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POINTWISE ESTIMATES IN THE FILIPPOV LEMMA AND FILIPPOV–WAŻEWSKI THEOREM FOR FOURTH ORDER DIFFERENTIAL INCLUSIONS

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Abstract. In this work we give a generalization of the Filippov–Ważewski Theorem to the fourth order differential inclusions in a separable complex Banach space $\mathbb{X}$

$$\mathcal{D}y = y'''' - (A^2 + B^2)y'' + A^2B^2y \in F(t, y),$$

with the initial conditions in $c \in [0, T]$

$$y(c) = \alpha, \qquad y'(c) = \beta, \qquad y''(c) = \gamma, \qquad y'''(c) = \delta, \tag{0.1}$$

We assume that the multifunction $F : [0, T] \times \mathbb{X} \rightsquigarrow c(\mathbb{X})$ is Lipschitz continuous in y with the integrable Lipschitz constant $l(\,\cdot\,)$, while $A^2, B^2 \in B(\mathbb{X})$ are the infinitesimal generators of two cosine families of operators. The main result is the following version of Filippov Lemma:

Theorem. *Let $y_0 \in W^{4,1} = W^{4,1}([0, T], \mathbb{X})$ be such function with* (0.1) *that*

$$\operatorname{dist}(\mathcal{D}y_0(t), F(t, y_0(t))) \le p_0(t) \quad \textit{a.e. in } [c, d] \subset [0, T],$$

where $p_0 \in L^1[0, T]$. Then there are σ_0 (depending on p_0) and φ such that for each $\varepsilon > 0$ there exists a solution $y \in W^{4,1}$ of the above problem such that almost everywhere in $t \in [c, d]$ we have $|\mathcal{D}y(t) - \mathcal{D}y_0(t)| \le \sigma_0(t)$,

$$|y(t) - y_0(t)| \le (\varphi *_c \sigma_0)(t), \qquad |y'(t) - y_0'(t)| \le (\varphi' *_c \sigma_0 t)(t),$$
$$|y''(t) - y_0''(t)| \le (\varphi'' *_c \sigma_0)(t) \qquad |y'''(t) - y_0'''(t)| \le (\varphi''' *_c \sigma_0)(t),$$

where $*_c$ stands for the convolution started at c.

Our estimates are constructive and more precise then those in the known versions of Filippov Lemma.

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1. Introduction

The theory of the existence and properties of solution sets of differential inclusions

$$\mathcal{D}y \in F(t, y), \tag{1.1}$$

where $\mathcal{D}$ is an ordinary differential operator, has been developed for many years and most of the facts have already been discovered. This concernes mainly the initial value problems. However, the boundary value problems still are not well examined. Recently we observe an increase of interest in this field, especially in the field of ordinary differential inclusions of higher order; in particular evolution inclusions [25], differential inclusions of the Sturm–Liouville type [3], the Schrödinger type [4], [5] and n-th order of the form $y^{(n)} - \lambda y \in F(t, y)$ in [11].

In this paper the attention is focused on the fourth order differential inclusions of the form

$$\mathcal{D}y = y'''' - (A^2 + B^2)y'' + A^2B^2y \in F(t, y) \tag{1.2}$$

in a separable complex Banach space $(\mathbb{X}, |\cdot|)$. Here $F\colon [0, T] \times \mathbb{X} \rightsquigarrow \mathbb{X}$ is a multifunction with the Lipschitz continuous right-hand side, i.e. there exists a positive integrable function $l(\,\cdot\,)$, such that for every $y_1, y_2 \in \mathbb{X}$ the inequality

$$\mathrm{d}_H(F(t, y_1), F(t, y_2)) \le l(t)|y_1 - y_2| \tag{1.3}$$

holds. Here A^2 and B^2 are the infinitesimal generators of two commuting cosine families of operators (see [30]). For (1.2) we impose the following initial conditions (IC)

$$y(a) = \alpha, \qquad y'(a) = \beta, \qquad y''(a) = \gamma, \qquad y'''(a) = \delta, \tag{1.4}$$

