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A DUAL REPRESENTATION RESULT FOR VALUE FUNCTIONS IN STOCHASTIC CONTROL OF INFINITE DIMENSIONAL GROUPS

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Dedicated to the memory of Professor Ioan Vrabie

Abstract. We study here the problem of dual representation of the value functions associated to linear-convex stochastic control problems in infinite dimensional Hilbert spaces. Since the dual state runs backwards in time, it turns out that the dual representation has the meaning of a classical (Markov) control problem only if the primal linear state equation is driven by the generator of a group. In the general case, a dual representation of the value function still holds, but such a representation cannot be reduced to solving a dual Hamilton–Jacobi–Bellman equation.

1. Introduction

It is well known since Bismut [3] that convex duality between spaces of martingales together with the Fenchel–Legendre transform of convex functions can be used as a powerful tool to solve stochastic control problems. This line of ideas has been used extensively, especially in the Mathematical Finance literature related to Optimal Investment. In this context, duality is not only used to get a representation of the value function in terms of its dual, but is actually an essential part in proving existence of optimizers in the primal problem. This is exactly the case in [5] and [6], where additional technical difficulties come from

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the fact that the dual and primal states lack square integrability conditions, so a duality theory over the set of non-negative processes has to be developed.

The present note contributes to the duality theory in stochastic control of *infinite dimensional* systems. Unlike the Optimal Investment literature mentioned above, here there is no difficulty in handling either the primal or the dual state, which are square integrable. In addition, existence of the optimal control in the primal can be easily proved using coercivity and weak compactness, as in Lemma 3.1. The main goal here is a dual representation formula for the primal value function, and primal optimal controls.

We first derive a dual representation formula for value functions in general linear convex control problems driven by C_0-semigroups, more precisely Theorem 3.4. This is widely expected and is well in line with the ideas following from Bismut together with duality results in deterministic control coupled with backward stochastic differential equations. It is worth noting that this representation is proved using a minimax result which follows quickly from the existence of the optimal control in the *primal problem* as well as the set of necessary conditions (Proposition 3.2). Therefore, as mentioned above, our result is only about the dual representation of the primal value function, since existence of the primal optimal is known *a-priori*.

At this level of generality, in *infinite dimensions*, the dual problem does not have the meaning of a Markov control problem. This is due to the fact that the dual to the primal value function depends on *random variables* rather than a deterministic state and the dual state runs *backwards* (see Remark 3.5).

However, if the linear primal stochastic equation is actually driven by a C_0 group, then the miminax result can be rewritten to lead to a dual representation of the primal value function in the form of a convex conjugate that corresponds to a classical control problem. This is actually our main result, Theorem 3.6. This easily implies that, for the case of a C_0 group, the generalized (viscosity) solution of the the primal HJB is the dual conjugate of the solution of the *dual* HJB. The theory reduces to the duality of solutions of Riccati equations for the case of linear-quadratic control problems in Subsection 3.2.

2. Linear state equations and dual states

Consider the linear state equation

$$\begin{cases} d_t y(t) = (Ay(t) + Bu(t))\,dt \\ \qquad\qquad + \displaystyle\sum_{i=1}^{\infty} \big(C_i y(t) + \sigma_{1,i}\big)\,d\beta_{1,i}(t) + \sum_{j=1}^{\infty} \big(D_j u(t) + \sigma_{2,j}\big)\,d\beta_{2,j}(t), \\ y(0) = x. \end{cases} \tag{2.1}$$

In equation (2.1) we assume that $A\colon D(A) \subset H \to H$ is the infinitesimal generator of a C_0 semigroup $\{\exp(tA),\ t \geq 0\}$ on the Hilbert space H (with norm $|\cdot|$ and product $\langle\,\cdot\,,\,\cdot\,\rangle$). The control takes values in a different Hilbert space U, and $B \in \mathcal{L}(U,H)$.

