

Adequacy Results for Some Priorean Modal Propositional Logics

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Abstract Standard possible world semantics for *propositional* modal languages ignore truth-value gaps. However, simple considerations suggest that it should not be so. In Section 1, I identify what I take to be a correct truth-clause for necessity under the assumption that some possible worlds are incomplete (i.e., “at” which some propositions lack a truth-value). In Section 2, I build a world semantics, the semantics of *TV-models*, for standard modal propositional languages, which agrees with the truth-clause for necessity previously identified. Sections 3–5 are devoted to systematic concerns. In particular, in Section 4, Prior’s system *Q* (propositional version) is given a TV-models semantics and proved adequate (i.e., sound and complete) with respect to it.

1 Incomplete worlds and modality Let a *proposition* be any statement that is actually true or false,¹ and let us say that possible world *w* is (i) *complete with respect to proposition p* if and only if *p* is true or false at *w*, and (ii) *complete* (tout court) if and only if it is complete with respect to all propositions. Then by definition the actual world is complete. And a classical assumption in possible worlds semantics for propositional modal logics is that *every* possible world is complete.

There are serious reasons to reject that assumption. Consider, for instance, the proposition ‘Socrates is mortal’, and assume (i) that there are possible worlds where Socrates does not exist, and (ii) that for ‘Socrates is mortal’ to have a truth-value at a world, Socrates must exist therein: two defensible assumptions, which jointly entail that there are worlds where ‘Socrates is mortal’ has no truth-value.

Once it is granted that some propositions have no truth-value at some worlds, it is still not decided if and how these truth-value gaps are transmitted to more complex propositions. In this paper, we shall adopt the *principle of contamination*, according to which if a proposition has no truth-value at a given world, then every proposition containing the first thereby has no truth-value at that world.

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Now, the admission of incomplete worlds, together with the acceptance of the principle of contamination, raises some difficulties when it comes to stating truth-clauses for necessity operators. Consider first the usual clause:

- (1) $\Box A$ is true iff A is true at every possible world.

Now, let w be a possible world, incomplete with respect to some proposition A . Then, by the principle of contamination, w is also incomplete with respect to $A \vee \sim A$. And since, of course, being true at a world entails having a truth-value at that world, a consequence of (1) is that $A \vee \sim A$ is not necessarily true: an unhappy result.

The classical modal logician has a ready-made solution to this problem. It consists in adopting the spirit, not the letter, of the classical truth-clause for the box:

- (2) $\Box A$ is true iff A is true at every complete world.

The problem with this proposal is that, as far as I can see, there is no good reason to accept the restriction to *complete* worlds. For consider some proposition A . Then intuitively, if A is necessarily true, then A is true *at every world where it has a truth-value*—and not only at every complete world. And conversely, if A is true at every world where it has a truth-value, then, plausibly, A is necessarily true. (Note here that if A is true at every world where it has a truth-value, it is true at every complete world, and so by the classical clause, it is necessarily true.) That is, intuitively the following biconditional holds:

- (3) $\Box A$ is true iff A is true at every possible world where it has a truth-value.

Let us now turn to possibility. Defining possibility in terms of necessity by the usual ' $\Diamond = \sim \Box \sim$ ', the following truth-clauses can be derived from (1), (2), and (3) respectively:

- (1') $\Diamond A$ is true iff there is a possible world at which A is not false;
 (2') $\Diamond A$ is true iff there is a complete possible world at which A is true;
 (3') $\Diamond A$ is true iff there is a possible world at which A is true.

(The derivations make use of the basic truth-condition for \sim , the fact that being true at a world entails having a truth-value at that world, and the principle that having a truth-value at world w and not being true at w entails being false at w .) Condition (1') is subject to the same type of problem as (1). For let w be a possible world, incomplete with respect to some proposition A . Then, by the principle of contamination, w is also incomplete with respect to $A \& \sim A$. And once again, since being false at a world entails having a truth-value at that world, a consequence of (1') is that $\Diamond(A \& \sim A)$ is true—an undesirable result. On the other hand, condition (2'), just like condition (2), seems ill-motivated. As to condition (3'), it sounds perfectly right.

Clause (3) is radically different from any usual truth-clause for the box. The difference is essentially this. Let \mathcal{W}_A be the set of all possible worlds at which proposition A must be true for $\Box A$ to be true. According to any classical truth-clause for necessity, $\mathcal{W}_A = \mathcal{W}_B$ for any two distinct propositions A and B : \mathcal{W}_A and \mathcal{W}_B are in both cases the set of all possible worlds (accessible from the actual world). On the other hand, according to condition (3), it may be the case that $\mathcal{W}_A \neq \mathcal{W}_B$; it is actually so as soon as there is some world at which only one of A or B has a truth-value.

In Section 2 below, I build a simple world semantics for standard languages containing a necessity operator which agrees on necessity with condition (3). Systematic matters and adequacy results are dealt with in Sections 3–5.

2 Modeling necessity Let \mathcal{L} be a formal language, whose vocabulary consists in (i) a denumerable set of propositional letters (the atoms), and (ii) the operators \sim (negation), $\&$ (conjunction), and \Box (necessity). What counts as a *formula* of \mathcal{L} is characterized in the usual way, and operators \vee (disjunction), \supset (material implication), \equiv (material equivalence), and \Diamond (possibility) are standardly defined.