where $a \in [0, T]$ and $\alpha, \beta, \gamma, \delta \in \mathbb{X}$. The operator $\mathcal{D}$ represents an abstract form of complex beam differential operator $\mathcal{D}_0 y = y'''' - (a^2 + b^2)y'' + a^2b^2y$, where $a, b \in \mathbb{C}$. On the other hand $\mathcal{D}$ is a formal composition of two different elliptic operators $\mathcal{D}_A y = -y'' + A^2y$ and $\mathcal{D}_B y = -y'' + B^2y$, namely, $\mathcal{D} = \mathcal{D}_A \circ \mathcal{D}_B$. By $L^1 = L^1([0, T], \mathbb{X})$ we mean the space of Bochner integrable functions $u\colon [0, T] \to \mathbb{X}$ with the usual norm $\|u\|_1 = \int_I |u(t)|\,dt$, while $W^{s,1} = W^{s,1}([0, T], \mathbb{X})$, s is a positive integer, stands for the space of $u \in L^1$ such that all weak derivatives $u^{(i)} = d^iu/dt^i$, $i = 1, \ldots, s$ belong to L^1. In the space $AC = W^{1,1}$ we use the representation $u(t) - u(c) = \int_c^t u'(s)\,ds$ for almost all $t, c \in [0, T]$.

By a solution of (1.2) with the initial conditions (1.4) we mean a function $y \in W^{4,1}$ satisfying (1.2) and (1.4).

Our aim is to obtain individual pointwise estimates in the Filippov type lemma for complex beam inclusions (1.2) from the qualitative and quantitative point of view. We apply the obtained results to the Filippov–Ważewski density result of the solution set of the problem (1.2) and (1.4) in the solution set of the so-called relaxed problem

$$\mathcal{D}y \in \operatorname{clco} F(t, y), \tag{1.5}$$

with (1.4), where $\operatorname{clco} S$ stands for the closed convex hull of a set S. The density is understood in terms of the uniform topology for the function y and its derivatives y', y'' and y'''. Let us also mention that analogues of the Filippov type Lemma (see cf. [1], [2], [4], [6]–[10], [12]–[17], [20]–[22], [26]–[29], [31]) play a crucial role in the usual proofs of relaxation (density) results ([14], [26], [23], [18]).

In the presented version of the Filippov Lemma for (1.2) with (1.4) we assume that infinitesimal the generators A^2 and B^2 are linear bounded operators. This leads to better (smaller) estimates for possible solutions and its derivatives then those in Frankowska [14] and Papageorgiou [26]. Our estimates are expressed in terms of $\|A\|$, $\|B\|$ and l. The methods used are based on the differential equation (see [19]) $\mathcal{D}y = f$, where $f \in L^1 = L^1([0,T], \mathbb{X})$. We present them in Section 2, while in Section 4 we formulate and demonstrate an analogue of the Filippov Lemma for the IVP (1.2) with IC's (1.4). In Section 3 we show a lemma concerning recursive Gronwall inequalities, while in Section 5 we present pointwise estimates in the Filippov–Ważewski Relaxation Theorem which are based on our more precise Filippov Lemma.

Preliminaries. Let $\mathbb{X}$ be a separable complex Banach space with the norm $|\cdot|$. For $I = [c,d]$ the Banach space of Lebesgue integrable functions $u\colon I \to \mathbb{X}$ with the norm $\|u\|_1 = \int\limits_I |u(t)| dt$ is denoted by $L^1(I, \mathbb{X})$. Let $B(\mathbb{X})$ be the Banach algebra of linear bounded operators in $\mathbb{X}$ with the usual norm

$$\|A\| = \sup\{|Ax| : |x| \le 1\}.$$

Consider a family $\{A(t) : t \in \mathbb{R}\} \subset B(\mathbb{X})$ of operators with the following properties:

1. for each $x \in \mathbb{X}$ the map $t \to A(t)x$ is strongly Lebesgue measurable on $\mathbb{R}$;
2. the map $t \to \|A(t)\|$ is locally essentially bounded.

