Notre Dame Journal of Formal Logic
Volume 39, Number 2, Spring 1998

# Interpolation and Preservation in $\mathcal{ML}_{\omega_1}$

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**Abstract** In this paper we deal with the logic $\mathcal{ML}_{\omega_1}$ which is the infinitary extension of propositional modal logic that has conjunctions and disjunctions only for countable sets of formulas. After introducing some basic concepts and tools from modal logic, we modify Makkai's generalization of the notion of consistency property to make it fit for modal purposes. Using this construction as a universal instrument, we prove, among other things, interpolation for $\mathcal{ML}_{\omega_1}$ as well as preservation results for universal, existential, and positive $\mathcal{ML}_{\omega_1}$-formulas.

***1 Introduction*** For a long time infinitary logics were widely ignored in the area of modality. This situation changed only quite recently. About five years ago logicians, computer scientists, and philosophers began to investigate infinitary modal logics more deeply, thereby concentrating on extensions of $\mathcal{ML}$, such as $\mathcal{ML}_\infty$ and $\mathcal{ML}_{\omega_1}$.

By $\mathcal{ML}$ we mean the polymodal version of standard propositional modal logic, that is, the logic one gets by adding several boxes and diamonds to the logical part of the language of propositional logic, and by using a Kripke-style semantics for interpreting its formulas. The infinitary modal logic $\mathcal{ML}_\infty$ is obtained from $\mathcal{ML}$ by adjoining conjunctions and disjunctions for arbitrary *sets* of formulas and by adapting the semantics correspondingly. If conjunctions and disjunctions are only defined for sets of cardinality smaller than $\kappa$, for a fixed regular cardinal $\kappa$, we have the logic $\mathcal{ML}_\kappa$.

One can make out several good reasons why infinitary modal logics should deserve our attention. In the first place, and this does not only apply to the modal case, infinitary logics provide a natural means for overcoming the expressive weakness of the corresponding finite systems; this concerns both aspects of expressiveness, the ability to express certain properties of structures, as well as the ability to characterize certain relations between structures and to distinguish pairs of structures between which these relations hold. Second, several interesting modal logics may be regarded—*via* suitable translations—as fragments of infinitary extensions of $\mathcal{ML}$;

*Received April 16, 1998; revised December 2, 1998*

the most popular ones are certainly *propositional dynamic logic* and the *logic of common knowledge*. There is legitimate hope that a deeper understanding of infinitary logics will lead to important insights into their respective fragments. Third, infinitary modal logics themselves might be analyzed as fragments of other logics, namely, as fragments of infinitary versions of first-order logic. This can be done with the aid of a straightforward adaptation of van Benthem's standard translation to the infinite; obviously, in the case of $\mathcal{ML}_\infty$ we get $\mathcal{L}_{\infty\omega}$ as the target logic, and $\mathcal{L}_{\kappa\omega}$ in the case of $\mathcal{ML}_\kappa$. From a logical point of view these fragments show well-behavior: quite a few metalogical properties are hereditary from logics to their modal fragments. Last but not least, in a recent book Barwise and Moss [2] have pointed out interesting connections between infinitary modal logics and the theory of non-wellfounded sets.

The first results that were proved with regard to infinitary modal logics were centered around the notion of bisimulation and its linguistic characterizability. Bisimulations are a special kind of equivalence relation between models that has turned out to play an important role in the model theory of modal logic. What makes them suitable for modal purposes is this: modal formulas cannot distinguish between bisimilar models, that is, if two models are bisimilar, then the same $\mathcal{ML}$-formulas hold in them. On the other hand, there are modally equivalent models that are not bisimilar. To bridge the gap, one needs infinitary tools. Around 1990, several authors made the observation that bisimilarity and elementary equivalence with respect to $\mathcal{ML}_\infty$ coincide. This result has since been improved upon and developed in a number of different directions. In [12], for instance, van Benthem and Bergstra obtained a modal variant of Scott's isomorphism theorem; they proved that every countable model (over a countable vocabulary) can be characterized up to bisimilarity by a single $\mathcal{ML}_{\omega_1}$-formula. In the same paper the authors extended van Benthem's bisimulation theorem to $\mathcal{L}_{\omega_1\omega}$; they showed that a $\mathcal{L}_{\omega_1\omega}$-formula is equivalent to (the translation of) a $\mathcal{ML}_{\omega_1}$-formula if and only if it is invariant for bisimulations. For $\mathcal{L}_{\infty\omega}$ such a characterization was given in van Benthem [9]. Further results on infinitary modal logics are contained in Barwise and van Benthem [1], Barwise and Moss [2], van Benthem [10], van Benthem [11], de Rijke [3], Sturm [6], and Sturm [7].

In [1], Barwise and van Benthem obtained a number of interpolation and preservation results for $\mathcal{ML}_\infty$ as corollaries to an abstract interpolation theorem. A more down-to-earth presentation of these results may be found in [11]. In spite of the wide applicability of their techniques, it is far from obvious how to apply them to $\mathcal{ML}_{\omega_1}$. This might give the reader some further motivation for paying attention to the method developed in the present paper.

Our paper exclusively deals with the logic $\mathcal{ML}_{\omega_1}$. Together with its companion piece [7], it provides an analysis of $\mathcal{ML}_{\omega_1}$ with respect to its most basic metalogical properties. The paper has the following structure: in the next section, we introduce basic concepts of the syntax and semantics of modal logic. Section 3 is devoted to two model constructions, a polymodal version of unraveling and a kind of modal amalgamation. The paper's main results are all obtained by a single method. This method may be described as a modal variant of a proof construction as introduced by Makkai [4] in the context of $\mathcal{L}_{\omega_1\omega}$. The key to this construction is what we will call the notion of an interpolation property, a concept that generalizes the notion of a consistency property, one of the main tools in the model theory of infinitary logics. The purpose of Section 4 is to give an informal description of how the whole construction works. It should help the reader to understand the things to come, and moreover, should give him some motivation for working through the formal details. An interpolation property is a more concrete version of an inductive property, a notion that forms the topic of Section 5. In Section 6 we introduce pseudo-complete theories and recall some basic facts on them. Roughly speaking, pseudo-complete theories are to $\mathcal{L}_{\omega_1\omega}$ what maximal consistent theories are to first-order logic. To complete our preliminary work, we have to introduce a wider class of modal formulas, so-called generalized modal formulas; this is done in Section 7. Section 8 finally defines interpolation properties and develops their main features; from the methodological point of view, this section forms the center of the article. By an application of the results from Sections 8 and 3, we obtain a proof of Craig's interpolation theorem for $\mathcal{ML}_{\omega_1}$ in Section 9. Slight variations of this proof yield preservation results for universal, existential, and positive formulas in the final section, 10. In addition, Section 10 contains an alternative proof of van Benthem and Bergstra's bisimulation theorem.

***2 Syntax and semantics*** A vocabulary $\tau$ is a set of relation symbols and individual constants. Every relation symbol is equipped with a positive natural number, its arity. We use $\mathcal{P}_\tau$ for the set of unary relation symbols, and $\mathcal{R}_\tau$ for the set of relation symbols of greater arity. Throughout this paper we assume that $\mathcal{R}_\tau$ only contains binary relation symbols. A vocabulary with no individual constants is called a *relational* vocabulary. If a vocabulary contains exactly one individual constant it is said to be a *modal* vocabulary.

In this paper we will heavily exploit the fact that via standard translation, $\mathcal{ML}_{\omega_1}$ may be considered as a fragment of $\mathcal{L}_{\omega_1\omega}$. In order to emphasize this relationship even on the syntactical level, we describe modal languages as based on (certain) first-order vocabularies. We choose vocabularies with a unique constant for two reasons. First, first-order languages over such a vocabulary and modal languages are interpreted on models of the same signature. Second, we prefer to correlate modal formulas with first-order *sentences* rather than with first-order formulas (with one free variable).

**Definition 2.1** Let $\tau$ be a modal vocabulary. The set $\mathcal{ML}_{\omega_1}(\tau)$ of $\mathcal{ML}_{\omega_1}$-formulas is defined as the smallest set $X$ such that

1. for every $P \in \mathcal{P}_\tau$, the propositional letter $p^P$ is in $X$,
2. if $\varphi$ is in $X$, then $\neg\varphi$ is in $X$,
3. if $\Phi \subseteq X$ is countable, then $\bigwedge \Phi$ and $\bigvee \Phi$ are in $X$,
4. if $R \in \mathcal{R}_\tau$ and $\varphi$ is in $X$, then $\Diamond_R\varphi$ and $\Box_R\varphi$ are in $X$.

By "countable," we mean finite or of cardinality $\omega$. We also allow $\Phi$ to be the empty set. In this case $\bigwedge \Phi$ is the verum and $\bigvee \Phi$ the falsum. Accordingly, we use $\top$ as an abbreviation for the empty conjunction and $\bot$ for the empty disjunction.

An $\mathcal{ML}_{\omega_1}$-formula (over $\tau$) is called a *universal* formula if and only if it is built up from propositional letters and negated propositional letters using $\bigvee$, $\bigwedge$ and $\Box_R$, with $R \in \mathcal{R}_\tau$. By $\Pi(\tau)$ we denote the set of universal formulas (over $\tau$). Analogously, the set $\Sigma(\tau)$ of *existential* formulas is defined as the smallest subset of $\mathcal{ML}_{\omega_1}(\tau)$ which contains all propositional letters as well as their negations and is closed under

$\bigwedge$, $\bigvee$, and $\Diamond_R$. A formula that is built up without using the negation symbol is called a *positive* formula. $\Upsilon(\tau)$ denotes the set of positive formulas over the vocabulary $\tau$.

The reader might have noticed that the above definitions only take modal formulas into account that are negation normal, that is, formulas in which the negation symbol only occurs in front of propositional letters. However, this restriction is not a serious one: it can easily be proved that for every modal formula, there is an equivalent formula which is negation normal. Through the following, we make the assumption that we have fixed a mapping which correlates each formula $\varphi$ with an equivalent negation normal formula $\varphi^{nf}$.

The semantics of modal logic is based on the same type of models as the semantics of first-order logic (and its infinitary extensions), restricted to modal vocabularies. Given a modal vocabulary $\tau$, a $\tau$-model consists of the following ingredients: a nonempty set $A$, a binary relation $R^{\mathfrak{A}}$ on $A$ for each $R \in \mathcal{R}_\tau$, a subset $P^{\mathfrak{A}}$ of $A$ for each $P \in \mathcal{P}_\tau$, and an element $a \in A$ which interprets the unique constant of $\tau$. To adjust the notation to the modal setting, we use $(\mathfrak{A}, a), (\mathfrak{B}, b), \ldots$ to designate $\tau$-models.

**Definition 2.2** Let $\tau$ be a modal vocabulary. The truth of an $\mathcal{ML}_{\omega_1}$-formula with respect to a $\tau$-model is defined inductively:

$$
\begin{aligned}
(\mathfrak{A}, a) \models p^P \quad &:\Longleftrightarrow P^{\mathfrak{A}}a, \text{ for } P \in \mathcal{P}_\tau,\\
(\mathfrak{A}, a) \models \neg\varphi \quad &:\Longleftrightarrow (\mathfrak{A}, a) \not\models \varphi,\\
(\mathfrak{A}, a) \models \bigvee \Phi \quad &:\Longleftrightarrow \text{there is a } \varphi \in \Phi\text{: } (\mathfrak{A}, a) \models \varphi,\\
(\mathfrak{A}, a) \models \bigwedge \Phi \quad &:\Longleftrightarrow \text{for all } \varphi \in \Phi\text{: } (\mathfrak{A}, a) \models \varphi,\\
(\mathfrak{A}, a) \models \Diamond_R\varphi \quad &:\Longleftrightarrow \exists a' \in A(R^{\mathfrak{A}}aa' \,\&\, (\mathfrak{A}, a') \models \varphi),\\
(\mathfrak{A}, a) \models \Box_R\varphi \quad &:\Longleftrightarrow \forall a' \in A(R^{\mathfrak{A}}aa' \Rightarrow (\mathfrak{A}, a') \models \varphi).
\end{aligned}
$$

In the last two clauses of the next definition, the variable $x$ is assumed to be the first variable from a list of variables that do not occur in $St(\varphi)$.