There are many ways one can provide \mathcal{L} with a world semantics which agrees with condition (3) of Section 1. However, the most straightforward way to model \mathcal{L} is by means of what I shall call *TV-models*. TV-models are essentially **S5**-models without accessibility relation, modified so as to take into account the possibility of truth-value gaps. More precisely, a TV-model for language \mathcal{L} is a quadruple $\langle @, \mathcal{W}, \text{TV}, \models \rangle$, where \mathcal{W} is a set, $@$ is in \mathcal{W} , and TV and \models are two-place relations between worlds and atoms, meeting conditions:

- [TV-@] for every atom p , $\text{TV}(@, p)$, and
- [\models -TV] for every w in \mathcal{W} and every atom p , if $w \models p$ then $\text{TV}(w, p)$.

Under the intended interpretation, \mathcal{W} is the set of all possible worlds, $@$ is the actual world, ' $\text{TV}(w, p)$ ' is read ' p has a truth-value at w ', and ' $w \models p$ ' is read ' p is true at w '. The first condition amounts to the claim that the atoms of \mathcal{L} stand for propositions in the sense introduced at the beginning of Section 1 and the second condition speaks for itself.

Given an arbitrary TV-model $\langle @, \mathcal{W}, \text{TV}, \models \rangle$, we must specify how TV and \models extend to relations between worlds and complex formulas. The conditions on truth-valuedness I choose are

- [TV. \sim] $\text{TV}(w, \sim A)$ iff $\text{TV}(w, A)$,
- [TV. $\&$] $\text{TV}(w, A \& B)$ iff $\text{TV}(w, A)$ and $\text{TV}(w, B)$, and
- [TV. \Box] $\text{TV}(w, \Box A)$ iff $\text{TV}(w, A)$.

The idea behind these three conditions is that (i) a complex formula has no truth-value at a world if some of its subformulas have no truth-value at that world, and (ii) a complex formula has a truth-value at a world if its subformulas all have a truth-value at that world. (i) is motivated by the principle of contamination and (ii) seems reasonable.

The clauses for \models are

- [\models . \sim] $w \models \sim A$ iff $\text{TV}(w, A)$ and $w \not\models A$,
- [\models . $\&$] $w \models A \& B$ iff $w \models A$ and $w \models B$,
- [\models . \Box] $w \models \Box A$ iff $\text{TV}(w, A)$ and for every v in \mathcal{W} such that $\text{TV}(v, A)$, $v \models A$.

For an arbitrary TV-model we then have

- $@ \models \sim A$ iff $@ \not\models A$;
- $@ \models A \& B$ iff $@ \models A$ and $@ \models B$;
- $@ \models \Box A$ iff for every w in \mathcal{W} such that $\text{TV}(w, A)$, $w \models A$.

The clauses for \sim and $\&$ are standard and the condition for \Box is as foreshadowed. We also have, for every world w and every formula A ,

1. $\text{TV}(@, A)$;
2. if $w \models A$ then $\text{TV}(w, A)$;
3. $\text{TV}(w, A)$ iff $w \models A$ or $w \models \sim A$;
4. $w \models \Diamond A$ iff $\text{TV}(w, A)$ and for some v in \mathcal{W} , $v \models A$;
5. $@ \models \Diamond A$ iff for some v in \mathcal{W} , $v \models A$.

Let us turn finally to validity. Formula A will be said to be *valid in TV-model M* if and only if for every world w of M at which A has a truth-value, A is true at w . And formula A will be said to be *valid* if and only if A is valid in every TV-model.

As one can easily check, all \mathcal{L} -instances of axiom (schema) T ($\Box A \supset A$) and axiom E ($\Diamond A \supset \Box \Diamond A$) are valid, and the rule necessitation ($A/\Box A$) is validity-preserving. On the other hand, some \mathcal{L} -instances of axiom K ($(\Box(A \supset B) \& \Box A) \supset \Box B$) are not valid. In fact, let p and q be two atoms. Then $(\Box(p \supset q) \& \Box p) \supset \Box q$ is false at the actual world of any TV-model $\langle @, \{w, @\}, \text{TV}, \models \rangle$ where p and q are both true at $@$, p has no truth-value at w and q is false at w . The logics to be presented below, in particular Prior's Q , diverge from system **S5** essentially in that each contains as a theorem a modified version of axiom K. (Here it should be noted that while some \mathcal{L} -instances of axiom K are not valid, every instance of $(\Box(A \supset (B \& (A \vee \sim A))) \& \Box A) \supset \Box(B \& (A \vee \sim A))$ is valid, even though $B \equiv (B \& (A \vee \sim A))$ has all its instances valid.)²

3 System $\text{S5}^>$ The first system I shall envisage is $\text{S5}^>$. Like the systems to be defined in Section 4, it is formulated in a language richer than \mathcal{L} .

3.1 $\text{S5}^>$ and its semantics The language for $\text{S5}^>$ is $\mathcal{L}^>$ —that is, \mathcal{L} with extra two-place operator $>$. We define the TV-models for $\mathcal{L}^>$ in the same way as the TV-models for \mathcal{L} , and the semantical clauses for the new operator are given by

- $$\begin{aligned} [\text{TV}.>] \quad & \text{TV}(w, A > B) \text{ iff } \text{TV}(w, A) \text{ and } \text{TV}(w, B), \text{ and} \\ [\models .>] \quad & w \models A > B \text{ iff } \text{TV}(w, A) \text{ and } \text{TV}(w, B) \text{ and for every } v \text{ in } \mathcal{W}, \\ & \text{if } \text{TV}(v, A), \text{ then } \text{TV}(v, B). \end{aligned}$$

Validity is defined as before. For an arbitrary TV-model, we have

$$@ \models A > B \text{ iff for every } w \text{ in } \mathcal{W}, \text{ if } \text{TV}(w, A), \text{ then } \text{TV}(w, B).$$

Thus, ' $A > B$ ' is to be read as 'at every world where A has a truth-value, B has a truth-value', or as 'for A to have a truth-value B must also have a truth-value'. ' $A > B$ ' can be seen as expressing the idea that there is some kind of relevance link between A and B or between the "information" conveyed by A and by B . System $\text{S5}^>$ is defined by the following axioms (schemas) and rules.