We further assume that $C_i \in \mathcal{L}(H)$, $\sigma_{1,i} \in H$ for each $i \in \mathbb{N}$ and $D_j \in L(U,H)$, $\sigma_{2,j} \in H$ for $j \in \mathbb{N}$. In addition,

$$\sum_{i=1}^{\infty} \big(|C_i|^2_{\mathcal{L}(H)} + \sigma^2_{1,i}\big) + \sum_{j=1}^{\infty} \big(|D_j|^2_{\mathcal{L}(U,H)} + \sigma^2_{2,j}\big) < +\infty.$$

Finally, the countable set $\{\beta_{1,i}, \beta_{2,j} : i,j \in \mathbb{N}\}$ consists of independent standard Brownian motions defined on the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The filtration is assumed to satisfy the usual conditions of right continuity and completeness. Given any Hilbert space X, we denote by $M^2_{\mathcal{P}}(0,T;X)$ the set of all X-valued processes ζ adapted to $\{\mathcal{F}_t : t \geq 0\}$ such that:

$$\|\zeta\|^2_{M^2_{\mathcal{P}}(0,T;X)} = \mathbb{E}\bigg[\int_0^T |\zeta(s)|^2_X \, ds\bigg] < +\infty.$$

We also denote by $C^2_{\mathcal{P}}(0,T;X)$ the space of all processes $\zeta \in M^2_{\mathcal{P}}(0,T;X)$ such that $\zeta \in C([0,T], L^2(\Omega, \mathcal{F}, \mathbb{P}; X))$.

As it is well known (see for instance [4] for the proof in a much more general situation) for any initial data $x \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and any control $u \in M^2_{\mathcal{P}}(0,T;H)$ there exists a unique mild solution $y \in C^2_{\mathcal{P}}(0,T;H)$ of (2.1) (the definition of mild solution is standard, see [4]). When needed, we will denote the solution by $y(\,\cdot\,, x, u)$.

We now introduce the dual state p corresponding to system (2.1). It turns out that this is the solution to the backward stochastic differential equation:

$$\begin{cases} d_t p(t) = \Big[-\Big(A^* p(t) + \displaystyle\sum_{i=1}^{\infty} C_i^* q_{1,i}(t) \Big) + v \Big]\, dt \\ \qquad\qquad + \displaystyle\sum_{i=1}^{\infty} q_{1,i}(t) d\beta_{1,i}(t) + \sum_{j=1}^{\infty} q_{2,j}(t) d\beta_{2,j}(t), \\ p(T) = \xi, \end{cases} \tag{2.2}$$

where $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$ and $v \in M^2_{\mathcal{P}}(0,T;H)$. In order to be able to solve the backward SDE, we assume that $\mathcal{F}_t$ is the smallest filtration which satisfies the usual conditions and such that all Brownian motions are adapted. More precisely, we assume that $\mathcal{F}_t = \overline{\mathcal{F}^o_t}$, where $\mathcal{F}^o_t = \sigma\{\beta_{1,i}(s), \beta_{2,j}(s) : s \in [0,t],\ i = 1,2,\ldots,\ j = 1,2,\ldots\}$. We have to emphasize here that $L^2(\Omega, \mathcal{F}_0, \mathbb{P}, H) = H$, since $\mathcal{F}_0$ is trivial. Proceeding exactly as in [8] we get that for all $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$ and $v \in M^2_{\mathcal{P}}(0,T;H)$ there exists a unique mild solution

$$(p(\xi,v), q(\xi,v)) = \{p, q_{1,i}, q_{2,j} : i = 1,2,\ldots,\ j = 1,2,\ldots\}$$

of (2.2) with $p \in C^2_{\mathcal{P}}(0,T;H)$, $q_{1,i} \in M^2_{\mathcal{P}}(0,T;H)$ for $i = 1,2,\ldots$, $q_{2,j} \in M^2_{\mathcal{P}}(0,T;H)$ for $j = 1,2,\ldots$ and

$$\sum_{i=1}^{\infty} \mathbb{E}\bigg[\int_0^T |q_{1,i}(s)|^2\,ds\bigg] + \sum_{j=1}^{\infty} \mathbb{E}\bigg[\int_0^T |q_{2,j}(s)|^2\,ds\bigg] < +\infty.$$