**Definition 2.3** Let $\tau$ be a modal vocabulary and $c$ be the unique individual constant in $\tau$. The following clauses define a mapping $St$ from $\mathcal{ML}_{\omega_1}(\tau)$ into the set of $\mathcal{L}_{\omega_1\omega}$-sentences over $\tau$:

$$
\begin{aligned}
St(p^P) \quad &:= Pc, \text{ for } P \in \mathcal{P}_\tau,\\
St(\neg\varphi) \quad &:= \neg St(\varphi),\\
St(\bigvee \Phi) \quad &:= \bigvee\{St(\varphi) \mid \varphi \in \Phi\},\\
St(\bigwedge \Phi) \quad &:= \bigwedge\{St(\varphi) \mid \varphi \in \Phi\},\\
St(\Diamond_R\varphi) \quad &:= \exists x(Rcx \wedge St(\varphi)[x/c]), \text{ for } R \in \mathcal{R}_\tau,\\
St(\Box_R\varphi) \quad &:= \forall x(Rcx \rightarrow St(\varphi)[x/c]), \text{ for } R \in \mathcal{R}_\tau.
\end{aligned}
$$

**Notation 2.4** For a formula $\psi$ and individual terms $t_1$ and $t_2$, $\psi[t_1/t_2]$ denotes the formula we obtain by replacing every occurrence of $t_2$ in $\psi$ by $t_1$.

The following lemma tells us that a modal formula and its translation are true in the same models. For this reason it is legitimate to regard $\mathcal{ML}_{\omega_1}$ as a fragment of $\mathcal{L}_{\omega_1\omega}$.

**Lemma 2.5** *Let $\tau$ be a modal vocabulary and $(\mathfrak{A}, a)$ a $\tau$-model. Then for every $\varphi \in \mathcal{ML}_{\omega_1}(\tau)$, $(\mathfrak{A}, a) \models \varphi$ if and only if $(\mathfrak{A}, a) \models St(\varphi)$.*

*Proof:* By induction on the complexity of $\varphi$. □

**Definition 2.6** Let $\tau$ and $\tau'$ be modal vocabularies such that $\tau \subseteq \tau'$, and let $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ be $\tau'$-models. A relation $Z \subseteq A \times B$ is a $\tau$-bisimulation between $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$, abbreviated by $Z : (\mathfrak{A}, a) \sim^{\tau}_{bs} (\mathfrak{B}, b)$, if and only if it satisfies the following conditions:

B0 $Zab$,

B1 $\forall a', b'(Za'b' \Longrightarrow \forall P \in \mathcal{P}_\tau(P^{\mathfrak{A}}a' \Longleftrightarrow P^{\mathfrak{B}}b'))$,

B2a $\forall a', b'\Big(Za'b' \Longrightarrow \forall R \in \mathcal{R}_\tau \forall a''(R^{\mathfrak{A}}a'a'' \Longrightarrow \exists b''(R^{\mathfrak{B}}b'b'' \,\&\, Za''b''))\Big)$,

B2b $\forall a', b'\Big(Za'b' \Longrightarrow \forall R \in \mathcal{R}_\tau \forall b''(R^{\mathfrak{B}}b'b'' \Longrightarrow \exists a''(R^{\mathfrak{A}}a'a'' \,\&\, Za''b''))\Big)$.

$(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ are said to be $\tau$-bisimilar, denoted by $(\mathfrak{A}, a) \sim^{\tau}_{bs} (\mathfrak{B}, b)$, if and only if there exists a $\tau$-bisimulation between $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$.

**Lemma 2.7** *Let $\tau$ be a modal vocabulary and suppose $(\mathfrak{A}, a) \sim^{\tau}_{bs} (\mathfrak{B}, b)$. Then for every $\varphi \in \mathcal{ML}_{\omega_1}(\tau)$, $(\mathfrak{A}, a) \models \varphi$ if and only if $(\mathfrak{B}, b) \models \varphi$.*

*Proof:* By induction on the complexity of $\varphi$. □

When we drop clause B2b in Definition 2.6, we obtain the definition of a $\tau$-*simulation*. $\tau$-simulations are to existential formulas what $\tau$-bisimulations are to modal formulas in general. By induction it can be shown that existential formulas are preserved under $\tau$-simulations, that is, if there is a $\tau$-simulation from $(\mathfrak{A}, a)$ to $(\mathfrak{B}, b)$—$(\mathfrak{A}, a) \rightsquigarrow^{\tau} (\mathfrak{B}, b)$ for short—then every existential formula which is true in $(\mathfrak{A}, a)$ is also true in $(\mathfrak{B}, b)$. This raises the following question: is it possible to convert the above observation into a full preservation result, that is, can we characterize $\Sigma(\tau)$ as the set of exactly those modal formulas that are preserved under $\tau$-simulations? In Section 10 we will give a a positive answer to this question. In fact, we will do a slightly better job. The first part of Theorem 10.2 tells us that a modal formula is equivalent to an existential formula if and only if it is preserved under extensions, where "extension" is understood in the sense of classical model theory. From this we also will obtain a preservation result for universal formulas: a modal formula will be shown to be equivalent to a universal formula if and only if it is preserved under submodels.

Looking for relations suitable for positive formulas, we come across *positive* $\tau$-bisimulations. A *positive* $\tau$-bisimulation is a relation $Z$ which satisfies B0, B2a, B2b from Definition 2.6 as well as the following weakening of clause B1: $\forall a', b'(Za'b' \Longrightarrow \forall P \in \mathcal{P}_\tau(P^{\mathfrak{A}}a' \Longrightarrow P^{\mathfrak{B}}b'))$. By $(\mathfrak{A}, a) \preceq^{\tau}_{+} (\mathfrak{B}, b)$ we mean that there exists a positive $\tau$-bisimulation from $(\mathfrak{A}, a)$ to $(\mathfrak{B}, b)$. It is easy to verify that positive formulas are preserved under these relations. However, as in the preceding case, the corresponding preservation result in Section 10 is stated and proved with respect to a different notion, which is again taken from classical model theory. This time we make use of the notion of a *weak* extension, where $(\mathfrak{B}, b)$ is said to be a *weak extension* of $(\mathfrak{A}, a)$—$(\mathfrak{A}, a) \subseteq_w (\mathfrak{B}, b)$ for short—if and only if $A = B$, $a = b$, the two models agree with respect to each $R \in \mathcal{R}_\tau$, and for every $P \in \mathcal{P}_\tau$, $P^{\mathfrak{A}} \subseteq P^{\mathfrak{B}}$.

***3 Unraveling and amalgamation*** In [5], Sahlqvist introduced a universal method of transforming a Kripke model into a treelike Kripke model which is bisimilar and hence modally equivalent to the original model. In this section we first generalize the construction of unraveling to the polymodal case. Then we prove two lemmas in which unraveled models are essentially involved and which will be needed in the proofs of our main results. Lemma 3.4 will play an important role in the proof of Craig's interpolation theorem in Section 9, whereas Lemma 3.5 enables us to sharpen our preservation results in Section 10 in the way it was described in the foregoing section.

**Definition 3.1** Let $\tau$ be a modal vocabulary and $(\mathfrak{A}, a)$ be a $\tau$-model. We define the sets of paths through $(\mathfrak{A}, a)$ of length $n$, $Path_n^\tau(\mathfrak{A}, a)$ for short, by induction.

1. $Path_\tau^0(\mathfrak{A}, a) := \{\langle a \rangle\}$,
2. $Path_\tau^{n+1}(\mathfrak{A}, a) := \{\sigma \circ \langle R, a' \rangle \mid \sigma \in Path_\tau^n(\mathfrak{A}, a) \,\&\, R \in \mathcal{R}_\tau \,\&\, R^{\mathfrak{A}}[\sigma]_{\ell(\sigma)} a'\}$.

**Notation 3.2** '$\circ$' denotes the concatenation operation on sequences. If $\sigma$ is a finite sequence, $\ell(\sigma)$ designates the length of $\sigma$, and if $0 < i \leq \ell(\sigma)$, then $[\sigma]_i$ designates the $i$-th item of $\sigma$.

We use $Path_\tau(\mathfrak{A}, a)$ to designate the set of paths through $(\mathfrak{A}, a)$, that is the union of the sets $Path_\tau^n(\mathfrak{A}, a)$.

**Definition 3.3** Let $\tau$ be a modal vocabulary and $(\mathfrak{A}, a)$ be a $\tau$-model. The unraveling of $(\mathfrak{A}, a)$, denoted by $(\mathfrak{A}^u, a^u)$, is defined as follows.

1. $A^u := \{\sigma \mid \sigma \in Path_\tau(\mathfrak{A}, a)\}$,
2. $a^u := \langle a \rangle$,
3. for $P \in \mathcal{P}_\tau$ and $\sigma \in A^u$, set $P^{\mathfrak{A}^u}\sigma :\Longleftrightarrow P^{\mathfrak{A}}[\sigma]_{\ell(\sigma)}$,
4. for $R \in \mathcal{R}_\tau$ and $\sigma, \sigma' \in A^u$, set $R^{\mathfrak{A}^u}\sigma\sigma' :\Longleftrightarrow \sigma' = \sigma \circ \langle R, [\sigma']_{\ell(\sigma')} \rangle$.

A model $(\mathfrak{A}, a)$ is said to be unraveled if it is isomorphic to its unraveling, that is, if $(\mathfrak{A}, a) \cong (\mathfrak{A}^u, a^u)$. For each element $a'$ of an unraveled model, there is exactly one $\sigma \in Path_\tau(\mathfrak{A}, a)$ that ends in $a'$. We use $\sigma_{a'}$ to designate this unique path. It is easy to verify that the following clause defines a bisimulation between a $\tau$-model $(\mathfrak{A}, a)$ and its unraveling: for every $a' \in A$ and $\sigma \in A^u$, $Za'\sigma :\Longleftrightarrow [\sigma]_{\ell(\sigma)} = a'$.

**Lemma 3.4** (Amalgamation) *Let $\tau, \tau_1, \tau_2$ be modal vocabularies such that $\tau = \tau_1 \cap \tau_2$. Suppose $(\mathfrak{A}, a)$ is an unraveled $\tau_1$-model and $(\mathfrak{B}, b)$ an unraveled $\tau_2$-model such that $(\mathfrak{A}, a) \sim_{bs}^{\tau} (\mathfrak{B}, b)$. Then there exists a $(\tau_1 \cup \tau_2)$-model $(\mathfrak{C}, e)$ with $(\mathfrak{A}, a) \sim_{bs}^{\tau_1} (\mathfrak{C}, e)$ and $(\mathfrak{B}, b) \sim_{bs}^{\tau_2} (\mathfrak{C}, e)$.*

*Proof:* Without loss of generality we may assume that $A$ and $B$ are disjoint sets. By assumption there is a $\tau$-bisimulation $Z$ between $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$. Utilizing $Z$ we define the model $(\mathfrak{C}, e)$ as follows: for $E$ we choose the union of the following three sets.

1. $\{\langle a', b' \rangle \mid Za'b' \,\&\, \sigma_{a'} \in Path_\tau(\mathfrak{A}, a) \,\&\, \sigma_{b'} \in Path_\tau(\mathfrak{B}, b)\}$
2. $\{a' \in A \mid \sigma_{a'} \notin Path_\tau(\mathfrak{A}, a)\}$
3. $\{b' \in B \mid \sigma_{b'} \notin Path_\tau(\mathfrak{B}, b)\}$.