For such family by the primitive for $A(t)$ on $\mathbb{R}$ we mean an operator-valued function $F(t)$ given by

$$F(t)x = \int_c^t [A(s)x]\, ds.$$

For given $u \in L^1(I, \mathbb{X})$ by $A *_c u$ we denote the convolution operator

$$(A *_c u)(t) = \int_c^t A(t-s)u(s)\, ds.$$

Observe that $A *_c u \in L^1(I, \mathbb{X})$. Moreover, if $F(t)$ is the primitive for $A(t)$ then the convolution $F *_c u \in AC$ and for almost all $t \in I$

$$(F *_c u)'(t) = (A *_c u)(t) + F(0)u(t).$$

In particular, if $F(0) = 0$ then for almost all $t \in I$

$$(F *_c u)'(t) = (A *_c u)(t). \tag{1.6}$$

Recall that the set of operators $\{C(t) : t \in \mathbb{R}\} \subset B(\mathbb{X})$ is called *the strongly continuous cosine family* if

(a) $C(0) = I$;
(b) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$;
(c) the map $t \to C(t)x$ is strongly continuous for all $t \in \mathbb{R}$ and $x \in \mathbb{X}$.

Remark 1.1. The conditions (a)–(c) are equivalent to (a′), (b), (c′), where

(a′) there is t_0 in which $C(t_0)$ is invertible in $B(\mathbb{X})$;
(c′) the map $t \to C(t)x$ is strongly Lebesgue measurable.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$ associated with a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by $S(t)x = \int_0^t C(z)x\,dz$. The infinitesimal generator $A^2 \colon \mathbb{X} \to \mathbb{X}$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by $A^2x = \frac{d^2}{dt^2}C(t)x\big|_{t=0}$. Our notation A^2 is justified by the fact that for $\mathbb{X}$, being a Hilbert space, the positive, self-adjoint operator $\frac{d^2}{dt^2}C(t)x\big|_{t=0}$ possess the square root. Observe that for any $A \in B(\mathbb{X})$

$$C_A(t) = \cosh(At) = I + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\, A^{2n} \tag{1.7}$$

is the strongly continuous cosine family $\{C_A(t) : t \in \mathbb{R}\}$. Note, that this family is generated by the operator A^2.

Thus, if $A \in \mathrm{GL}(\mathbb{X})$ (i.e. has the bounded inverse), we have the representation

$$S_A(t) = A^{-1}\sinh(At) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\, A^{2n}. \tag{1.8}$$

More detailes on cosine and sine families of operators can be found in the paper by Travis and Webb [30]. We restrict ourselves to bounded generators and at the present moment we can not avoid this assumption. Similar difficulties were met by other authors, e.g. [11].

2. Some ODE's of higher order

2.1. An IVP of second order. Consider the differential equation

$$\mathcal{D}_A y = y'' - A^2 y = f, \tag{2.1}$$

where $A^2 \in B(\mathbb{X})$ is the generator of a cosine family of operators $\{C_A(t)\}_{t\in\mathbb{R}}$ given by (1.7). Let $S_A(t)x = \int_0^t C_A(x)\,dx$. By a mild solution of (2.1) on $I = [c,d\,]$ with the initial conditions

$$y(c) = \alpha, \qquad y'(c) = \beta \tag{2.2}$$

we mean a continuous function $y \in C\,[c,d]$ satisfying, for each $t \in I$, the relation

$$y(t) = C_A(t-c)\alpha + S_A(t-c)\beta + (S_A *_c f)(t).$$

Denote $\mathcal{R}_A f = S_A *_c f$ and

$$y_A(t) = C_A(t-c)\alpha + S_A(t-c)\beta. \tag{2.3}$$

Observe that $\mathcal{R}_A f$ is a solution of (2.1) with the zero initial conditions

$$y(c) = 0, \qquad y'(c) = 0, \tag{2.4}$$

while $y_A(t)$ solves $\mathcal{D}_A y = 0$ with (2.2). Moreover, we have a representation of the solution for (2.1) with $IC's$ (2.2) of the form $y(t) = (\mathcal{R}_A f)(t) + y_A(t)$. In what follows we shall assume that $[a,b] \subset [0,T]$. Observe that for any $f \in L^1([a,b],\mathbb{X})$ we have for all $t \in [a,b]$ the inequality

$$|A^{-1}(\sinh(Ax) *_a f)(t)| \leq \bigg(\frac{\sinh(\|A\|x)}{\|A\|} *_a |f|\bigg)(t). \tag{2.5}$$