We remark that the infinite dimensionality of the noise does not cause any problems because of the assumption

$$\sum_{i=1}^{\infty} |C_i|^2_{\mathcal{L}(H)} < +\infty.$$

Lemma 2.1. *Fix $x \in H$, $u \in M^2_{\mathcal{P}}(0,T;U)$ and $y \in C^2_{\mathcal{P}}(0,T;H)$. Then y is the solution of the state equation (2.1), i.e. $y = y(x,u)$ if and only if, for any $\xi \in L^2(\mathcal{F}_T, H)$ and $v \in M^2_{\mathcal{P}}(0,T;H)$, we have $Q(y,u,p,v,q) = 0$, where*

$$(p,q) = (p(\xi,v), q(\xi,v))$$

is the unique solution of the BSDE (2.2) *and Q is defined by*

$$\begin{aligned}(2.3)\quad Q(y,u,p,v,q) &\triangleq \mathbb{E}\big[\langle y(T),p(T)\rangle\big] - \langle x,p(0)\rangle \\ &- \mathbb{E}\bigg[\int_0^T \bigg(\langle y(t),v(t)\rangle + \bigg\langle u(t), B^*p(t) + \sum_{j=1}^{\infty} D_j^* q_{2,j}(t)\bigg\rangle \\ &\qquad + \sum_{i=1}^{\infty}\langle \sigma_{1,i}, q_{1,i}(t)\rangle + \sum_{j=1}^{\infty}\langle \sigma_{2,j}, q_{2,j}(t)\rangle\bigg)\,dt\bigg].\end{aligned}$$

Proof. The proof is an easy exercise involving Itô's lemma for the product $\langle y(t),p(t)\rangle$. We note that, since the filtration is the enlarged filtration generated by $\{\beta_{1,i}(s),\beta_{2,j}\}$, then $p(0)$ is deterministic. □

Remark 2.2. If $\sigma_{1,i} = 0$, $i = 1,2,\ldots$ and $\sigma_{2,j} = 0$, $j = 1,2,\ldots$, we can define the linear bounded operator

$$U\colon H \times M^2_{\mathcal{P}}(0,T;U) \to L^2(\mathcal{F}_T,H) \times M^2_{\mathcal{P}}(0,T;H)$$

by

$$U(x,u(\,\cdot\,)) = (y(T), -y(\,\cdot\,)).$$

The previous lemma ensures that the adjoint

$$U^*\colon L^2(\mathcal{F}_T,H) \times M^2_{\mathcal{P}}(0,T;H) \to H \times M^2_{\mathcal{P}}(0,T;U)$$

is represented by

$$U^*(\xi,v) = \bigg(p(0), B^*p(\,\cdot\,) + \sum_{j=1}^{\infty} D_j^* q_{2,j}(\,\cdot\,)\bigg),$$

where (p,q) is the solution of (2.2).

3. Duality for value functions in linear-convex control problems

Consider the convex, proper and lower semi-continuous functions

$$h\colon U \to (-\infty, \infty], \quad g, l\colon H \to (-\infty, \infty],$$

with the additional assumption that h is coercive. More precisely, we assume that there exists $k > 0$ such that $h(u) \geq k|u|^2$ for $u \in U$.

Consider now the stochastic control problem of minimizing

$$\mathcal{M}^2_{\mathcal{P}}(0,T,U) \ni u \to \mathbb{E}\bigg[\int_0^T (h(u(s)) + g(y(s)))\,ds + l(y(T))\bigg], \tag{3.1}$$

where y is the solution to (2.1). We denote by φ the value function. More precisely,

$$\varphi(0,x) = \min_u \mathbb{E}\bigg[\int_0^T (h(u(s)) + g(y(s)))\,ds + l(y(T))\bigg]. \tag{3.2}$$

We define the value function for time $t = 0$ only in order to avoid the discussion on filtrations. The assumptions at hand ensure that the optimal control problem is well posed.