Let $e$ be $\langle a, b \rangle$. For $P \in \mathcal{P}_{\tau_1 \cup \tau_2}$ and $e' \in E$ we put $P^{\mathfrak{C}}e'$, if $e' \in A$ and $P^{\mathfrak{A}}e'$, or $e' \in B$ and $P^{\mathfrak{B}}e'$, or there are $a' \in A$ and $b' \in B$ such that $e' = \langle a', b' \rangle$ and $P^{\mathfrak{A}}a'$ or $P^{\mathfrak{B}}b'$. Finally, for $R \in \mathcal{R}_{\tau_1 \cup \tau_2}$ and $e', e'' \in E$ we put $R^{\mathfrak{C}}e'e''$, if one of the following holds:

1. $\exists a' \exists b' (e' = \langle a', b' \rangle \,\&\, \exists a'', b'' (R^{\mathfrak{A}}a'a'' \,\&\, R^{\mathfrak{B}}b'b'' \,\&\, e'' = \langle a'', b'' \rangle))$,
2. $\exists a' \exists b' (e' = \langle a', b' \rangle \,\&\, e'' \in A \,\&\, R^{\mathfrak{A}}a'e'')$,

3. $\exists a'\exists b'(e' = \langle a', b'\rangle \,\&\, e'' \in B \,\&\, R^{\mathfrak{B}}b'e'')$,
4. $e' \in A \,\&\, e'' \in A \,\&\, R^{\mathfrak{A}}e'e''$,
5. $e' \in B \,\&\, e'' \in B \,\&\, R^{\mathfrak{B}}e'e''$.

To prove that $(\mathfrak{A}, a) \sim^{\tau_1}_{bs} (\mathfrak{E}, e)$ we define a relation $Z_1$ by

$$\forall a'\forall e'(Z_1 a'e' :\Longleftrightarrow e' = a' \vee \exists a''\exists b''(e' = \langle a'', b''\rangle \,\&\, a' = a'')),$$

and verify that $Z_1$ forms a $\tau_1$-bisimulation between $(\mathfrak{A}, a)$ and $(\mathfrak{E}, e)$. The conditions B0 and B1 are obvious. For B2a suppose $Z_1 a'e'$ and $R^{\mathfrak{A}}a'a''$. Two cases need to be distinguished.

*Case 1:* $e' = a'$. By the definition of $E$ it follows that $\sigma_{a'} \notin Path_\tau(\mathfrak{A}, a)$, hence $\sigma_{a''} \notin Path_\tau(\mathfrak{A}, a)$, hence $a'' \in E$. Together with the definitions of $Z_1$ and $\mathfrak{E}$, this yields $R^{\mathfrak{E}}e'a''$ and $Z_1 a''a''$, closing the first case.

*Case 2:* $e' = \langle a', b'\rangle$. If $R \notin \mathcal{R}_\tau$, then $a'' \in E$, and the desired result follows easily. If $R$ is in $\mathcal{R}_\tau$, we reason as follows. By the definition of $E$ we obtain $Za'b'$, thus there is a $b'' \in B$ such that $R^{\mathfrak{B}}b'b''$ and $Za''b''$. From the latter we obtain $\langle a'', b''\rangle \in E$. This, together with $R^{\mathfrak{A}}a'a''$ and $R^{\mathfrak{B}}b'b''$, implies $Z_1 a''\langle a''b''\rangle$ as well as $R^{\mathfrak{E}}\langle a', b'\rangle\langle a'', b''\rangle$. This concludes the proof for B2a. B2b is proved in a similar way.

For $(\mathfrak{B}, b) \sim^{\tau_2}_{bs} (\mathfrak{E}, e)$ we use exactly the same argument. □

**Lemma 3.5** *Let $\tau$ be a modal vocabulary and let $(\mathfrak{A}, a)$, $(\mathfrak{B}, b)$ be unraveled $\tau$-models.*

1. *If $(\mathfrak{A}, a) \rightsquigarrow^\tau (\mathfrak{B}, b)$, then there exists an unraveled $\tau$-model $(\mathfrak{B}', b')$ such that $(\mathfrak{B}', b') \sim^\tau_{bs} (\mathfrak{B}, b)$ and $(\mathfrak{A}, a) \subseteq (\mathfrak{B}', b')$.*
2. *If $(\mathfrak{A}, a) \preceq^\tau_+ (\mathfrak{B}, b)$, then there exist unraveled $\tau$-models $(\mathfrak{A}', a')$ and $(\mathfrak{B}', b')$ such that $(\mathfrak{A}', a') \sim^\tau_{bs} (\mathfrak{A}, a)$, $(\mathfrak{B}', b') \sim^\tau_{bs} (\mathfrak{B}, b)$ and $(\mathfrak{A}', a') \subseteq_w (\mathfrak{B}', b')$.*

*Proof:* The proofs of the two claims are based on a tedious though straightforward copying procedure which goes back to van Benthem [9]. For lack of space we dispense with their presentation. The reader interested in (detailed) proofs should consult [6]. □

***4 Interpolation properties: the informal account*** The purpose of this section is to give an informal presentation of the method of interpolation properties as it will be carried out in the following sections. Our approach adapts a method as introduced by Makkai (see [4]) in the context of $\mathcal{L}_{\omega_1\omega}$ to the modal setting. Moreover, it generalizes the method of consistency properties. For that reason we start our informal account by recalling the basic structure of the latter. Roughly, a consistency property $S$ (for $\mathcal{L}_{\omega_1\omega}$) is a set of finite consistent sets $s$ of $\mathcal{L}_{\omega_1\omega}$-formulas which satisfies certain closure conditions. With a few exceptions, these closure conditions show the following pattern: if $s \in S$ contains a formula of the shape ..., then there is a set $s' \in S$ which extends $s$ and contains a formula of the shape $\cdots$. In the next section we define such closure conditions as pairs $(I_1, I_2)$ of subsets of $S$. To say that $S$ is closed under $(I_1, I_2)$ means that for every $s \in S \cap I_1$, $S$ contains a set $s' \in I_2$ which extends $s$.

**Example 4.1** For a concrete example, consider the condition regarding existential formulas. It requires that for every set $s \in S$ containing a formula of the shape $\exists x\varphi$, there is a set $s' \in S$ with $s \subseteq s'$ and a constant $c$ such that $\varphi[c/x] \in s'$.

The closure conditions are chosen so that the following claim becomes provable: let $S$ be a consistency property and suppose $s \in S$, then there exists an $\omega$-sequence $\langle s_n \mid n \in \omega\rangle$ such that

1. $\forall n \in \omega : s_n \in S$,
2. $s_0 = s$,
3. $\forall n \in \omega : s_n \subseteq s_{n+1}$,
4. the limit $\Gamma := \bigcup_{n\in\omega} s_n$ of this sequence is a pseudo-complete theory, that is, a set of sentences that uniquely determines a model $\mathfrak{A}_\Gamma$,
5. $\mathfrak{A}_\Gamma \models \Gamma$.

For an application, consider completeness. Suppose we have a calculus that is sound with respect to the semantics of $\mathcal{L}_{\omega_1\omega}$. To prove completeness, it suffices to show that the set $U$ of all finite consistent subsets of $\mathcal{L}_{\omega_1\omega}$ is a consistency property. For if this has been shown we can reason as follows: let $\varphi$ be consistent, then the set $\{\varphi\}$ is an element of $U$, hence there is a sequence as described above. Since $\mathfrak{A}_\Gamma$ is a model of $\Gamma$ it is a model of $\varphi$ as well; therefore $\varphi$ has a model, which concludes the proof.

The role that is played by (finite) sets of sentences in the context of consistency properties will in our construction be overtaken by triples of the form $s = \langle\Theta_1, \Theta_2, z\rangle$—this is not quite correct: for technical reasons we will use objects of a slightly different sort; the triples were only chosen to make our informal presentation more suggestive—where $\Theta_1$ and $\Theta_2$ are finite consistent sets of sentences over two (possibly distinct) vocabularies $\tau_1$ and $\tau_2$, and where $z$ is the finite approximation of a suitable relation between $\tau_1$- and $\tau_2$-models. To be more precise, such an interpolation property $S$ is a set of triples $\langle\Theta_1, \Theta_2, z\rangle$ which again satisfies certain closure conditions, this time ensuring that for every triple $s = \langle\Theta_1, \Theta_2, z\rangle$ in $S$ there exists an $\omega$-sequence $\langle\langle\Theta_1^n, \Theta_2^n, z_n\rangle \mid n \in \omega\rangle$ such that

1. $\forall n \in \omega : \langle\Theta_1^n, \Theta_2^n, z_n\rangle \in S$,
2. $\langle\Theta_1^0, \Theta_2^0, z_0\rangle = s$,
3. $\forall n \in \omega : \Theta_1^n \subseteq \Theta_1^{n+1}, \Theta_2^n \subseteq \Theta_2^{n+1}, z_n \subseteq z_{n+1}$,
4. $\Gamma_1 := \bigcup_{n\in\omega} \Theta_1^n$ uniquely determines a $\tau_1$-model $\mathfrak{A}_{\Gamma_1}$ such that $\mathfrak{A}_{\Gamma_1} \models \Gamma_1$,
5. $\Gamma_2 := \bigcup_{n\in\omega} \Theta_2^n$ uniquely determines a $\tau_2$-model $\mathfrak{A}_{\Gamma_2}$ such that $\mathfrak{A}_{\Gamma_2} \models \Gamma_2$,
6. $Z := \bigcup_{n\in\omega} z_n$ defines a suitable structural relation between $\mathfrak{A}_{\Gamma_1}$ and $\mathfrak{A}_{\Gamma_2}$.

For an illustration of how this construction works, we give a quite simplified sketch of the proof of Craig's interpolation theorem (for a detailed proof see Section 9). The theorem is proved by contraposition. Suppose $\varphi$ and $\chi$ are two $\mathcal{ML}_{\omega_1}$-formulas which have no interpolant, meaning that there is no modal formula $\vartheta$ over their common vocabulary such that $\varphi \models \vartheta$ and $\vartheta \models \chi$. In other words, the sets $\{\varphi\}$ and $\{\neg\chi\}$ are modally nonseparable. Let $\tau_1$ be the vocabulary of $\varphi$, $\tau_2$ of $\chi$ and $\tau = \tau_1 \cap \tau_2$. Let $\Theta_1 := \{St(\varphi)\}$, $\Theta_2 := \{St(\neg\chi)[d/c]\}$ for a fresh constant $d$, and $z := \langle c, d\rangle$. It can be shown that the triple $\langle\Theta_1, \Theta_2, z\rangle$ is an element of an interpolation property.

Hence there is a sequence $\langle\langle\Theta_1^n, \Theta_2^n, z_n\rangle \mid n \in \omega\rangle$, satisfying the above conditions 1–6; in this special situation the relation $Z$ will turn out to be a $\tau$-bisimulation between $(\mathfrak{A}_{\Gamma_1}, c^{\mathfrak{A}_{\Gamma_1}})$ and $(\mathfrak{A}_{\Gamma_2}, d^{\mathfrak{A}_{\Gamma_2}})$. From this we infer $(\mathfrak{A}_{\Gamma_1}, c^{\mathfrak{A}_{\Gamma_1}}) \models \varphi$ and $(\mathfrak{A}_{\Gamma_2}, d^{\mathfrak{A}_{\Gamma_2}}) \models \neg\chi$. By an application of Lemma 3.4 we obtain a model $(\mathfrak{E}, e)$ such that $(\mathfrak{E}, e) \sim_{bs}^{\tau_1} (\mathfrak{A}_{\Gamma_1}, c^{\mathfrak{A}_{\Gamma_1}})$ and $(\mathfrak{E}, e) \sim_{bs}^{\tau_2} (\mathfrak{A}_{\Gamma_2}, d^{\mathfrak{A}_{\Gamma_2}})$. Finally, an application of Lemma 2.7 yields $(\mathfrak{E}, e) \models \varphi$ and $(\mathfrak{E}, e) \models \neg\chi$, which means that $\chi$ does not follow from $\varphi$. This completes the proof.

***5 Inductive properties*** The fact that each element of an interpolation property $S$ may serve as the first item of a suitable $\omega$-sequence, relies on features of $S$ that can be described in a much more abstract setting. By utilizing the notion of an inductive property this will be done in the present section. Apart from notation, the following is strongly inspired by [4].