Classical axioms

Every PC-valid \mathcal{L} -formula

Axioms for $>$

- $A > B$ if each atom in B is in A
- $(A > B \ \& \ B > C) \supset A > C$
- $(A > B \ \& \ A > C) \supset A > (B \ \& \ C)$
- $(A > B \ \& \ A > C) \supset A > (B > C)$
- $A > B \supset \Box(A > B)$

Modal axioms

- (K[>]) $(B > A \ \& \ \Box(A \supset B) \ \& \ \Box A) \supset \Box B$
- (T) $\Box A \supset A$
- (E) $\Diamond A \supset \Box \Diamond A$

Rules

- (modus ponens) If $\vdash A$ and $\vdash A \supset B$ then $\vdash B$
- (necessitation) If $\vdash A$ then $\vdash \Box A$

3.2 Adequacy System $\mathbf{S5}^>$ is adequate (i.e., sound + complete) with respect to the semantics of TV-models. For soundness, it is more or less routine to show that each axiom of $\mathbf{S5}^>$ is valid and that necessitation is validity-preserving. The case for modus ponens, though, is not standard: the rule does not preserve validity-in-a-model (if it did, then axiom K would have all its instances valid). However, modus ponens is validity-preserving, as the following argument shows. Let A, B be formulas, $M = \langle @, \mathcal{W}, \text{TV}, \models \rangle$ a TV-model, and w_0 a world of M such that $\text{TV}(w_0, B)$. We want to prove that if A and $A \supset B$ are valid, then B is true at w_0 . Suppose that every atom in A is in B . Then $\text{TV}(w_0, A)$ and $\text{TV}(w_0, A \supset B)$. So, if both A and $A \supset B$ are valid, they are true at w_0 in M , and therefore so is B . Suppose now that some atom in A is not in B . Consider the model $N = \langle @, \mathcal{W}', \text{TV}', \models' \rangle$ defined by

1. $\text{TV}'(w_0, p)$ for every atom p in A not in B ;
2. $\text{TV}'(w, p)$ iff $\text{TV}(w, p)$ for every atom p and every w in \mathcal{W}' such that $w \neq w_0$ or p is not an atom in A not in B ;
3. $w_0 \models' p$ for every atom p in A not in B ;
4. $w \models' p$ iff $w \models p$ for every atom p and every w in \mathcal{W}' such that $w \neq w_0$ or p is not an atom in A not in B .

By the definition of N , for every atom p in B and for every w in \mathcal{W}' ,

- (i) $\text{TV}'(w, p)$ iff $\text{TV}(w, p)$, and
- (ii) $w \models' p$ iff $w \models p$.

From this fact, it follows that given any formula C whose atoms are all in B , for every w in \mathcal{W}' , $w \models' C$ if and only if $w \models C$ (the proof is by induction on the complexity of C). As a consequence, $w_0 \models' B$ if and only if $w_0 \models B$. Now by construction, both A and $A \supset B$ have a truth-value at w_0 in N . So, if both A and $A \supset B$ are valid, they both are true at w_0 in N and so, by the properties of truth-at-a-world, B is true at w_0 in N .

By the previous result then, B is true at w_0 in M . Let us turn now to completeness. Useful for what follows is the following proposition.

Proposition 3.1

1. If $\vdash \Box(A \supset B)$ and $\vdash B > A$, then $\vdash \Box A \supset \Box B$.
2. $A > B \supset A > C$, provided all atoms in C are in B .
3. $A > B \equiv A > C$, provided B and C contain exactly the same atoms.
4. If B_1, \dots, B_n are all the atoms in B , then $\vdash (A > B_1 \& \dots \& A > B_n) \equiv A > B$.
5. $\sim(A > B) \supset \Box \sim(A > B)$.
6. $\Box A \supset \Box \Box A$.
7. $(\Box A \& \Box B) \supset \Box(A \& B)$.

Proof:

1. By axiom $K^>$.
2. Let C_1, \dots, C_n be all the atoms in C . If each is in B , then by the first axiom for $>$, $\vdash B > C_1, \dots, \vdash B > C_n$. So by the transitivity of $>$, $\vdash A > B \supset A > C_1, \dots, \vdash A > B \supset A > C_n$. So, $\vdash A > B \supset (A > C_1 \& \dots \& A > C_n)$. As a consequence of the third axiom for $>$, $\vdash A > B \supset A > (C_1 \& \dots \& C_n)$. Now, each atom in C is in $C_1 \& \dots \& C_n$, and so, $\vdash (C_1 \& \dots \& C_n) > C$. By the transitivity of $>$, it follows that $\vdash A > B \supset A > C$.
3. By the previous result.
4. Let B_1, \dots, B_n be all the atoms in B . By Proposition 3.1(3), $\vdash A > B \equiv A > (B_1 \& \dots \& B_n)$. Now we prove that $\vdash A > (B_1 \& \dots \& B_n) \equiv (A > B_1 \& \dots \& A > B_n)$.
 - (i) $\vdash A > (B_1 \& \dots \& B_n) \supset (A > B_1 \& \dots \& A > B_n)$ follows from Proposition 3.1(2).
 - (ii) $\vdash (A > B_1 \& \dots \& A > B_n) \supset A > (B_1 \& \dots \& B_n)$ follows from the third axiom for $>$.
5. (a) By axiom T, $\vdash \Box A \supset \Diamond \Box A$.
 (b) By axiom E, $\vdash \Diamond \Box A \supset \Box \Diamond \Box A$.
 (c) By axiom E, $\vdash \Diamond \Box A \supset \Box A$.
 By necessitation then, $\vdash \Box(\Diamond \Box A \supset \Box A)$. But $\vdash \Box A > \Diamond \Box A$. So by Proposition 3.1(1), $\vdash (\Box \Diamond \Box A) \supset \Box \Box A$. Points (a), (b), and (c) yield the result.
6. By classical logic and necessitation, $\vdash \Box(A \supset (B \supset (A \& B)))$. But $\vdash (B \supset (A \& B)) > A$ and $\vdash A \& B > B$. We then have the result by Proposition 3.1(1).
7. $\vdash \sim \Box(A > B) \supset \sim(A > B)$. By necessitation then, $\vdash \Box(\sim \Box(A > B) \supset \sim(A > B))$. But by the fifth axiom for $>$, $\vdash \sim(A > B) > \sim \Box(A > B)$. So by Proposition 3.1(1), $\vdash \Box \sim \Box(A > B) \supset \Box \sim(A > B)$. As a consequence, $\vdash \Diamond(A > B) \supset \Diamond \Box(A > B)$. The result follows from axioms E and T. \square

Now for the completeness proof, let α be a nontheorem, and let $@$ be a maximal consistent extension of $\{\sim\alpha\}$ (use a Lindenbaum-type construction to prove the existence of $@$). (I use a standard definition of consistency and inconsistency: a set of formulas

Γ is *inconsistent* with respect to a given system if and only if there is a finite collection A_1, \dots, A_n of members of Γ such that $\neg(A_1 \wedge \dots \wedge A_n)$ is a theorem of that system; and Γ is *consistent* with respect to a given system if and only if it is not *inconsistent* with respect to that system.)

Proposition 3.2 *Every theorem is in @, and for all formulas A and B , if $A \in @$ and $A \supset B \in @$ then $B \in @$.*

This proposition will be used without explicit mention. Its proof is standard.

Let At be the set of all atoms, and let χ (constituency) be the function from the set of all formulas to $\mathcal{P}(At)$ such that $\chi(A)$ is the set of all atoms in A .

Let X be a nonempty subset of At . Then the *closure* of X , cX , is $\{p \in At \mid \text{there are } p_1, \dots, p_n \text{ in } X \text{ such that } (p_1 \& \dots \& p_n) > p \in @\}$. Note that for X and Y any subsets of At , $X \subseteq cX$, $ccX = cX$, and if $X \subseteq Y$ then $cX \subseteq cY$. A nonempty subset X of At will be said to be *closed* if and only if $X = cX$.

Proposition 3.3 *For all formulas A and B , $A > B \in @$ if and only if $\chi(B) \subseteq c\chi(A)$.*

Proof: Let A and B be formulas, let $\chi(A)$ be $\{A_1, \dots, A_n\}$, and let $\chi(B)$ be $\{B_1, \dots, B_m\}$.

- (i) Let B_j be in $\chi(B)$. Suppose $A > B_j \in @$. Then since $\vdash (A_1 \& \dots \& A_n) > A$, $(A_1 \& \dots \& A_n) > B_j \in @$. So by definition of closure, $B_j \in c\chi(A)$. Conversely, suppose that $B_j \in c\chi(A)$. By definition of closure, there are $\alpha_1, \dots, \alpha_k$ in $\chi(A)$ such that $(\alpha_1 \& \dots \& \alpha_k) > B_j \in @$. But $\vdash A > (\alpha_1 \& \dots \& \alpha_k)$. So, $A > B_j \in @$. As a conclusion, $A > B_j \in @$ iff $B_j \in c\chi(A)$.
- (ii) By Proposition 3.1(4), $\vdash A > B \equiv (A > B_1 \& \dots \& A > B_m)$. So $A > B \in @$ if and only if $A > B_1 \in @$ and \dots and $A > B_m \in @$. So by (i) above, $A > B \in @$ if and only if $\chi(B) \subseteq c\chi(A)$. \square

Where X is a closed set of atoms, let $@[X]$ be the set of all formulas A such that $\Box A \in @$ and $\chi(A) \subseteq X$. Note that $@[X]$ is never empty (by definition, a closed set is never empty, and if X contains, say p , then $@[X]$ contains $p \vee \sim p$). Also note that by axiom T, $@[X] \subseteq @$ for every closed set of atoms X . As a consequence, each $@[X]$ is consistent.

For every set of formulas S and every closed set of atoms X , say that S is *X-maximal* in case (i) for every A in S , $\chi(A) \subseteq X$ and (ii) for every formula A such that $\chi(A) \subseteq X$, either $A \in S$ or $\sim A \in S$. Clearly, every consistent set of formulas satisfying (i), in particular every $@[X]$, has some X -maximal consistent extension (adapt the usual Lindenbaum-type construction).

