protracted. For example, both of these approaches might be frustrated in the attempt to determine from the information given in lines one through five below: How many of a university's philosophy majors are not in the class?

1.	$S \geq 65$	There are at least 65 students in the class.
2.	$M \leq 50$	There are at most 50 math majors (in all).
3.	$P \leq 30$	There are at most 30 philosophy majors (in all).
4.	15SAM	All but 15 of the students in the class are math majors.
5.	10MIP	At least 10 of the math majors are also philosophy majors.

However, the problem is quite simple when addressed as a numerically expanded syllogism with presuppositions.

6.	40MAP	2, 5 IP	All but 40 math majors are also philosophy majors.
7.	55SAP	4, 6 Ba	All but 55 students in the class are philosophy majors.
8.	10SIP	1, 7 IP	At least 10 students in the class are philosophy majors.
9.	10PIS	8 Cv	At least 10 philosophy majors are students in the class.
10.	20PAS	3, 9 IP	All but 20 philosophy majors are students in the class.
11.	$20\mathrm{PE}\overline{\mathrm{S}}$	10 Ob	At most 20 philosophy majors are not students in the class.

Even far more complicated problems are readily solved by this method. For example, proofs for sorites—including additional presuppositions for the terms of the additional premises—can be constructed in the same fashion. Another way of characterizing the expansion presented above is that it identifies a certain set of problems and offers a practical way of dealing with them from within the framework of the numerically expanded rendition of the traditional syllogism.

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