Let $(S, \leq)$ be a partial order. A pair $I = (I_1, I_2)$ of subsets of $S$ is called a condition on $(S, \leq)$ if and only if for every $s \in I_1$ and for every $s' \in S$, if $s \leq s'$ then $s' \in I_1$. A partial order $(S, \leq)$ is closed under the condition $I = (I_1, I_2)$ if and only if for every $s_1 \in I_1$ there is an element $s_2$ of $S$ such that $s_1 \leq s_2$ and $s_2 \in I_2$.

**Definition 5.1** A triple $(S, \leq, \mathcal{I})$ is an inductive property if and only if $(S, \leq)$ is a partial order, $\mathcal{I}$ a set of conditions on $(S, \leq)$ under which $(S, \leq)$ is closed, and for each $s \in S$ the set $\{I \in \mathcal{I} \mid s \in I_1\}$ is countable.

**Definition 5.2** Let $(S, \leq, \mathcal{I})$ be an inductive property. We call a sequence $\langle s_n \mid n \in \omega\rangle$ closed with respect to $(S, \leq, \mathcal{I})$ if and only if

1. for every $n \in \omega$: $s_n \in S$,
2. for every $n \in \omega$: $s_n \leq s_{n+1}$,
3. for every $n \in \omega$ and $I \in \mathcal{I}$: if $s_n \in I_1$, then there is $m \geq n$ with $s_m \in I_2$.

**Lemma 5.3** *Let $(S, \leq, \mathcal{I})$ be an inductive property and $s \in S$. Then there is a sequence $\langle s_n \mid n \in \omega\rangle$ which is closed with respect to $(S, \leq, \mathcal{I})$ such that $s_0 = s$.*

*Proof:* For each $s \in S$ let $L_s$ be an enumeration of the set $\{I \in \mathcal{I} \mid s \in I_1\}$. By Definition 5.1, we can assume that the length of $L_s$ does not exceed $\omega$. Let $g$ be a bijection between $\omega \times \omega$ and $\omega \setminus \{0\}$ such that for every $j, n \in \omega$: $n < g(j, n)$.

The sequence is defined by induction: for the start put $s_0 := s$. Suppose $\langle s_0, \ldots, s_n\rangle$ has already been defined. By assumption there exist unique $m, j \in \omega$ with $g(j, m) = n + 1$. By the choice of $g$ it holds that $m < n + 1$. Consider $s_m$; we need to distinguish two cases.

*Case 1:* If $j$ is not smaller than the cardinality of $\{I \in \mathcal{I} \mid s_m \in I_1\}$, we just put $s_{n+1} := s_n$.

*Case 2:* For the other case, let $I$ be the $j$th item of $L_{s_m}$. From $s_m \in I_1$, $s_m \leq s_n$ and Definition 5.1, it follows that $s_n \in I_1$, hence there is a set $s' \in S$ such that $s_n \leq s'$ and $s' \in I_2$. Define $s_{n+1} := s'$.

It remains to check that the defined sequence $\langle s_n \mid n \in \omega\rangle$ is closed with respect to $(S, \leq, \mathcal{I})$. Suppose $s_m \in I_1$ for $I \in \mathcal{I}$. Then there is a $j$ smaller than the cardinality of $\{I \in \mathcal{I} \mid s_m \in I_1\}$ such that $I$ is the $j$th item of $L_{s_m}$. Again, by the definition of $g$ there is a unique $n \in \omega$ with $m < g(j, m) = n + 1$. By construction we obtain $s_{n+1} \in I_2$; this completes the proof. □

**6 *Pseudo-complete theories*** The present section is devoted to pseudo-complete theories. Pseudo-complete theories occupy a central position in the model theory of $\mathcal{L}_{\omega_1\omega}$. This is due to the fact that the internal structure of such a theory enables us to create a Henkin-style model out of it. So we may regard them as the legitimate infinitary counterparts of maximal consistent theories. Though the notion seems to be older, the wording of the definition as well as the name trace back to [4].

**Definition 6.1** Let $\tau$ be a countable relational vocabulary and $C$ a set of individual constants of cardinality $\omega$. A set $\Gamma$ of $\mathcal{L}_{\omega_1\omega}$-sentences over $\tau \cup C$ is called a pseudo-complete $(\tau, C)$-theory if and only if

- T0 each $\varphi \in \Gamma$ is negation normal,
- T1 if $\varphi \in \Gamma$, then $(\neg\varphi)^{nf} \notin \Gamma$,
- T2 for every *valid* first-order sentence $\varphi$ over $(\tau \cup C)$, $\varphi^{nf}$ is in $\Gamma$,
- T3 if $\bigwedge \Phi$ is in $\Gamma$, then every $\varphi \in \Phi$ is in $\Gamma$,
- T4 if $\forall x\varphi$ is in $\Gamma$, then for every $c \in C$ the sentence $\varphi[c/x]$ is in $\Gamma$,
- T5 for every $\bigvee \Phi$ in $\Gamma$, there is a $\varphi \in \Phi$ such that $\varphi$ is in $\Gamma$,
- T6 for every $\exists x\varphi$ in $\Gamma$, there is a $c \in C$ such that $\varphi[c/x]$ is in $\Gamma$.

Suppose $\Gamma$ is a pseudo-complete $(\tau, C)$-theory. The model $\mathfrak{A}_\Gamma$ is then defined as follows. As the carrier of $\mathfrak{A}_\Gamma$ we choose the set of constants $C$. The elements of $C$ are interpreted by themselves. For $R \in \mathcal{R}_\tau$ and $c_0, c_1 \in C$ we put $R^{\mathfrak{A}_\Gamma} c_0 c_1$ if and only if $Rc_0c_1 \in \Gamma$. The interpretation of $P \in \mathcal{P}_\tau$ is defined analogously by $P^{\mathfrak{A}_\Gamma} := \{c \in C \mid Pc \in \Gamma\}$.

**Lemma 6.2** *Let $\Gamma$ be a pseudo-complete $(\tau, C)$-theory.*

1. $\mathfrak{A}_\Gamma \models \Gamma$.
2. *For every first-order sentence $\varphi$ over $(\tau \cup C)$: $\mathfrak{A}_\Gamma \models \varphi \iff \varphi^{nf} \in \Gamma$.*

*Proof:* The first claim is proved by induction. The atomic case is an immediate consequence of the construction of $\mathfrak{A}_\Gamma$. Together with T1 this implies the claim for negated atomic formulas. As the elements of $\Gamma$ are negation normal, negations of more complex formulas need not be considered. For the remaining cases we make use of T3 to T6. For the second claim we reason as follows: by T2, T5, and T1 we either have $\varphi^{nf} \in \Gamma$ or $(\neg\varphi)^{nf} \in \Gamma$ for each first-order sentence $\varphi$. An application of the first claim leads to the desired result. □

**7 *Generalized modal formulas*** In defining the notion of an interpolation property in Section 8 we will make use of a special class of $\mathcal{L}_{\omega_1\omega}$-formulas. For reasons that will appear in Lemma 7.4 we call them *generalized modal formulas*. In the context of $\mathcal{ML}$ they trace back to van Benthem's [8].

From now on we are operating with two countable disjoint sets of individual variables, *Var* and $V$; the elements of $V$ are exclusively used as the free variables in the generalized modal formulas.

**Definition 7.1** Let $\tau$ be a relational vocabulary. The set of generalized modal formulas over $(\tau, V)$, designated by $\mathcal{ML}^G_{\omega_1}(\tau, V)$, is defined as the smallest set $X$ such that

1. for every $P \in \mathcal{P}_\tau$ and every $v \in V$, $Pv \in X$,
2. if $\varphi$ is in $X$, then $\neg\varphi$ is in $X$,
3. if $\Phi \subseteq X$ is countable, then $\bigwedge \Phi$ and $\bigvee \Phi$ are in $X$,
4. if $\varphi \in X$, $R \in \mathcal{R}_\tau$, $v, v' \in V$, $x \in Var$ and $x$ does not occur in $\varphi$, then the formulas $\exists x(Rvx \wedge \varphi[x/v'])$ and $\forall x(Rvx \rightarrow \varphi[x/v'])$ are in $X$.

In the next definition we assemble a number of closure conditions, which will be used to fix special subclasses of $\mathcal{ML}^G_{\omega_1}(\tau, V)$.

**Definition 7.2** Let $\tau$ be a relational vocabulary and $R \in \mathcal{R}_\tau$. A set $\Delta$ of generalized modal formulas over $(\tau, V)$ is said to be closed under

$AT$ if and only if for every $P \in \mathcal{P}_\tau$ and every $v \in V$, $Pv$ is in $\Delta$.

$\neg_{AT}$ if and only if for every $P \in \mathcal{P}_\tau$ and every $v \in V$,$\neg Pv$ is in $\Delta$.

$\bigwedge$ if and only if for every countable $\Phi \subseteq \Delta$, $\bigwedge \Phi$ is in $\Delta$.

$\bigvee$ if and only if for every countable $\Phi \subseteq \Delta$, $\bigvee \Phi$ is in $\Delta$.

$\exists_R$ if and only if for every $\varphi \in \Delta$, $v, v' \in V$, and every $x \in Var$ which does not occur in $\varphi$, the formula $\exists x(Rvx \wedge \varphi[x/v'])$ is in $\Delta$.

$\forall_R$ if and only if for every $\varphi \in \Delta$, $v, v' \in V$, and every $x \in Var$ which does not occur in $\varphi$, the formula $\forall x(Rvx \rightarrow \varphi[x/v'])$ is in $\Delta$.

In the following definition we introduce generalized versions of existential, universal, and positive $\mathcal{ML}_{\omega_1}$-formulas.

**Definition 7.3** Let $\tau$ be a relational vocabulary.

1. $\Sigma^G(\tau, V)$ is defined as the smallest subset of $\mathcal{ML}^G_{\omega_1}(\tau, V)$ that is closed under $AT$, $\neg_{AT}$, $\bigwedge$, $\bigvee$ and $\exists_R$, for each $R \in \mathcal{R}_\tau$.
2. $\Pi^G(\tau, V)$ is defined as the smallest subset of $\mathcal{ML}^G_{\omega_1}(\tau, V)$ that is closed under $AT$, $\neg_{AT}$, $\bigwedge$, $\bigvee$ and $\forall_R$, for each $R \in \mathcal{R}_\tau$.
3. $\Upsilon^G(\tau, V)$ is defined as the smallest subset of $\mathcal{ML}^G_{\omega_1}(\tau, V)$ that is closed under $AT$, $\bigwedge$, $\bigvee$, $\forall_R$ and $\exists_R$, for each $R \in \mathcal{R}_\tau$.

**Lemma 7.4** *Let $\tau \cup \{c\}$ be a modal vocabulary. Suppose $\varphi$ is a generalized modal formula over $(\tau, V)$ and the free variables of $\varphi$ are contained in $\{v_0\}$. Then*

1. *there is a $\varphi^* \in \mathcal{ML}_{\omega_1}(\tau \cup \{c\})$ such that $\varphi[c/v_0]$ and $St(\varphi^*)$ are equivalent; moreover,*
2. *if $\varphi \in \Sigma^G(\tau, V)$, then $\varphi^* \in \Sigma(\tau \cup \{c\})$;*
3. *if $\varphi \in \Pi^G(\tau, V)$, then $\varphi^* \in \Pi(\tau \cup \{c\})$;*
4. *if $\varphi \in \Upsilon^G(\tau, V)$, then $\varphi^* \in \Upsilon(\tau \cup \{c\})$.*

*Proof:* The first claim is proved by induction on the complexity of $\varphi$. A careful examination of this proof verifies the remaining claims as well. □

***8 Interpolation properties: the formal account*** Let $\tau$, $\tau_1$, and $\tau_2$ be countable relational vocabularies with $\tau = \tau_1 \cap \tau_2$, let $C_1$ and $C_2$ be two disjoint sets of individual constants of cardinality $\omega$, and let $\Delta$ be a subset of $\mathcal{ML}^G_{\omega_1}(\tau, V)$. $S[\Delta, \tau_1, \tau_2]$ is then defined as the set of all quadruples $(\Theta_1, \Theta_2, g_1, g_2)$ such that

P1 $\Theta_i$ is a finite subset of $\mathcal{L}_{\omega_1\omega}(\tau_i \cup C_i)$ which contains only finitely many constants from $C_i$, for $i \in \{1, 2\}$,

P2 the sentences in $\Theta_1$ and $\Theta_2$ are negation normal,

P3 $g_1$ and $g_2$ are *finite* functions with $dom(g_1) = dom(g_2) \subseteq V$, $rng(g_1) \subseteq C_1$, and $rng(g_2) \subseteq C_2$,

P4 there is no $\psi \in \Delta$ such that
the free variables of $\psi$ are contained in $dom(g_1)$,
$\Theta_1 \models \psi(g_1)$, and
$\Theta_2 \models \neg\psi(g_2)$.