LEMMA 3.1. *The optimization problem* (3.1) *has a minimizer* $u^* \in \mathcal{M}_{\mathcal{P}}(0,T,U)$. *If the function h is strictly convex then the minimizer is unique.*

PROOF. The function

$$\mathcal{M}^2_{\mathcal{P}}(0,T,U) \ni u \to \mathbb{E}\bigg[\int_0^T (h(u(s)) + g(y(s)))\,ds + l(y(T))\bigg]$$

is convex, lower semi-continuous and coercive, therefore admits a minimizer. The minimizer is unique if the function above is strictly convex. This is exactly the case if h is strictly convex, so the proof is complete. $\square$

The next result is a set of necessary (and sufficient, because of convexity) conditions, i.e. a stochastic maximum principle.

PROPOSITION 3.2. *Let* $u^* \in \mathcal{M}^2_{\mathcal{P}}(0,T,U)$ *be a minimizer in* (3.1). *Then, there exist* $(\xi^*, v^*) \in L^2(\mathcal{F}_T, H) \times \mathcal{M}_{\mathcal{P}}(0,T,H)$ *such that*

$$\begin{cases} \xi^* \in \partial l(y^*(T)), \\ B^* p^*(t) + \displaystyle\sum_{j=1}^{\infty} D_j^* q_{2,j}^*(t) \in -\partial h(u^*(t)) & \text{for } 0 \leq t \leq T, \\ v^*(t) \in -\partial g(y^*(t)) & \text{for } 0 \leq t \leq T, \end{cases} \tag{3.3}$$

where (p^*, q^*) *solve* (2.2).

PROOF. If l and h are smooth the proof is standard and immediate. Otherwise, we use the adapted penalty method used in Barbu and Precupanu [2] in the deterministic case. We omit the details. $\square$

According to Lemma 2.1 the state equation (2.1), together with its initial condition can be written as a set of linear constraints. We therefore define the infinite dimensional Lagrangian

$$L(y,u,p,v,q) \triangleq \mathbb{E}\bigg[\int_0^T (h(u(s)) + g(y(s)))\,ds + l(y(T))\bigg] - Q(y,u,p,v,q),$$

where $Q(y,u,p,v,q)$ was defined in (2.3).

The next results is a minimax theorem. We would like to point out that its proof is straightforward, as long as we have Lemma 3.1 and Proposition 3.2. We therefore avoid a direct proof of the minimax theorem. Since compactness or coercivity are missing, one cannot simply cite a classical result for the minimax.

Proposition 3.3. *Using the notation above, we have the minimax relation*

$$\varphi(0,x) = \min_{y,u}\max_{\xi,v} L(y,u,p(\xi,v),v,q(\xi,v))) = \max_{\xi,v}\min_{y,u} L(y,u,p(\xi,v),v,q(\xi,v)).$$

Proof. The proof is almost trivial as soon as we observe that Lemma 3.1 and Proposition 3.2 provide a saddle point for the Lagrangian. More precisely, using the pairs (u^*,y^*) and (ξ^*,v^*) we have

$$L(y,u,p^*,v^*,q^*) \geq L(y^*,u^*,p^*,v^*,q^*) \geq L(y^*,u^*,p(\xi,v),v,q(\xi,v))$$

for any $(u,y) \in \mathcal{M}_P^2(0,T,U)\times\mathcal{C}_P^2(0,T,H)$ and $(\xi,v) \in L^2(\mathcal{F}_T,H)\times\mathcal{M}_P^2(0,T,H)$. The first inequality follows from the properties of the sub-differentials together with relations (3.3) and the definition of the Lagrangian L. The second inequality is actually an equality, and follows from Lemma 2.1, since