Let \mathcal{W} be $\{w \mid w \text{ is an } X\text{-maximal consistent extension of } @[X] \text{ for some closed set of atoms } X\}$. Note that $@$ is in \mathcal{W} , since At is closed and $@$ is trivially an At -maximal extension of $@[At]$. For every w in \mathcal{W} , there is only one closed set of atoms X such that w is an X -maximal consistent extension of $@[X]$. Call it ' $D(w)$ '. In the other direction, for every closed set of atoms X , there is some world w such that $D(w) = X$. The reason is that $@[X]$ is never empty.

Now, where p is any atom, put ‘ $\text{TV}(w, p)$ ’ for ‘ $p \in D(w)$ ’, and ‘ $w \models p$ ’ for ‘ $p \in w$ ’. We have, for every atom p and every w in \mathcal{W} ,

1. $\text{TV}(@, p)$ (since $D(@) = At$), and
2. if $w \models p$ then $\text{TV}(w, p)$ (by maximality).

The 4-tuple $\langle @, \mathcal{W}, \text{TV}, \models \rangle$ is then a TV-model. The aim is now to prove that for every formula A and every world w , $w \models A$ if and only if $A \in w$, which will give us completeness.

Proposition 3.4 *Let w be in \mathcal{W} . Then*

1. every theorem A such that $\chi(A) \subseteq D(w)$ is in w , and for all formulas A and B , if $A \in w$ and $(A \supset B) \in w$ then $B \in w$;
2. for all formulas A and B , if $\chi(A) \subseteq D(w)$ and $A > B \in @$ then $\chi(B) \subseteq D(w)$.

Proof: The proof for (1) is quite standard. For (2), let A, B be formulas. Suppose that $\chi(A) \subseteq D(w)$. Then $c\chi(A) \subseteq cD(w)$, and since $D(w)$ is closed, $c\chi(A) \subseteq D(w)$. Now suppose that $A > B \in @$. By Proposition 3.3, it follows that $\chi(B) \subseteq c\chi(A)$. So, $\chi(B) \subseteq D(w)$. \square

Proposition 3.5 *For every w in \mathcal{W} and for every formula A , $\Box A \in w$ if and only if $\Box A \in @$ and $\chi(A) \subseteq D(w)$.*

Proof: Let A be a formula.

1. Suppose $\Box A \in @$. Then by Proposition 3.1(6), $\Box \Box A \in @$. So, for every w in \mathcal{W} , if $\chi(A) \subseteq D(w)$ then $\Box A \in @[D(w)]$, and consequently $\Box A \in w$.
2. Suppose $\Box A \notin @$. By maximality, $\sim \Box A \in @$. So by axiom E, $\Box \sim \Box A \in @$. Consequently, for every w in \mathcal{W} such that $\chi(A) \subseteq D(w)$, $\sim \Box A \in @[D(w)]$. Thus, $\sim \Box A \in w$, and so by consistency, $\Box A \notin w$. \square

Proposition 3.6 *For every w in \mathcal{W} and for all formulas A and B ,*

1. $\sim A \in w$ if and only if $\chi(A) \subseteq D(w)$ and $A \notin w$;
2. $A \& B \in w$ if and only if $A \in w$ and $B \in w$;
3. $A > B \in w$ if and only if $\chi(A) \cup \chi(B) \subseteq D(w)$ and for every v in \mathcal{W} such that $\chi(A) \subseteq D(v)$, $\chi(B) \subseteq D(v)$.

Proof: Let A and B be formulas and let w be in \mathcal{W} .

1. (i) Suppose $\sim A \in w$. Then $\chi(A) \subseteq D(w)$ and by consistency $A \notin w$.
(ii) By maximality, if $\chi(A) \subseteq D(w)$ and $A \notin w$ then $\sim A \in w$.
2. (i) Suppose $A \& B \in w$. Then $\chi(A \& B) \subseteq D(w)$, and thus $\chi(A \& B \supset A)$ and $\chi(A \& B \supset B)$ are subsets of $D(w)$. So since $\vdash A \& B \supset A$ and $\vdash A \& B \supset B$, by Proposition 3.4(1) $A \in w$ and $B \in w$.
(ii) Suppose $A \in w$ and $B \in w$. Then $\chi(A) \cup \chi(B) \subseteq D(w)$, and so, $\chi(A \supset (B \supset (A \& B))) \subseteq D(w)$. Thus since $\vdash A \supset (B \supset (A \& B))$, by Proposition 3.4(1) $A \& B \in w$.

3. (i) Suppose $A > B \in w$. Then $\chi(A > B) \subseteq D(w)$. (a) A consequence is that $\chi(A) \cup \chi(B) \subseteq D(w)$. (b) Another consequence is that $\chi(A > B \supset \Box(A > B)) \subseteq D(w)$. But since $\vdash A > B \supset \Box(A > B)$, it follows by Proposition 3.4(1) that $\Box(A > B) \in w$. By Proposition 3.5, then $\Box(A > B) \in @$. So by axiom T, $A > B \in @$. We have then by Proposition 3.4(2): for every v in \mathcal{W} if $\chi(A) \subseteq D(v)$ then $\chi(B) \subseteq D(v)$.
- (ii) Suppose $A > B \notin w$ and $\chi(A > B) \subseteq D(w)$. Then by maximality, $\sim(A > B) \in w$. By Propositions 3.1(5) and 3.4(1), it follows that $\Box \sim(A > B) \in w$. So, by Proposition 3.5 and axiom T $\sim(A > B) \in @$, and as a consequence, $A > B \notin @$. By Proposition 3.3, then, $\chi(B)$ is not a subset of $c\chi(A)$. Let v be any world with $D(v) = c\chi(A)$. We have: $\chi(A)$ but not $\chi(B)$ is a subset of $D(v)$.