**Notation 8.1** For a formula $\psi$ and a function $g$ from a set of variables into a set of constants, $\psi(g)$ denotes the sentence we obtain by replacing every $v$ free in $\psi$ by the constant $g(v)$. Whenever we use this notation it is presupposed that the domain of $g$ contains all free variables of $\psi$.

**Definition 8.2** If $s = (\Theta_1, \Theta_2, g_1, g_2) \in S[\Delta, \tau_1, \tau_2]$ and $\varphi \in \mathcal{L}_{\omega_1\omega}(\tau_1 \cup C_1)$, then we use $s +_1 \varphi$ as an abbreviation for $(\Theta_1 \cup \{\varphi\}, \Theta_2, g_1, g_2)$. For $\varphi \in \mathcal{L}_{\omega_1\omega}(\tau_2 \cup C_2)$ $s +_2 \varphi$ is defined in a similar way.

**Lemma 8.3** *Let $\Delta$ be closed under $\bigwedge, \bigvee$. Then for every $s = (\Theta_1, \Theta_2, g_1, g_2)$ in $S[\Delta, \tau_1, \tau_2]$ and every $i \in \{1, 2\}$ the following hold:*

1. *$\Theta_i$ is consistent,*
2. *if $\varphi$ is a valid first-order sentence over $\tau_i \cup C_i$, then $s +_i \varphi^{nf}$ is in $S[\Delta, \tau_1, \tau_2]$,*
3. *if $\bigwedge \Phi$ is in $\Theta_i$, then for every $\varphi \in \Phi$, $s +_i \varphi$ is in $S[\Delta, \tau_1, \tau_2]$,*
4. *if $\forall x\varphi$ is in $\Theta_i$, then for every $c \in C_i$, $s +_i \varphi[c/x]$ is in $S[\Delta, \tau_1, \tau_2]$,*
5. *if $\bigvee \Phi$ is in $\Theta_i$, then there is a $\varphi \in \Phi$ such that $s +_i \varphi$ is in $S[\Delta, \tau_1, \tau_2]$,*
6. *if $\exists x\varphi$ is in $\Theta_i$, then there is a $c \in C_i$ such that $s +_i \varphi[c/x]$ is in $S[\Delta, \tau_1, \tau_2]$.*

*Proof*:
*Case 1:* Suppose $\Theta_1$ is inconsistent, hence $\Theta_1 \models \bot$. Since $\Delta$ is closed under $\bigvee$, it holds that $\bot := \bigvee \varnothing \in \Delta$. Taken together this contradicts $s \in S[\Delta, \tau_1, \tau_2]$. For $\Theta_2$ we argue in a similar way, this time using $\top := \bigwedge \varnothing \in \Delta$.

To prove the remaining statements of the lemma it suffices to concentrate on condition P4; P1 to P3 are satisfied by definition. By using the assumption $s \in S[\Delta, \tau_1, \tau_2]$ the statements 2–4 are easily shown; the proofs are left to the reader.

*Case 5:* Let $\bigvee \Phi \in \Theta_1$. Suppose there is no $\varphi \in \Phi$ such that $s +_1 \varphi$ satisfies P4. Then for every $\varphi \in \Phi$ there is a $\psi_\varphi \in \Delta$ such that $\Theta_1 \cup \{\varphi\} \models \psi_\varphi(g_1)$ and $\Theta_2 \models \neg\psi_\varphi(g_2)$. This implies (i) $\Theta_1 \cup \{\bigvee \Phi\} \models \bigvee\{\psi_\varphi \mid \varphi \in \Phi\}(g_1)$ as well as (ii) $\Theta_2 \models \neg \bigvee\{\psi_\varphi \mid \varphi \in \Phi\}(g_2)$. Moreover, $\bigvee \Phi \in \Theta_1$ and (i) imply (iii) $\Theta_1 \models \bigvee\{\psi_\varphi \mid \varphi \in \Phi\}(g_1)$. By the closure properties of $\Delta$ it holds that $\bigvee\{\psi_\varphi \mid \varphi \in \Phi\}$ is in $\Delta$. But this, together with

(iii) and (ii), contradicts the assumption $s \in S[\Delta, \tau_1, \tau_2]$. In the case of $\bigvee \Phi \in \Theta_2$ we make use of the fact that $\Delta$ is closed under $\bigwedge$; the rest is analogous.

*Case 6:* Let $\exists x\varphi \in \Theta_1$. By definition, $\Theta_1$ only contains *finitely* many constants from $C_1$, hence there is a $c \in C_1$ that does not occur in $\Theta_1$. For such a $c$ it is easy to verify that $s +_1 \varphi[c/x]$ is in $S[\Delta, \tau_1, \tau_2]$. Assume to the contrary. Then there is a $\psi \in \Delta$ such that (i) $\Theta_1 \cup \{\varphi[c/x]\} \models \psi(g_1)$, and (ii) $\Theta_2 \models \neg\psi(g_2)$. As $c$ does not occur in $\Theta_1$, (i) implies $\Theta_1 \cup \{\exists x\varphi\} \models \psi(g_1)$. To complete the proof we argue as in case 5. The case $\exists x\varphi \in \Theta_2$ is proved in a similar way. □

We are now ready to introduce the main notion of this paper.

**Definition 8.4** (Interpolation property) An inductive property $(S, \leq, I)$ is called an interpolation property if and only if there are $\tau, \tau_1, \tau_2, C_1, C_2$, and $\Delta$ such that $S[\Delta, \tau_1, \tau_2] = S$, and $\leq$ is defined as follows: $(\Theta_1, \Theta_2, g_1, g_2) \leq (\Theta_1', \Theta_2', g_1', g_2')$ if and only if $\Theta_i \subseteq \Theta_i'$ and $g_i = g_i' \restriction dom(g_i)$ for $i \in \{1, 2\}$.

There are only three concrete $\Delta$'s in this paper: the collection of all generalized modal formulas (in Theorems 9.1 and 10.1) and its two subcollections $\Sigma^G$ and $\Upsilon^G$ (in the first respectively third part of Theorem 10.2).

In the following list we assemble a number of sets of conditions. In the second part of this section we use these sets in order to define certain interpolation properties. Let $S[\Delta, \tau_1, \tau_2]$ and $i \in \{1, 2\}$ be fixed.

$I^{\top,i}$ The set of all pairs $I = (I_1, I_2)$ such that there is a valid first-order sentence $\varphi$ over $\tau_i \cup C_i$ in negation normal form with:
$I_1 := S[\Delta, \tau_1, \tau_2]$,
$I_2 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid \varphi \in \Theta_i\}$.

$I^{\wedge,i}$ The set of all pairs $I = (I_1, I_2)$ such that there is a $\mathcal{L}_{\omega_1\omega}$-sentence $\bigwedge \Phi$ over $\tau_i \cup C_i$ and $\varphi \in \Phi$ with:
$I_1 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid \bigwedge \Phi \in \Theta_i\}$,
$I_2 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid \varphi \in \Theta_i\}$.

$I^{\forall,i}$ The set of all pairs $I = (I_1, I_2)$ such that there is a $\mathcal{L}_{\omega_1\omega}$-sentence $\forall x\varphi$ and $c \in C_i$ with:
$I_1 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid \forall x\varphi \in \Theta_i\}$,
$I_2 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid \varphi[c/x] \in \Theta_i\}$.

$I^{\vee,i}$ The set of all pairs $I = (I_1, I_2)$ such that there is a $\mathcal{L}_{\omega_1\omega}$-sentence $\bigvee \Phi$ with:
$I_1 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid \bigvee \Phi \in \Theta_i\}$,
$I_2 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid \exists\varphi \in \Phi : \varphi \in \Theta_i\}$.

$I^{\exists,i}$ The set of all pairs $I = (I_1, I_2)$ such that there is a $\mathcal{L}_{\omega_1\omega}$-sentence $\exists x\varphi$ with:
$I_1 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid \exists x\varphi \in \Theta_i\}$,
$I_2 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid \exists c \in C_i : \varphi[c/x] \in \Theta_i\}$.

$I^{R,1}$ The set of all pairs $I = (I_1, I_2)$ such that there are $R \in \mathcal{R}_\tau$, $c, c' \in C_1$ and $d \in C_2$ with:
$I_1 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid Rcc' \in \Theta_1 \,\&\, \exists v \in dom(g_1)(g_1(v) = c \,\&\, g_2(v) = d)\}$,
$I_2 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid \exists d' \in C_2(Rdd' \in \Theta_2 \,\&$
$\exists v \in dom(g_1) : g_1(v) = c' \,\&\, g_2(v) = d')\}$.

$I^{R,2}$ The set of all pairs $I = (I_1, I_2)$ such that there are $R \in \mathcal{R}_\tau$, $d, d' \in C_2$ and $c \in C_1$ with:
$I_1 := \{(\Theta_1, \Theta_2, g_1, g_2) | Rdd' \in \Theta_2 \,\&\, \exists v \in dom(g_1)(g_1(v) = c \,\&\, g_2(v) = d)\}$,
$I_2 := \{(\Theta_1, \Theta_2, g_1, g_2) \mid \exists c' \in C_1(Rcc' \in \Theta_1 \,\&$
$\exists v \in dom(g_1) : g_1(v) = c' \,\&\, g_2(v) = d')\}$.

**Definition 8.5** Let $(S[\Delta, \tau_1, \tau_2], \leq, I)$ be an interpolation property, and suppose $\langle(\Theta_1^n, \Theta_2^n, g_1^n, g_2^n) \mid n \in \omega\rangle$ is closed with respect to $(S[\Delta, \tau_1, \tau_2], \leq, I)$. The limit $l$ of this sequence is the quadruple $(\Gamma_1, \Gamma_2, h_1, h_2)$ defined by

$\Gamma_1 := \bigcup_{n\in\omega} \Theta_1^n$ and $\Gamma_2 := \bigcup_{n\in\omega} \Theta_2^n$,
$h_1 := \bigcup_{n\in\omega} g_1^n$ and $h_2 := \bigcup_{n\in\omega} g_2^n$.

**Theorem 8.6**

1. *Let $I := \bigcup\{I^{*,i} \mid * \in \{\top, \wedge, \forall, \vee, \exists\} \,\&\, i \in \{1, 2\}\}$, and let $\Delta$ be closed under $\bigwedge$ and $\bigvee$, then $(S[\Delta, \tau_1, \tau_2], \leq, I)$ is an interpolation property.*
2. *Let $I := I^{R,1} \cup \bigcup\{I^{*,i} \mid * \in \{\top, \wedge, \forall, \vee, \exists\} \,\&\, i \in \{1, 2\}\}$, and let $\Delta$ be closed under $\bigwedge$, $\bigvee$ and $\exists_R$ (for every $R \in \mathcal{R}_\tau$), then $(S[\Delta, \tau_1, \tau_2], \leq, I)$ is an interpolation property.*
3. *Let $I := I^{R,2} \cup \bigcup\{I^{*,i} \mid * \in \{\top, \wedge, \forall, \vee, \exists\} \,\&\, i \in \{1, 2\}\}$, and let $\Delta$ be closed under $\bigwedge$, $\bigvee$ and $\forall_R$ (for every $R \in \mathcal{R}_\tau$), then $(S[\Delta, \tau_1, \tau_2], \leq, I)$ is an interpolation property.*
4. *Let $I := \bigcup\{I^{*,i} \mid * \in \{R, \top, \wedge, \forall, \vee, \exists\} \,\&\, i \in \{1, 2\}\}$, and let $\Delta$ be closed under $\bigwedge$, $\bigvee$, $\exists_R$ and $\forall_R$ (for every $R \in \mathcal{R}_\tau$), then $(S[\Delta, \tau_1, \tau_2], \leq, I)$ is an interpolation property.*

*Proof:* We only prove the fourth claim; it should then be clear how the remaining cases go. So let $\Delta$ and $I$ be as in 4. Obviously, $(S[\Delta, \tau_1, \tau_2], \leq)$ is a partial order and $I$ forms a set of conditions on this partial order. Suppose $s = (\Theta_1, \Theta_2, g_1, g_2)$ is an element of $S[\Delta, \tau_1, \tau_2]$. As the vocabularies are countable and the $\Theta_i$'s are finite, there are at most countably many $I \in I$ such that $s \in I_1$. What remains to be shown is that $S[\Delta, \tau_1, \tau_2]$ is closed under every $I \in I$. Suppose $I \in I$. We must distinguish several cases.