As a consequence of the estimate (2.5) one can derive that

Proposition 2.1. $|\mathcal{R}_A u - \mathcal{R}_A v| \leq \mathcal{R}_{\|A\|}|u - vt|$.

2.2. An IVP of fourth order.

2.2.1. General case. Consider the fourth order differential equation on I

$$\mathcal{D}y = \mathcal{D}_{AB}y = y'''' - (A^2 + B^2)y'' + A^2B^2y = f \tag{2.6}$$

with the initial conditions

$$y(c) = \alpha, \qquad y'(c) = \beta, \qquad y''(c) = \gamma, \qquad y'''(c) = \delta. \tag{2.7}$$

We shall assume that the commutative operators $A^2, B^2 \in B(\mathbb{X})$ are the generators of two cosine families of operators $\{C_A(t) : t \in \mathbb{R}\}$ and $\{C_B(t) : t \in \mathbb{R}\}$ and $(A^2 - B^2) \in \mathrm{GL}(\mathbb{X})$. Our task is to find a representation of solutions of the above problem by the use of solutions of $\mathcal{D}y = 0$ with (2.7) and $\mathcal{D}y = f$ with the initial conditions

$$y(c) = 0, \qquad y'(c) = 0, \qquad y''(c) = 0, \qquad y'''(c) = 0. \tag{2.8}$$

We begin with the following observation based on Travis and Webb [30]:

Proposition 2.2. *Assume that the generators A^2 and B^2 are commutative with $(A^2 - B^2) \in \mathrm{GL}(\mathbb{X})$. Let $u, v \in W^{2,1}$ be solutions of the equations*

$$u'' = A^2 u + f \quad \text{and} \quad v'' = B^2 v + f',$$

respectively. Then

(a) *the function $y = (A^2 - B^2)^{-1}(u - v)$ solves (2.6). Moreover,*

$$y' = (A^2 - B^2)^{-1}(u' - v'), \qquad y'' = (A^2 - B^2)^{-1}(A^2 u - B^2 v),$$
$$y''' = (A^2 - B^2)^{-1}(A^2 u' - B^2 v'), \qquad y'''' = (A^2 - B^2)^{-1}(A^2 u'' - B^2 v'').$$

(b) *If u and v satisfy the IC's*

$$u(c) = u'(c) = 0 \quad \text{and} \quad v(c) = v'(c) = 0$$

then

$$y = (A^2 - B^2)^{-1}(u - v) = (A^2 - B^2)^{-1}((S_A - S_B) *_c f)$$

is a solution of $\mathcal{D}y = f$ with the IVP's (2.8). Moreover,

$$\begin{aligned} y' &= (A^2 - B^2)^{-1}((C_A - C_B) *_c f), \\ y'' &= (A^2 - B^2)^{-1}(A^2 S_A - B^2 S_B) *_c f, \\ y''' &= (A^2 - B^2)^{-1}(A^2 C_A - B^2 C_B) * f, \\ y'''' &= (A^2 - B^2)^{-1}(A^4 S_A - B^4 C_B) * f + f. \end{aligned}$$

(c) *If u and v satisfy the* IC*'s*

$$u(c) = \gamma - B^2\alpha, \qquad u'(c) = \delta - B^2\beta$$

and

$$v(c) = \gamma - A^2\alpha, \qquad v'(c) = \delta - A^2\beta$$

then $y = (A^2 - B^2)^{-1}(u - v)$ is a solution of $\mathcal{D}y = f$ with the IVP*'s (2.7).*