□

Proposition 3.7 *For every w in \mathcal{W} and for every formula A , $\Box A \in w$ if and only if $\chi(A) \subseteq D(w)$ and for every v in \mathcal{W} such that $\chi(A) \subseteq D(v)$, $A \in v$.*

Proof: Let A be a formula and let w be in \mathcal{W} .

1. Suppose $\Box A \in w$. A first consequence is that $\chi(A) \subseteq D(w)$. A second consequence is that $\Box A \in @$ by Proposition 3.5. From this it follows that for every v in \mathcal{W} such that $\chi(A) \subseteq D(v)$, $A \in @[D(v)]$. So, for every v in \mathcal{W} such that $\chi(A) \subseteq D(v)$, $A \in v$.
2. Suppose $\Box A \notin w$ and $\chi(A) \subseteq D(w)$. By Proposition 3.5, then $\Box A \notin @$. Now let us prove that $\{\sim A\} \cup @[c\chi(A)]$ is consistent. Suppose it is not. Then one can find B_1, \dots, B_n in $@[c\chi(A)]$ such that $\vdash (B_1 \& \dots \& B_n) \supset A$. We have then the following:
 - a. $\Box B_1, \dots, \Box B_n$ are in $@$, and so by Proposition 3.1(7), $\Box(B_1 \& \dots \& B_n)$ is in $@$.
 - b. By necessitation, $\Box[(B_1 \& \dots \& B_n) \supset A]$ is in $@$.
 - c. Since each B_i is in $@[c\chi(A)]$, each $\chi(B_i)$ is included in $c\chi(A)$. So, by Proposition 3.3, each $A > B_i$ is in $@$. Now by the third axiom for $>$, $\vdash (A > B_1 \& \dots \& A > B_n) \supset A > (B_1 \& \dots \& B_n)$. So, $A > (B_1 \& \dots \& B_n)$ is in $@$.

These three points plus axiom $K^>$ entail that $\Box A$ is in $@$. So, since by hypothesis $\Box A$ is not in $@$, we must conclude that $\{\sim A\} \cup @[c\chi(A)]$ is consistent. Now, let v be a $c\chi(A)$ -maximal extension of $\{\sim A\} \cup @[c\chi(A)]$. v is, of course, a $c\chi(A)$ -maximal extension of $@[c\chi(A)]$, and so v is in \mathcal{W} . Moreover, (a) $\chi(A) \subseteq D(v)$, and (b) $\sim A \in w$, which by consistency entails that $A \notin w$. □

Proposition 3.8 *For every formula A and every world w , $TV(w, A)$ if and only if $\chi(A) \subseteq D(w)$.*

Proof: Easy. □

Proposition 3.9 *For every formula A and every world w , $w \models A$ if and only if $A \in w$.*

Proof: By induction on the length of the formulas, using Propositions 3.6, 3.7, and 3.8. \square

This ends the completeness proof.

4 System Q^3 Prior was aware that the possibility that a proposition has no truth-value at some possible world has to be taken into account in a correct treatment of propositional modal logic. Accordingly, he developed a system, Q , and gave some indications as to how to provide it with a world semantics (see Prior and Fine [1], pp. 85–86). These indications show that, essentially, Prior agrees with the TV-modeling of necessity presented in Section 2. In the present section, two results are achieved. First, it is shown that system Q can be seen as a fragment of a mild extension of system $S5^>$. Second, Q is given a TV-model semantics and proved adequate with respect to it.

4.1 System Q Prior formulates system Q in a language with primitive operators \sim , $\&$, \diamond , and \mathcal{S} —where \mathcal{S} is a one-place operator intended to express necessary statability (a proposition is necessarily statable if and only if it is statable (i.e., has a truth-value) at every possible world).⁴ For the sake of uniformity, I will rather formulate Q in language $\mathcal{L}^{\mathcal{S}}$, namely, \mathcal{L} augmented by operator \mathcal{S} .

System Q can then be defined thus, with \diamond standing for $\sim \Box \sim$ as before (see [1], pp. 84–85).

Classical axioms

Every PC-valid formula

Axioms for \mathcal{S}

$SA \supset Sp$, for any atom p in A

$(Sp_1 \& \cdots \& Sp_n) \supset SA$, where p_1, \dots, p_n are all the atoms in A

$\diamond SA \supset SA$

Modal axioms

(K ^{\mathcal{S}}) $(Sp_1 \& \cdots \& Sp_n \& \Box(A \supset B) \& \Box A) \supset \Box B$, where p_1, \dots, p_n are all the atoms of A not in B

(T) $\Box A \supset A$

(E) $\diamond A \supset \Box \diamond A$

Rules

(modus ponens) If $\vdash A$ and $\vdash A \supset B$ then $\vdash B$

(necessitation) If $\vdash A$ then $\vdash \Box A$

4.2 Q in $S5^{>t}$ Let $\mathcal{L}^{>t}$ be $\mathcal{L}^{>}$ augmented by a special atom, t . The TV-models for $\mathcal{L}^{>t}$ are like those for $\mathcal{L}^{>}$, except that we impose the following:

- [TV. t] $\text{TV}(w, t)$ for any world w of any TV-model, and
 $[\models .t]$ $w \models t$ for any world w of any TV-model.