*Case 1:* In case $I$ is taken from one of the sets $I^{*,i}$, with $* \in \{\top, \wedge, \forall, \vee, \exists\}$ and $i \in \{1, 2\}$, we apply the relevant part of Lemma 8.3. For an example consider $I \in I^{\vee,1}$. By definition there is an $\mathcal{L}_{\omega_1\omega}$-sentence $\bigvee \Phi$ over the vocabulary $\tau_1 \cup C_1$ such that $I_1 = \{(\Theta_1, \Theta_2, g_1, g_2) \mid \bigvee \Phi \in \Theta_1\}$ and $I_2 = \{(\Theta_1, \Theta_2, g_1, g_2) \mid \exists\varphi \in \Phi : \varphi \in \Theta_1\}$. Now, assume $s = (\Theta_1, \Theta_2, g_1, g_2)$ is in $I_1$, hence $\bigvee \Phi \in \Theta_1$. By an application of 5 in Lemma 8.3 there is a $\varphi \in \Phi$ such that $s +_1 \varphi$ is in $S[\Delta, \tau_1, \tau_2]$; by the definition of $s +_1 \varphi$ we obtain $s \leq s +_1 \varphi$ as well as $s +_1 \varphi \in I_2$, which completes the proof.

The most interesting cases are $I \in I^{R,1}$ and $I \in I^{R,2}$.

*Case 2:* At first assume $I \in I^{R,1}$. By definition there are $R \in \mathcal{R}_\tau$, $c, c' \in C_1$ and $d \in C_2$ such that

$$I_1 = \{(\Theta_1, \Theta_2, g_1, g_2) \mid Rcc' \in \Theta_1 \,\&\, \exists v \in dom(g_1)(g_1(v) = c \,\&\, g_2(v) = d)\},$$

and

$$I_2 = \{(\Theta_1, \Theta_2, g_1, g_2) \mid \exists d' \in C_2(Rdd' \in \Theta_2 \,\&$$

$$\exists v \in dom(g_1)(g_1(v) = c' \,\&\, g_2(v) = d'))\}.$$

Suppose $s = (\Theta_1, \Theta_2, g_1, g_2) \in I_1$. Choose $d'$ to be a constant from $C_2$ that does not occur in $\Theta_2$, and choose $v'$ to be a variable from $V$ such that $v' \notin dom(g_1)$. It suffices to show that

$$s' := (\Theta_1, \Theta_2 \cup \{Rdd'\}, g_1 \cup \{\langle v', c'\rangle\}, g_2 \cup \{\langle v', d'\rangle\})$$

is an element of $S[\Delta, \tau_1, \tau_2]$: since $s \leq s'$ and $s' \in I_2$ this would complete the proof. Assume to the contrary that $s'$ is not in $S[\Delta, \tau_1, \tau_2]$. By definition— P1 to P3 are satisfied by $s'$—there is a $\psi \in \Delta$ such that

(i) the free variables of $\psi$ are contained in $dom(g_1) \cup \{v'\}$,
(ii) $\Theta_1 \models \psi(g_1 \cup \{\langle v', c'\rangle\})$, and
(iii) $\Theta_2 \cup \{Rdd'\} \models \neg\psi(g_2 \cup \{\langle v', d'\rangle\})$.

Let $x \in Var$ be new, and $v$ be that variable from $V$ which satisfies $g_1(v) = c$ and $g_2(v) = d$. Then (i), (ii) and "$Rcc' \in \Theta_1$" imply

(iv) the free variables of $\exists x(Rvx \wedge \psi[x/v'])$ are contained in $dom(g_1)$, and
(v) $\Theta_1 \models \exists x(Rvx \wedge \psi[x/v'])(g_1)$.

Furthermore, as $d'$ does not occur in $\Theta_2$, (iii) yields

(vi) $\Theta_2 \models \neg\exists x(Rvx \wedge \psi[x/v'])(g_2)$

by easy logical reasoning. From the fact that $\Delta$ is closed under $\exists_R$ we also get $\exists x(Rvx \wedge \psi[x/v']) \in \Delta$. This, together with (vi) and (iv), leads to $s \notin S[\Delta, \tau_1, \tau_2]$, in contradiction to our assumption. Therefore it is shown that $s' \in S[\Delta, \tau_1, \tau_2]$. This completes the case $I \in I^{R,1}$.

*Case 3:* For $I \in I^{R,2}$ we reason in a similar way, taking advantage of the fact that $\Delta$ is closed under $\forall_R$. □

**Lemma 8.7** *Let $(S[\Delta, \tau_1, \tau_2], \leq, I)$ be one of the interpolation properties from Theorem 8.6, and suppose $l = (\Gamma_1, \Gamma_2, h_1, h_2)$ is the limit of a sequence that is closed with respect to $(S[\Delta, \tau_1, \tau_2], \leq, I)$. Then the following statements hold:*

1. *$\Gamma_i$ is a pseudo-complete $(\tau_i \cup C_i)$-theory, for $i \in \{1, 2\}$.*
2. *$dom(h_1) = dom(h_2) \subseteq V$.*
3. *$h_i[dom(h_i)] \subseteq C_i$, for $i \in \{1, 2\}$.*

*Proof:* For the first claim we only make a few simple observations. First, $\Gamma_i$ is the union of sets of sentences over the vocabulary $\tau_i \cup C_i$; hence $\Gamma_i$ belongs to the right language. Second, all $\Delta$'s under consideration are closed under $\bigwedge$ and $\bigvee$; hence the prerequisites of Lemma 8.3 are satisfied. Third, all the relevant $I$'s are subsets of $\bigcup\{I^{*,i} \mid * \in \{\top, \wedge, \forall, \vee, \exists\} \,\&\, i \in \{1,2\}\}$. Exploiting the fact that $l$ is the limit of a sequence which is closed with respect to the corresponding $(S[\Delta, \tau_1, \tau_2], \leq, I)$, the first claim follows by an application of Lemma 8.3. The details may easily be filled in by the reader. The two remaining claims are obvious. □

Now, this seems to be a good place for a short break. Let's reconsider what we have done so far in this section, and compare it with our informal sketch from section 4. First, in Theorem 8.6 we saw that all the objects $(S[\Delta, \tau_1, \tau_2], \leq, I)$ under consideration are interpolation properties. By Lemma 5.3 we obtain for each $s = (\Theta_1, \Theta_2, g_1, g_2) \in S[\Delta, \tau_1, \tau_2]$ a sequence that is closed with respect to the corresponding property, and with $s$ as its first item. Furthermore, Lemma 8.7 tells us that the limit of such a sequence supplies two pseudo-complete theories, $\Gamma_1$ and $\Gamma_2$, where the first belongs to the language $\tau_1 \cup C_1$ and the second to the language $\tau_2 \cup C_2$. Moreover, in section 6 it was shown that a pseudo-complete theory uniquely determines a model which satisfies the theory; therefore $l$ fixes—via $\Gamma_1$ and $\Gamma_2$—two models $\mathfrak{A}_{\Gamma_1}$ and $\mathfrak{A}_{\Gamma_2}$. Finally, it is an immediate consequence of the definition of $l$ that $\mathfrak{A}_{\Gamma_1}$ satisfies $\Theta_1$ and $\mathfrak{A}_{\Gamma_2}$ satisfies $\Theta_2$. What remains to be taken into account in order to complete our formal development is the structural relation that should hold between the two constructed models. This will be done in the remainder of this section.

**Definition 8.8** Let $(S[\Delta, \tau_1, \tau_2], \leq, I)$ be one of the interpolation properties from Theorem 8.6, and let $l = (\Gamma_1, \Gamma_2, h_1, h_2)$ be the limit of a sequence which is closed with respect to that property. The relation $Z_l$ between $A_{\Gamma_1}$ and $A_{\Gamma_2}$ is then defined as follows:

$$\forall c \in C_1 \forall d \in C_2 (Z_l cd :\Longleftrightarrow \exists v \in dom(h_1)(h_1(v) = c \,\&\, h_2(v) = d)).$$

**Lemma 8.9** *Let $(S[\Delta, \tau_1, \tau_2], \leq, I)$ and $l$ be as in the preceding definition.*

1. *Suppose $\Delta$ is closed under $AT$. Then for every $c \in A_{\Gamma_1}$ and $d \in A_{\Gamma_2}$, if $Z_l cd$ then $\forall P \in \mathcal{P}_\tau (P^{\mathfrak{A}_{\Gamma_1}} c \Longrightarrow P^{\mathfrak{A}_{\Gamma_2}} d)$.*
2. *Suppose $\Delta$ is closed under $\neg_{AT}$. Then for every $c \in A_{\Gamma_1}$ and $d \in A_{\Gamma_2}$, if $Z_l cd$ then $\forall P \in \mathcal{P}_\tau (P^{\mathfrak{A}_{\Gamma_2}} d \Longrightarrow P^{\mathfrak{A}_{\Gamma_1}} c)$.*

*Proof:* For the first claim assume that $\Delta$ contains all atomic formulas of the form $Pv$, with $P \in \mathcal{P}$ and $v \in V$. Let $P \in \mathcal{P}$, $c \in A_{\Gamma_1}$ and $d \in A_{\Gamma_2}$ such that $Z_l cd$ and $P^{\mathfrak{A}_{\Gamma_1}} c$. By an application of the second statement of Lemma 6.2, we obtain $Pc \in \Gamma_1$. Therefore, there is a natural number $n$ with $Pc \in \Theta_1^n$, where $s_n = (\Theta_1^n, \Theta_2^n, g_1^n, g_2^n)$ is the $n$-th item of the sequence that has $l$ as its limit. Moreover, $Z_l cd$ implies that there are $s_m = (\Theta_1^m, \Theta_2^m, g_1^m, g_2^m)$ and $v \in dom(g_1^m)$ with $g_1^m(v) = c$ and $g_2^m(v) = d$. Let $k$ be the supremum of $\{m, n\}$. For $s_k$ it holds that $Pc \in \Theta_1^k$, $v \in dom(g_1^k)$, $g_1^k(v) = c$ and $g_2^k(v) = d$. From $Pv \in \Delta$ we infer $\neg Pd \notin \Theta_2^{n'}$, for each $n' \geq k$, hence $\neg Pd \notin \Gamma_2$. On the other hand, by an application of the second claim in Lemma 8.3 we obtain $(Pd \vee \neg Pd) \in \Gamma_2$. Taken together this yields $Pd \in \Gamma_2$. Finally, by the first part of Lemma 6.2 we obtain $P^{\mathfrak{A}_{\Gamma_2}} d$, which completes the proof of the first claim.