$$Q(y^*,u^*,p^*,v^*,q^*) = Q(y^*,u^*,p(\xi,v),v,q(\xi,v)) = 0. \qquad \square$$

The whole point of Proposition 3.3 is to consider the representation

$$\varphi(0,x) = \max_{\xi,v}\min_{y,u} L(y,u,p(\xi,v),v,q(\xi,v)),$$

of the value function. Once $(\xi,v) \in L^2(\mathcal{F}_T,H)\times\mathcal{M}_{\mathcal{P}}^2(0,T;H)$ (therefore (p,v,q)) are fixed, the Lagrangian can now be optimized *point-wise in time* over y,u to obtain

$$\begin{aligned}\min_{y,u} L(y,u,p,v,q) &= \langle x, p(0)\rangle \\ &\quad - \mathbb{E}\bigg[\int_0^T \bigg\{h^*\bigg(-B^*p(t) - \sum_{j=1}^{\infty} D_j^* q_{2,j}(t)\bigg) + g^*(-v(s)) \\ &\qquad - \bigg(\sum_{i=1}^{\infty}\langle \sigma_{1,i}, q_{1,i}(t)\rangle + \sum_{j=1}^{\infty}\langle \sigma_{2,j}, q_{2,j}(t)\rangle\bigg)\bigg\}\,ds + l^*(p(T))\bigg].\end{aligned}$$

Using approximations, if needed, it is clear that the point-wise optimization of the Lagrangian above coincides with the optimization over $(u,y) \in \mathcal{M}_{\mathcal{P}}^2(0,T;U)$

$\times\mathcal{C}^2_{\mathcal{P}}(0,T;H)$ for *fixed* (ξ, v). Therefore we have proved the following dual representation result:

Theorem 3.4. *Define the convex function* $L^2(\mathcal{F}_T, H) \ni \xi \to \psi(0,\xi)$ *by*

$$\psi(0,\xi) \triangleq \min_{v\in\mathcal{M}^2_{\mathcal{P}}(0,T;H)} E\Big[\int_0^T \Big\{h^*\Big(-B^*p(t) - \sum_{j=1}^{\infty} D_j^* q_{2,j}(t)\Big) + g^*(-v(s)) - \Big(\sum_{i=1}^{\infty}\langle\sigma_{1,i}, q_{1,i}(t)\rangle + \sum_{j=1}^{\infty}\langle\sigma_{2,j}, q_{2,j}(t)\rangle\Big)\Big\}\,ds\Big] - \langle x, p(0)\rangle.$$

Then

$$\varphi(0,x) = \max_{\xi\in L^2(\mathcal{F}_T,H)} \{-\psi(0,\xi) - l^*(\xi)\}.$$

Remark 3.5. Theorem 3.4 is, indeed, a dual representation for $\varphi(0,\,\cdot\,)$. This corresponds to the well known Lax–Oleinik–Hopf-type representation of the value function in the deterministic case. We refer the reader to [1] for this classic result. In the stochastic case studied here, the function ψ depends on the random variable $\xi \in L^2(\mathcal{F}_T, H)$ rather then a deterministic dual state, and therefore it cannot satisfy a dual HJB equation.

3.1. Dual representation for groups. Assume now that A generates a group. The main observation is that, choosing (ξ, v) and solving equation (2.2) *backwards* is the same as choosing $p(0) = y$, and (v, q) and solving (2.2) *forward*. As before, we denoted here by $q = \{q_{1,i},\ q_{2,j} : i = 1,2,\ldots,\ j = 1,2,\ldots\}$ In other words, we have a bijective correspondence

$$\left.\begin{array}{r} p(T) = \xi \\ v \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} p(0) = y \\ v \\ q \end{array}\right.$$

Having this in mind, we can *fix* $p(0) = y \in H$ and use both v and q as *controls* for the *forward* controlled state equation (2.2). A similar idea was used in [7] for a linear quadratic problem related to the null-controllability of the state equation. With this observation, we can obtain the main result of the paper, namely:

Theorem 3.6. *Define the value function of the* (*classical*) *control problem*

$$\Psi(0,y) \triangleq \min_{v,q} \mathbb{E}\Big[\int_0^T \Big\{h^*\Big(-B^*p(t) - \sum_{j=1}^{\infty} D_j^* q_{2,j}(t)\Big) + g^*(-v(s)) - \Big(\sum_{i=1}^{\infty}\langle\sigma_{1,i}, q_{1,i}(t)\rangle + \sum_{j=1}^{\infty}\langle\sigma_{2,j}, q_{2,j}(t)\rangle\Big)\Big\}\,ds + l^*(p(T))\Big],$$

subject to (2.2)*, for each* $y \in H$*. Then*

$$\varphi(0,x) = \max_{y\in H}\{\langle x,y\rangle - \Psi(0,y)\} = \Psi^*(0,x), \quad \textit{for all } \ x \in H.$$

While the value functions φ and Ψ were only defined at time $t = 0$ for convenience, they can be defined at all later times $0 \le t < T$, provided we make the appropriate modifications for the filtrations. In this case, both φ and Ψ satisfy the appropriate Hamilton–Jacobi–Bellman equations, at least in an appropriate weak sense (usually viscosity). We obtain the HJB equation for the primal problem applying Itô to $\varphi(t,y(t))$:

$$\begin{aligned}
&d(\varphi(t,y(t)) \ + \text{“running cost”})\\
&\qquad = \big\{\varphi_t + \langle Ay(t) + Bu(t), \varphi_x\rangle + h(u(t)) + g(y(t))\big\}\,dt\\
&\qquad\quad + \frac{1}{2}\bigg(\sum_{i=1}^{\infty}\big\langle \varphi_{xx}\big(C_iy(t)+\sigma_{1,i}\big), \big(C_iy(t)+\sigma_{1,i}\big)\big\rangle\\
&\qquad\quad + \sum_{j=1}^{\infty}\big\langle \varphi_{xx}\big(D_ju(t)+\sigma_{2,j}\big), \big(D_ju(t)+\sigma_{2,j}\big)\big\rangle\bigg)dt\\
&\qquad\quad + \text{“martingale terms”}
\end{aligned}$$

and then minimizing over u:

$$\left\{\begin{aligned}
&\varphi_t + \langle Ax, \varphi_x\rangle - H^*(-B^*\varphi_x, \varphi_{xx})\\
&\qquad + \frac{1}{2}\bigg(\sum_{i=1}^{\infty}\big\langle \varphi_{xx}\big(C_ix+\sigma_{1,i}\big), \big(C_ix+\sigma_{1,i}\big)\big\rangle\bigg) + g(x) = 0,\\
&\varphi(T,x) = l(x), \quad x \in H.
\end{aligned}\right. \tag{3.4}$$

Here, for fixed P symmetric and non-negative the convex (in u) function H is defined by

$$H(u,P) \triangleq h(u) + \frac{1}{2}\bigg(\sum_{j=1}^{\infty}\big\langle P\big(D_ju+\sigma_{2,j}\big), \big(D_ju+\sigma_{2,j}\big)\big\rangle\bigg),$$

and $H^*(\,\cdot\,,P)$ is the Fenchel–Legendre transform of $H(\,\cdot\,,P)$. Similarly, we obtain the dual HJB for Ψ by applying Itô to $\Psi(t,p(t))$:

$$\begin{aligned}
&d(\Psi(t,p(t)) \ + \text{“running cost”}\\
&\qquad = \bigg\{\Psi_t + \bigg\langle\bigg[-\bigg(A^*p(t) + \sum_{i=1}^{\infty}C_i^*q_{1,i}(t)\bigg) + v\bigg], \Psi_p\bigg\rangle\bigg\}\,dt\\
&\qquad\quad + \bigg\{h^*\bigg(-B^*p(t) - \sum_{j=1}^{\infty}D_j^*q_{2,j}(t)\bigg) + g^*(-v(s))\\
&\qquad\quad - \bigg(\sum_{i=1}^{\infty}\langle\sigma_{1,i}, q_{1,i}(t)\rangle + \sum_{j=1}^{\infty}\langle\sigma_{2,j}, q_{2,j}(t)\rangle\bigg)\bigg\}\,dt
\end{aligned}$$