Validity is defined as before and system $\mathbf{S5}^{>t}$ is $\mathbf{S5}^>$ plus t as an axiom and $A > t$ as an axiom schema. It should be clear that $\mathbf{S5}^{>t}$ is adequate with respect to the semantics of TV-models for $\mathcal{L}^{>t}$. (Soundness is straightforward. For completeness, the major modification from the proof of Section 3 consists in requiring that every closed set of atoms contain t .)

Now, put SA for $t > A$. As is readily shown, we have for an arbitrary TV-model

$$w \models SA \text{ iff for every } w \text{ in } \mathcal{W}, \text{TV}(w, A).$$

That is, SA expresses the necessary stability of A . Moreover, where A is any formula of \mathcal{L}^S , let A^* be the result of replacing each occurrence of S in A by $t >$. Then it is easy to show that for every formula A of \mathcal{L}^S , A is a theorem of Q if and only if its translation A^* in $\mathcal{L}^{>t}$ is a theorem of $\mathbf{S5}^{>t}$.

4.3 Semantics for Q and adequacy We can do better: we can prove that system Q is adequate with respect to the obvious TV-model semantics for language \mathcal{L}^S . The TV-models for \mathcal{L}^S are defined in the same way as the TV-models for \mathcal{L} , and the semantic clauses for operator S are

1. $\text{TV}(w, SA)$ iff $\text{TV}(w, A)$, and
2. $w \models SA$ iff for every w in \mathcal{W} , $\text{TV}(w, A)$.

As is easily checked, Q is sound with respect to the semantics of TV-models for \mathcal{L}^S . For completeness, a slight adaptation of the completeness proof of Section 3 gives the result. The proof for Q is even a bit simpler. Propositions 3.1(6) and 3.1(7) hold in the present context, and we have the following proposition.

Proposition 4.1 $\vdash SA \supset \Box SA$.

Proof: $\vdash \Diamond SA \supset SA$. By necessitation, then $\vdash \Box(\Diamond SA \supset SA)$. So by axiom K^S , $\vdash \Box \Diamond SA \supset \Box SA$. As a consequence, $\vdash \Diamond \sim SA \supset \Diamond \Box \sim SA$. From axioms E and T, it follows that $\vdash \Diamond \sim SA \supset \sim SA$. Hence the result. \square

Let α be a nontheorem and let $@$ be a maximal consistent extension of $\{\sim \alpha\}$. Proposition 3.2 still holds.

Let At be now the set of all atoms of \mathcal{L}^S . Function χ is defined as before. Let $S(At)$ be the set of all atoms p of \mathcal{L}^S such that S_p is in $@$. Where X is a nonempty subset of At , the *closure* of X , cX , is now $X \cup S(At)$. As before, a subset X of At will be said to be *closed* if and only if $X = cX$.

Where X is a closed set of atoms, $@[X]$ and X -maximality are defined as in Section 3. As before, each $@[X]$ is consistent and has some X -maximal consistent extension.

We finally define the TV-model $\langle @, \mathcal{W}, \text{TV}, \models \rangle$ as in Section 3, and the aim is to prove now that for every formula A and every world w , $w \models A$ if and only if $A \in w$.

Proposition 4.2 *Let A and B be formulas, with p_1, \dots, p_n all the atoms of B not in A (we suppose there are such atoms). Then $(S_{p_1} \& \dots \& S_{p_n}) \in @$ if and only if $\chi(B) \subseteq c\chi(A)$.*

Proof: Let A, B, p_1, \dots, p_n be as stated and let X be all the atoms of B in A .

- (i) Suppose $(Sp_1 \& \dots \& Sp_n) \in @$. Then each Sp_i is in $@$ and so each p_i is in $S(At)$. Since $\chi(B)$ is $X \cup \{p_1, \dots, p_n\}$, and $X \subseteq \chi(A)$, it follows that $\chi(B) \subseteq \chi(A) \cup S(At)$. But $\chi(A) \cup S(At)$ is $c\chi(A)$.
- (ii) Suppose $\chi(B) \subseteq c\chi(A)$. Then each p_i is in $S(At)$. So, each Sp_i is in $@$ and therefore, $(Sp_1 \& \dots \& Sp_n) \in @$. \square

Proposition 3.4(1) is easily proved and we have the following.

Proposition 4.3 *Let A be a formula. Then if $SA \in @$, for every w in \mathcal{W} , $\chi(A) \subseteq D(w)$.*

Proof: Let A be a formula and suppose $SA \in @$. Then $\chi(A) \subseteq S(At)$. Now let w be a world. Since $D(w)$ is closed, $S(At) \subseteq D(w)$. So, $\chi(A) \subseteq D(w)$. \square

Propositions 3.5, 3.6(1), and 3.6(2) still hold, and we have the following proposition.

Proposition 4.4 *For every w in \mathcal{W} and for every formula A , $SA \in w$ if and only if for every v in \mathcal{W} , $\chi(A) \subseteq D(v)$.*

Proof: Let A be a formula, and let w be in \mathcal{W} .