The second claim is proved in a similar way. □

**Lemma 8.10** *Let $(S[\Delta, \tau_1, \tau_2], \leq, I)$ and $l = (\Gamma_1, \Gamma_2, h_1, h_2)$ be as in the preceding lemma.*

1. *Suppose $I^{R,1} \subseteq I$ and $\Delta$ is closed under $\bigwedge$, $\bigvee$ and $\exists_R$ (for every $R \in \mathcal{R}_\tau$), then $Z_l$ satisfies the forth-condition (clause B2a) in Definition 2.6.*
2. *Suppose $I^{R,2} \subseteq I$ and $\Delta$ is closed under $\bigwedge$, $\bigvee$ and $\forall_R$ (for every $R \in \mathcal{R}_\tau$), then $Z_l$ satisfies the back-condition (clause B2b) in Definition 2.6.*

*Proof:*

*Case 1:* Assume $Z_l cd$ and $R^{\mathfrak{A}_{\Gamma_1}} cc'$, for $c, c' \in C_1$, $d \in C_2$ and $R \in \mathcal{R}_\tau$. By Lemma 6.2 we obtain $Rcc' \in \Gamma_1$. Hence, there is an item $s_n = (\Theta_1^n, \Theta_2^n, g_1^n, g_2^n)$ in the sequence of which $l$ is the limit such that $Rcc' \in \Theta_1^n$. Moreover, $Z_l cd$ yields the existence of a $s_m = (\Theta_1^m, \Theta_2^m, g_1^m, g_2^m)$ and a variable $v \in dom(g_1^m)$ with $g_1^m(v) = c$ and $g_2^m(v) = d$. Again, let $k$ be the supremum of $\{m, n\}$. For $s_k$ we easily obtain $Rcc' \in \Theta_1^k$, $v \in dom(g_1^k)$, $g_1^k(v) = c$ and $g_2^k(v) = d$. Now, let $I$ be the element of $I^{R,1}$ such that $I_1 = \{(\Theta_1, \Theta_2, g_1, g_2) \mid Rcc' \in \Theta_1 \,\&\, \exists v \in dom(g_1) : g_1(v) = c \,\&\, g_2(v) = d\}$: Obviously, $s_k \in I_1$. As $l$ is the limit of a sequence that is closed with respect to $(S[\Delta, \tau_1, \tau_2], \leq, I)$, and $I$ contains $I$ by assumption, there is a $k' \geq k$ such that $s_{k'} = (\Theta_1^{k'}, \Theta_2^{k'}, g_1^{k'}, g_2^{k'}) \in I_2$. Hence $Rdd' \in \Theta_2^{k'}$ and there is a $v' \in dom(g_1^{k'})$ with $g_1^{k'}(v) = c'$ and $g_2^{k'}(v') = d'$. From this we obtain $Rdd' \in \Gamma_2$, $h_1(v') = c'$ as well as $h_2(v') = d'$. To complete the proof we apply Lemma 6.2 and make use of the definition of $Z_l$. This leads to $R^{\mathfrak{A}_{\Gamma_2}} dd'$ and $Z_l c'd'$.

The second case is shown by a similar argument. □

**9 *Interpolation*** For a modal vocabulary $\tau' \cup \{c\}$ and $\varphi \in \mathcal{ML}_{\omega_1}(\tau' \cup \{c\})$, $Voc(\varphi)$ designates the smallest relational vocabulary $\tau \subseteq \tau'$ such that $\varphi \in \mathcal{ML}_{\omega_1}(\tau \cup \{c\})$.

**Theorem 9.1** (Craig Interpolation) *Let $\tau' \cup \{c\}$ be a modal vocabulary and let $\varphi, \chi \in \mathcal{ML}_{\omega_1}(\tau' \cup \{c\})$ with $\varphi \models \chi$. Then there exists a formula $\vartheta \in \mathcal{ML}_{\omega_1}(\tau' \cup \{c\})$ such that*

$\varphi \models \vartheta$,
$\vartheta \models \chi$ *and*
$Voc(\vartheta) = Voc(\varphi) \cap Voc(\chi)$.

*Proof:* The theorem is proved by contraposition. Assume $\varphi$ and $\chi$ are two $\mathcal{ML}_{\omega_1}$-formulas which have no interpolant, that is, there is no $\mathcal{ML}_{\omega_1}$-formula $\vartheta$ which satisfies the conditions stated in the theorem. Under this assumption we are able to find a model $(\mathfrak{C}, e)$ in which $\varphi$ is true, but $\chi$ is false. Therefore $\varphi \models \chi$ does not hold.

To begin with, choose two disjoint sets of constants $C_1 = \{c_n \mid n \in \omega\}$ and $C_2 = \{d_n \mid n \in \omega\}$, and a set $V = \{v_n \mid n \in \omega\}$ of new variables. Furthermore, put $\tau_1 := Voc(\varphi)$, $\tau_2 := Voc(\chi)$, $\tau := \tau_1 \cap \tau_2$ and $\Delta := \mathcal{ML}^G_{\omega_1}(\tau, V)$, and define a quadruple $s$ by

$$s := (\{(St(\varphi)[c_0/c])^{nf}\}, \{(St(\neg\chi)[d_0/c])^{nf}\}, \{\langle v_0, c_0\rangle\}, \{\langle v_0, d_0\rangle\}).$$

It is easy to see that $s$ is an element of $(S[\Delta, \tau_1, \tau_2], \leq)$. P1 to P3 are obvious. For P4 assume, aiming for a contradiction, that there is a $\psi \in \Delta$ such that

1. the free variables of $\psi$ are contained in $\{v_0\}$,
2. $(St(\varphi)[c_0/c])^{nf} \models \psi(\langle v_0, c_0\rangle)$,
3. $(St(\neg\chi)[d_0/c])^{nf} \models \neg\psi(\langle v_0, d_0\rangle)$.

By the first claim in Lemma 7.4 there is a $\mathcal{ML}_{\omega_1}$-formula $\vartheta$ such that $\psi[c/v_0]$ and $St(\vartheta)$ are equivalent. Together with 2 and 3, this leads to $\varphi \models \vartheta$ and $\vartheta \models \chi$, in contradiction to the assumption that $\varphi$ and $\chi$ possess no interpolant. Thus it is shown that $s \in (S[\Delta, \tau_1, \tau_2], \leq)$.

For the next step define $I$ as the union of all sets $I^{*,i}$, with $* \in \{R, \top, \wedge, \forall, \vee, \exists\}$ and $i \in \{1, 2\}$. According to Theorem 8.6, $(S[\Delta, \tau_1, \tau_2], \leq, I)$ is an interpolation property; note that by assumption $\Delta$ satisfies the required closure conditions. Thus, by Lemma 5.3 there is a sequence $\langle s_n \mid n \in \omega\rangle$ that is closed with respect to $(S[\Delta, \tau_1, \tau_2], \leq, I)$ such that $s_0 = s$. Let $l = (\Gamma_1, \Gamma_2, h_1, h_2)$ be the limit of this sequence.

By Lemma 8.7, $\Gamma_i$ is a pseudo-complete $(\tau_i \cup C_i)$-theory, for $i \in \{1, 2\}$. Then Lemma 6.2 supplies models $\mathfrak{A}_{\Gamma_1}$ and $\mathfrak{A}_{\Gamma_2}$ such that

(i) $\mathfrak{A}_{\Gamma_1} \models \Gamma_1$,
(ii) $\mathfrak{A}_{\Gamma_1}$ is a $(\tau_1 \cup C_1)$-model,
(iii) $\mathfrak{A}_{\Gamma_2} \models \Gamma_2$, and
(iv) $\mathfrak{A}_{\Gamma_2}$ is a $(\tau_2 \cup C_2)$-model.

From (i), (iii) together with $(St(\varphi)[c_0/c])^{nf} \in \Gamma_1$ and $(St(\neg\chi)[d_0/c])^{nf} \in \Gamma_2$ we infer by an easy argument

(v) $(\mathfrak{A}_{\Gamma_1} \restriction \tau_1, c_0) \models \varphi$ and
(vi) $(\mathfrak{A}_{\Gamma_2} \restriction \tau_2, d_0) \models \neg\chi$.

Note that in Henkin-style models constants are interpreted by themselves.

Next, consider the relation $Z_l$ defined in Definition 8.8. Under the present conditions we can show that $Z_l$ forms a $\tau \cup \{c\}$-bisimulation between the models $(\mathfrak{A}_{\Gamma_1} \restriction \tau_1, c_0)$ and $(\mathfrak{A}_{\Gamma_2} \restriction \tau_2, d_0)$. As $\Delta (= \mathcal{ML}^G_{\omega_1}(\tau, V))$ contains both the atomic $\tau$-formulas and their negations, Lemma 8.9 ensures that $Z_l$ satisfies clause B1 in Definition 2.6. To verify B2a note that $I^{R,1} \subseteq I$ and that $\Delta$ is closed under $\exists_R$, for every $R \in \mathcal{R}_\tau$. An application of the first statement in Lemma 8.10 completes the case. B2b is proved by an analogous argument; this time we make use of the fact that $\Delta$ is closed under $\forall_R$, and apply the second claim of Lemma 8.10. Moreover, by $v_0 \in dom(g_1)$, $g_1(v_0) = c_0$ and $g_2(v_0) = d_0$ we obtain $Z_l c_0 d_0$. Putting everything together we obtain

(vii) $Z_l : (\mathfrak{A}_{\Gamma_1} \restriction \tau_1, c_0) \sim^{\tau}_{bs} (\mathfrak{A}_{\Gamma_2} \restriction \tau_2, d_0)$.

Let $(\mathfrak{B}_1, c_0)$ be the unraveling of $(\mathfrak{A}_{\Gamma_1} \restriction \tau_1, c_0)$ and $(\mathfrak{B}_2, d_0)$ the unraveling of $(\mathfrak{A}_{\Gamma_2} \restriction \tau_2, d_0)$. Obviously

(viii) $(\mathfrak{A}_{\Gamma_1} \restriction \tau_1, c_0) \sim^{\tau_1}_{bs} (\mathfrak{B}_1, c_0)$,
(ix) $(\mathfrak{A}_{\Gamma_2} \restriction \tau_2, d_0) \sim^{\tau_2}_{bs} (\mathfrak{B}_2, d_0)$,

and, by an application of (vii),

(x) $(\mathfrak{B}_1, c_0) \sim^{\tau}_{bs} (\mathfrak{B}_2, d_0)$.

So the prerequisites of the Amalgamation Lemma 3.4 are satisfied. An application of this lemma provides a $(\tau_1 \cup \tau_2 \cup \{c\})$-model $(\mathfrak{E}, e)$ with the following features:

(xi) $(\mathfrak{E}, e) \sim^{\tau_1}_{bs} (\mathfrak{B}_1, c_0)$,
(xii) $(\mathfrak{E}, e) \sim^{\tau_2}_{bs} (\mathfrak{B}_2, d_0)$.

As $\varphi \in \mathcal{ML}_{\omega_1}(\tau_1 \cup \{c\})$, $\varphi$ is preserved under $\tau_1 \cup \{c\}$-bisimulations. Therefore (v), (viii) and (xi) imply $(\mathfrak{E}, e) \models \varphi$. Similarly, by (vi), (ix), (xii) and $\neg\chi \in \mathcal{ML}_{\omega_1}(\tau_2 \cup \{c\})$ we obtain $(\mathfrak{E}, e) \models \neg\chi$. Thus it is shown that $\varphi \wedge \neg\chi$ has a model, hence $\varphi \models \chi$ does not hold. This completes the proof of the interpolation theorem. □

As an immediate consequence of the interpolation theorem we obtain a modal version of Beth's definability theorem. To state this result in a precise way, we first have to say what we mean by "explicitly definable" and by "implicitly definable" in the context of (infinitary) modal logic. This is done in the next definition.

**Definition 9.2** Let $\tau$ be a modal vocabulary and let $\varphi \in \mathcal{ML}_{\omega_1}(\tau)$.

1. $\varphi$ *implicitly* defines a propositional letter $p$ if and only if there is a propositional letter $q$, different from $p$, which does not occur in $\varphi$, such that $\varphi \wedge \varphi[q/p] \models (p \longleftrightarrow q)$.
2. $\varphi$ *explicitly* defines a propositional letter $p$ if and only if there is an $\mathcal{ML}_{\omega_1}$-formula $\psi$ in which $p$ does not occur, such that $\varphi \models (p \longleftrightarrow \psi)$.