$$+\frac{1}{2}\bigg(\sum_{i=1}^{\infty}\big\langle\Psi_{pp}q_{1,i},q_{1,i}\big\rangle+\sum_{j=1}^{\infty}\big\langle\Psi_{pp}q_{2,j},q_{2,j}\big\rangle\bigg)\,dt$$
$$+\text{“martingale terms”}$$

and then taking the formal minimum over v and q. We therefore obtain the equation

$$\tag{3.5}\begin{cases}\Psi_t-\langle A^*p,\Psi_p\rangle+H^*(-B^*p,\Psi_{pp}^{-1})\\ \qquad -\dfrac{1}{2}\bigg(\displaystyle\sum_{i=1}^{\infty}\big\langle\Psi_{pp}^{-1}\big(C_i\Psi_p+\sigma_{1,i}\big),\big(C_i\Psi_p+\sigma_{1,i}\big)\big\rangle\bigg)-g(\Psi_p)=0,\\ \Psi(T,p)=l^*(p),\quad x\in H.\end{cases}$$

Instead of minimizing over (v,q) we can also (formally) obtain the dual HJB (3.5) by applying the transformation

$$p=\varphi_x,\quad x=\Psi_p,\quad \Psi_{pp}=\varphi_{xx}^{-1}.$$

3.2. Duality for solutions of Riccati equations in LQ control of groups. We consider here the LQ case, which amounts to $\sigma_{1,i}=0$, $\sigma_{2,j}=0$ and

$$h(u)=\frac{1}{2}\langle Eu,u\rangle,\quad g(y)=\frac{1}{2}\langle Sy,y\rangle,\quad l(y)=\frac{1}{2}\langle Ry,y\rangle$$

for some symmetric and non-negative operators E, S and R. We then have $\varphi(t,x)=\langle P(t)x,x\rangle/2$ and $\Psi(t,p)=\langle Q(t)p,p\rangle/2$, where (at least formally)

$$Q=P^{-1}.$$

Following [9], the HJB's for φ and Φ translate into Riccati equations for P and Q. More precisely, since $\varphi_x(t,x)=P(t)x$ and $\varphi_{xx}(t,x)=P(t)$ we have that

$$H(u,P)=\frac{1}{2}\bigg\langle\bigg[E+\sum_{j=1}^{\infty}D_j^*PD_j\bigg]u,u\bigg\rangle$$

so

$$H^*(w,P)=\frac{1}{2}\bigg\langle\bigg[E+\sum_{j=1}^{\infty}D_j^*PD_j\bigg]^{-1}w,w\bigg\rangle.$$

Therefore the HJB (3.4) can be rewritten as the Riccati equation

$$\tag{3.6}\begin{cases}P'+A^*P+PA\\ \qquad -PB\bigg[E+\displaystyle\sum_{j=1}^{\infty}D_j^*PD_j\bigg]^{-1}B^*P+\sum_{i=1}^{\infty}C_i^*PC_i+S=0,\\ P(T)=R.\end{cases}$$

The dual Riccati equation corresponding to the HJB (3.5) is now

$$
\begin{cases}
Q' - AQ - QA^* + B\left[E + \sum_{j=1}^{\infty} D_j^* Q^{-1} D_j\right]^{-1} B^* \\
\qquad\qquad -Q\left(\sum_{i=1}^{\infty} C_i^* Q^{-1} C_i\right)Q - QSQ = 0, \\
Q(T) = R^{-1}.
\end{cases}
\tag{3.7}
$$

This can be either obtained from (3.5) or from the transformation $Q = P^{-1}$ in (3.6).

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