- (i) Suppose $SA \in w$. Then by maximality, $\chi(SA) \subseteq D(w)$. A consequence is that $\chi(SA \supset \Box SA) \subseteq D(w)$. But since by Proposition 4.1 $\vdash SA \supset \Box SA$, we have by Proposition 3.4(1) that $\Box SA \in w$. By Proposition 3.5, then $\Box SA \in @$. So by axiom T, $SA \in @$. We have then by Proposition 4.3: for every v in \mathcal{W} , $\chi(A) \subseteq D(v)$.
- (ii) Suppose $SA \notin w$. We have to prove that there is a world v such that $\chi(A)$ is not a subset of $D(v)$. First case: $\chi(A)$ is not a subset of $D(w)$. We directly have the result. Second case: $\chi(A)$ is a subset of $D(w)$. Then by maximality, $\sim SA \in w$. Since $\vdash \sim SA \supset \Box \sim SA$, we have by Proposition 3.4(1): $\Box \sim SA \in w$. So, by Proposition 3.5 and axiom T, $\sim SA \in @$, and as consequence, $SA \notin @$. So, there is an atom p in A such that $p \notin S(At)$. Let v be any world with $p \notin D(v)$. We have: $\chi(A)$ is not a subset of $D(v)$. \square

Proposition 3.7 also holds. The first half of the proof is the same as in Section 3. For the second half, minor modifications have to be made. Suppose that $\Box A \notin w$ and $\chi(A) \subseteq D(w)$. By Proposition 3.5, then $\Box A \notin @$. Now let us prove that $\{\sim A\} \cup @[c\chi(A)]$ is consistent. Suppose it is not. Then one can find B_1, \dots, B_n in $@[c\chi(A)]$ such that $\vdash (B_1 \& \dots \& B_n) \supset A$. We have then the following:

1. $\Box B_1, \dots, \Box B_n$ are in $@$, and so by Proposition 3.1(7), $\Box(B_1 \& \dots \& B_n)$ is in $@$.
2. By necessitation, $\Box[(B_1 \& \dots \& B_n) \supset A]$ is in $@$.
3. Since each B_i is in $@[c\chi(A)]$, each $\chi(B_i)$ is included in $c\chi(A)$. So, $\chi(B_1 \& \dots \& B_n) \subseteq c\chi(A)$. Let p_1, \dots, p_m be the atoms of $B_1 \& \dots \& B_n$ not in A (if there are such atoms). By Proposition 4.2, then $(Sp_1 \& \dots \& Sp_m) \in @$.

These three points plus axiom K^S entail that $\Box A$ is in $@$. So, since by hypothesis $\Box A$ is not in $@$, we must conclude that $\{\sim A\} \cup @[c\chi(A)]$ is consistent. The rest of the proof is as in Section 3.

Proposition 3.8 still holds. We conclude that for every formula A and every world w , $w \models A$ if and only if $A \in w$, as expected.

5 A simpler system All the systems considered so far are formulated in a language richer than the purely modal language \mathcal{L} . But consider system $S5^-$, whose rules are modus ponens and necessitation and whose axiom schemas are all PC-tautologies, T, E, and

(K^-) $\Box(A \supset B) \ \& \ \Box A \supset \Box B$, where all the atoms of A are in B .⁵

Clearly, K^- is a theorem of Q and $S5^>$ (and of $S5^{>t}$). So, $S5^-$ is sound with respect to the class of all TV-models for \mathcal{L} . Moreover, completeness is easily proved (adapt the completeness proof for $S5^>$ by defining the closure of a nonempty set as that very set; every proposition in the completeness proof for $S5^>$ which does not concern operator $>$ is provable as it stands).

6 Résumé Systems $S5^>$, $S5^{>t}$, Q , and $S5^-$ are all both sound and complete with respect to their respective semantics. $S5^- \subset S5^> \subset S5^{>t}$, $S5^- \subset Q$, and for every formula A in \mathcal{L}^S , A is a theorem of Q if and only if its translation A^* in $\mathcal{L}^{>t}$ is a theorem of $S5^{>t}$.

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NOTES

1. Following a standard assumption, I take a statement to be false at a world if and only if its negation is true at that world. I shall also suppose that no proposition can be *both* true and false at a world.
2. Before turning to systematic concerns, let me mention that Segerberg [2] proves completeness for modal systems (Q included), whose semantics is similar to the semantics of TV-models. The reader is invited to glance at this paper for a full comparison. Two big differences between Segerberg's systems/semantics and mine are: (1) his semantical clauses do not all respect the principle of contamination: formulas of type TA are true at worlds where A has no truth-value; (2) his systems are closed under a restricted version of modus ponens, not under full modus ponens.
3. In a previous version of the present paper, no section was devoted to system Q . It was a nice surprise for me to discover Prior's ideas on necessity once I obtained the previous results about $S5^>$ and its semantics.
4. Let a *constituent* of a proposition (in the sense introduced in Section 1) be any object rigidly denoted by some expression in that proposition. For Prior, a proposition is statable at a world if and only if either it has no constituent, or all its constituents exist in that world. And accordingly, a proposition is necessarily statable if and only if either it has no constituent, or all its constituents exist necessarily (see [1], pp. 93–94). Following Prior's account of statability, the operator $>$ introduced in Section 2 should be

considered as expressing a form of existential dependence. For in case propositions A and B both have some constituents, we should then read $A > B$ as something like ‘for the constituents of A to exist, those of B must exist’, or ‘the constituents of A cannot exist unless the constituents of B exist’.

5. I am indebted to an anonymous referee of the *Journal* for suggesting that I examine system **S5**⁻.

REFERENCES

- [1] Prior, A. N., and K. Fine, *Worlds, Times and Selves*, University of Massachusetts Press, Amherst, 1977. [Zbl 0198.01404](#) 4, 4.1, 6
- [2] Segerberg, K., “Some modal logics based on a three-valued logic,” *Theoria*, vol. 33 (1967), pp. 53–71. [Zbl 0189.00705](#) [MR 44:6460](#) 6

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