**Theorem 9.3** (Beth) *Let $\tau$ be a modal vocabulary and let $\varphi \in \mathcal{ML}_{\omega_1}(\tau)$. $\varphi$ implicitly defines a propositional letter $p$ if and only if $\varphi$ defines $p$ explicitly.*

*Proof:* The proof is standard and can be skipped here. The direction from right to left is fairly obvious. For the other direction consider the interpolant of the formulas $\varphi \wedge p$ and $\varphi[q/p] \rightarrow q$; the result follows by some easy logical manipulations. □

***10 Preservation*** In section 2 we saw (Lemma 2.7) that $\mathcal{ML}_{\omega_1}$-formulas are invariant for bisimulations. This observation suggests the following natural question: does van Benthem's bisimulation theorem also apply to $\mathcal{ML}_{\omega_1}$, that is, can we prove that an $\mathcal{L}_{\omega_1\omega}$-sentence is equivalent to the standard translation of an $\mathcal{ML}_{\omega_1}$-formula if and only if it is invariant for bisimulations? In [12] van Benthem and Bergstra gave a positive answer to this question. At the beginning of this last section we prove this result by an application of our own method.

By a careful examination of the two proofs the reader will probably come to the conclusion that the proof in [12] and our proof do not differ too much from each other. This is no surprise. For the construction van Benthem and Bergstra use in their proof may be described as a special case of our construction.

**Theorem 10.1** *Let $\tau \cup \{c\}$ be a modal vocabulary. For $\varphi \in \mathcal{L}_{\omega_1\omega}(\tau \cup \{c\})$ the following are equivalent:*

1. *There is a $\vartheta \in \mathcal{ML}_{\omega_1}(\tau \cup \{c\})$ such that $\varphi$ and $St(\vartheta)$ are equivalent.*
2. *$\varphi$ is invariant for $\tau \cup \{c\}$-bisimulations.*

*Proof:* The direction from 1 to 2 is an immediate consequence of Lemma 2.7 and Lemma 2.5. The other direction is proved by contraposition. Let $\varphi \in \mathcal{L}_{\omega_1\omega}(\tau \cup \{c\})$ and suppose there is no modal formula $\vartheta$ such that $\varphi$ and $St(\vartheta)$ are equivalent. We will construct two bisimilar models $(\mathfrak{B}_1, b_1)$ and $(\mathfrak{B}_2, b_2)$ for which $(\mathfrak{B}_1, b_1) \models \varphi$ and $(\mathfrak{B}_1, b_1) \models \neg\varphi$ hold; from this we can infer that $\varphi$ is not invariant under bisimulations, which concludes the proof.

Because a large part of the proof strongly resembles the proof of the interpolation theorem, we will be content here with a sketch. Choose $C_1$, $C_2$, $V$, $\Delta$ and $I$ as in the proof of Theorem 9.1, and let $\tau_1 := \tau$ and $\tau_2 := \tau$. Under the assumption that $\varphi$ has no modal equivalent it is easy to verify that

$$s := (\{(\varphi[c_0/c])^{nf}\}, \{(\neg\varphi[d_0/c])^{nf}\}, \{\langle v_0, c_0\rangle\}, \{\langle v_0, d_0\rangle\})$$

is an element of $(S[\Delta, \tau_1, \tau_2], \leq)$. Once again, by an application of Theorem 8.6 $(S[\Delta, \tau_1, \tau_2], \leq, I)$ is shown to be an interpolation property. Hence there is a suitable sequence $\langle s_n \mid n \in \omega\rangle$ with $s_0 = s$. Let $l$ be the limit of this sequence. Then its two sets $\Gamma_1$ and $\Gamma_2$ are pseudo-complete theories, this time over the vocabulary $\tau \cup C_1$ respectively $\tau \cup C_2$. For the corresponding models $\mathfrak{A}_{\Gamma_1}$ and $\mathfrak{A}_{\Gamma_2}$ it holds that $\mathfrak{A}_{\Gamma_1} \models \Gamma_1$ and $\mathfrak{A}_{\Gamma_2} \models \Gamma_2$. As $\varphi$ is contained in $\mathcal{L}_{\omega_1\omega}(\tau \cup \{c\})$ this implies $(\mathfrak{A}_{\Gamma_1} \restriction \tau_1, c_0) \models \varphi$ and $(\mathfrak{A}_{\Gamma_2} \restriction \tau_2, d_0) \models \neg\varphi$. Moreover, by an application of Lemmas 8.9 and 8.10—note that $\Delta$ contains all atomic formulas as well as their negations, and is closed under $\bigwedge$, $\bigvee$, $\exists_R$ and $\forall_R$—we obtain $Z_l : (\mathfrak{A}_{\Gamma_1} \restriction \tau_1, c_0) \sim_{bs}^{\tau\cup\{c\}} (\mathfrak{A}_{\Gamma_2} \restriction \tau_2, d_0)$. To complete the proof, define $(\mathfrak{B}_1, b_1) := (\mathfrak{A}_{\Gamma_1} \restriction \tau_1, c_0)$ and $(\mathfrak{B}_2, b_2) := (\mathfrak{A}_{\Gamma_2} \restriction \tau_2, d_0)$. □

The final result of this section, and of the whole paper, characterizes positive, universal and existential $\mathcal{ML}_{\omega_1}$-formulas by their preservation properties.

**Theorem 10.2** *Let $\tau \cup \{c\}$ be a modal vocabulary. For $\varphi \in \mathcal{ML}_{\omega_1}(\tau \cup \{c\})$ the following three equivalences hold:*

1. *$\varphi$ is preserved under extensions if and only if there is a $\psi \in \Sigma(\tau \cup \{c\})$ such that $\models \varphi \longleftrightarrow \psi$.*
2. *$\varphi$ is preserved under submodels if and only if there is a $\psi \in \Pi(\tau \cup \{c\})$ such that $\models \varphi \longleftrightarrow \psi$.*
3. *$\varphi$ is preserved under weak extensions if and only if there is a $\psi \in \Upsilon(\tau \cup \{c\})$ such that $\models \varphi \longleftrightarrow \psi$.*

*Proof:*

*Case 1:* A routine induction shows that existential formulas are preserved under extensions. For the other direction suppose $\varphi$ has no equivalent formula in $\Sigma(\tau \cup \{c\})$. The structure of the proof is again very similar to the proof of the interpolation theorem. In the present situation we will construct two models $(\mathfrak{B}_1, b_1)$, $(\mathfrak{B}_2, b_2)$ such that $(\mathfrak{B}_1, b_1) \models \varphi$, $(\mathfrak{B}_2, b_2) \models \neg\varphi$ and $(\mathfrak{B}_1, b_1) \subseteq (\mathfrak{B}_2, b_2)$, and from this we will conclude that $\varphi$ is not preserved under extensions.

$C_1$, $C_2$, $V$, $\tau_1$ and $\tau_2$ are chosen as in the proof of Theorem 10.1. For $\Delta$ we take $\Sigma^G(\tau, V)$ and for $I$ the union of $I^{R,i}$ and all the sets $I^{*,i}$, with $* \in \{\top, \wedge, \forall, \vee, \exists\}$ and $i \in \{1, 2\}$. By utilizing the assumption on $\varphi$ it is not hard to prove that

$$s := (\{(St(\varphi)[c_0/c])^{nf}\}, \{(St(\neg\varphi)[d_0/c])^{nf}\}, \{\langle v_0, c_0\rangle\}, \{\langle v_0, d_0\rangle\})$$

is contained in $S[\Delta, \tau_1, \tau_2]$. As always, only P4 requires some argument. So assume there is a $\psi \in \Delta$ such that

1. the free variables of $\psi$ are contained in $\{v_0\}$,
2. $(St(\varphi)[c_0/c])^{nf} \models \psi(\langle v_0, c_0\rangle)$,
3. $(St(\neg\varphi)[d_0/c])^{nf} \models \neg\psi(\langle v_0, d_0\rangle)$.

From 1 and the second part of Lemma 7.4, it follows that $\psi[c/v_0]$ is equivalent to the standard translation of an existential modal formula $\vartheta$. Together with 2 and 3 this implies that $\varphi$ is equivalent to an existential modal formula, in contradiction to what we have assumed. Thus $s \in S[\Delta, \tau_1, \tau_2]$.

The next steps in the proof may be taken from the proof of Theorem 8.4; so we skip them here. At a certain point we meet the following situation:

(i) $(\mathfrak{A}_{\Gamma_1} \upharpoonright \tau_1, c_0) \models \varphi$ and
(ii) $(\mathfrak{A}_{\Gamma_2} \upharpoonright \tau_2, d_0) \models \neg\varphi$.

Because $\Delta$ has the same closure properties as in the proof of Theorem 9.1, except $\forall_R$, $Z_l$ satisfies B1 and B2a from Definition 2.6. Hence $Z_l$ is a $\tau \cup \{c\}$-simulation, which means that

(iii) $Z_l : (\mathfrak{A}_{\Gamma_1} \upharpoonright \tau_1, c_0) \rightsquigarrow^{\tau\cup\{c\}} (\mathfrak{A}_{\Gamma_2} \upharpoonright \tau_2, d_0)$.

Let $(\mathfrak{A}_1, c_0)$ be the unraveling of $(\mathfrak{A}_{\Gamma_1} \upharpoonright \tau_1, c_0)$ and $(\mathfrak{A}_2, d_0)$ the unraveling of $(\mathfrak{A}_{\Gamma_2} \upharpoonright \tau_2, d_0)$. For these models we easily conclude

(iv) $(\mathfrak{A}_{\Gamma_1} \upharpoonright \tau_1, c_0) \sim_{bs}^{\tau\cup\{c\}} (\mathfrak{A}_1, c_0)$,
(v) $(\mathfrak{A}_{\Gamma_2} \upharpoonright \tau_2, d_0) \sim_{bs}^{\tau\cup\{c\}} (\mathfrak{A}_2, d_0)$

and, because of (iii),

(vi) $(\mathfrak{A}_1, c_0) \rightsquigarrow^{\tau\cup\{c\}} (\mathfrak{A}_2, d_0)$.

An application of the first claim of Lemma 3.5 supplies a model $(\mathfrak{A}_2', d_0)$ such that

(vii) $(\mathfrak{A}_1, c_0) \subseteq (\mathfrak{A}_2', d_0)$ and
(viii) $(\mathfrak{A}_2, d_0) \sim_{bs}^{\tau\cup\{c\}} (\mathfrak{A}_2', d_0)$.

Finally, set $(\mathfrak{B}_1, b_1) := (\mathfrak{A}_1, c_0)$ and $(\mathfrak{B}_2, b_2) := (\mathfrak{A}_2', d_0)$. To complete the proof, we argue as follows: (i) and (iv) imply $(\mathfrak{B}_1, b_1) \models \varphi$, whereas (ii), (v) and (viii) yield $(\mathfrak{B}_2, b_2) \models \neg\varphi$. Moreover, from (vii) we infer $(\mathfrak{B}_1, b_1) \subseteq (\mathfrak{B}_2, b_2)$. This shows that $\varphi$ is not preserved under extensions.

*Case 2:* The second claim of the Theorem is a consequence of the first claim. The argument goes as follows: Suppose $\varphi$ is preserved under submodels. Then $\neg\varphi$ is preserved under extensions. By the first claim there is an existential formula $\vartheta$ which is equivalent to $\neg\varphi$. Hence $\varphi$ is equivalent to $\neg\vartheta$, consequently to $(\neg\vartheta)^{nf}$. Now it is easy to check that the latter is a universal formula. For the other direction suppose $\varphi$ is equivalent to a universal formula $\vartheta$. Then $\neg\varphi$ is equivalent to the existential $(\neg\vartheta)^{nf}$. By 1, $\neg\varphi$ is preserved under extensions; therefore $\varphi$ is preserved under submodels.

*Case 3:* The proof is a duplicate of the proof of the first claim. Instead of using the first part of Lemma 3.5, one has to apply the second part of the lemma; the rest goes through without any change. So we can leave the details as an exercise to the (skeptical) reader. □

**Acknowledgments** The author would like to thank Stefan Iwan, Alexander Kurz, and the anonymous referee for helpful comments on an earlier version of this paper.

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