Proof. (a) Observe that

$$y'' = (A^2 - B^2)^{-1}(u'' - v'') = (A^2 - B^2)^{-1}(A^2 u - B^2 v) \in W^{2,1}.$$

Hence $y \in W^{4,1}$ and

$$y'''' = (A^2 - B^2)^{-1}(A^2 u'' - B^2 v'') = (A^2 - B^2)^{-1}(A^4 u - B^4 v) + f.$$

Therefore we end up with

$$\mathcal{D}y = y'''' - (A^2 + B^2)y'' + A^2 B^2 y = f.$$

(b) In this case we have $u = S_A *_c f$ and $v = S_B *_c f$. Take

$$y = (A^2 - B^2)^{-1}(u - v) = (A^2 - B^2)^{-1}((S_A - S_B) *_c f).$$

Since

$$(A^2 - B^2)^{-1}(S_A - S_B)(t) = S(t) = \int_0^t (A^2 - B^2)^{-1}(C_A(z) - C_B(z))\, dz$$

is the primitive of $(A^2-B^2)^{-1}(C_A-C_B)$ vanishing at 0 then by (1.6)

$$y' = (A^2-B^2)^{-1}((C_A-C_B)*_c f),$$

while

$$y'' = u''-v'' = (A^2u-B^2v) = (A^2-B^2)^{-1}(A^2S_A-B^2S_B)*_c f.$$

Similarly

$$\begin{aligned} y''' &= (A^2-B^2)^{-1}(A^2C_A-B^2C_B)*_c f,\\ y'''' &= (A^2-B^2)^{-1}(A^4S_A-B^4S_B)*_c f+f. \end{aligned}$$

Therefore one can again see that

$$y''''-(A^2+B^2)y''+A^2B^2y=f.$$

(c) According to the part (a) we need only to check that y satisfies the IC's. Doing that we see that

$$\begin{aligned} y(c) &= (A^2-B^2)^{-1}(u(c)-v(c))\\ &= (A^2-B^2)^{-1}((\gamma-B^2\alpha)-(\gamma-A^2\alpha))=\alpha,\\ y'(c) &= (A^2-B^2)^{-1}(u'(c)-v'(c))\\ &= (A^2-B^2)^{-1}((\delta-B^2\beta)-(\delta-A^2\beta))=\beta,\\ y''(c) &= (A^2-B^2)^{-1}(A^2u(c)-B^2v(c))\\ &= (A^2-B^2)^{-1}(A^2(\gamma-B^2\alpha)-B^2(\gamma-A^2\alpha))=\gamma,\\ y'''(c) &= (A^2-B^2)^{-1}(A^2u'(c)-B^2v'(c))\\ &= (A^2-B^2)^{-1}(A^2(\delta-B^2\beta)-B^2(\delta-A^2\beta))=\delta. \end{aligned}$$

This completes the proof. □

Let us denote

$$S_{AB} = (A^2-B^2)^{-1}(S_A-S_B) \tag{2.9}$$

$$C_{AB} = (A^2-B^2)^{-1}(C_A-C_B) \tag{2.10}$$

and

$$\begin{aligned} (\mathcal{R}_{AB}f)(t) &= (A^2-B^2)^{-1}(\mathcal{R}_Af-\mathcal{R}_Bf)((t))\\ &= (S_{AB}*_c f)(t) = \int_c^t S_{AB}(t-z)f(z)\,dz. \end{aligned} \tag{2.11}$$

Corollary 2.3. *Observe that $\mathcal{R}_{AB}f = S_{AB}*_c f$ solves the equation $\mathcal{D}y=f$ with the* IVP*'s* (2.8), *while the equation $\mathcal{D}y=0$ with the* IVP*'s* (2.7) *possesses* *the solution*

$$\begin{aligned}(2.12)\quad y_{AB}(t) &= (C_{AB}(t-c)\gamma + S_{AB}(t-c)\delta \\ &+ (C''_{AB} - (A^2+B^2)C_{AB})(t-c)\alpha + (S''_{AB} - (A^2+B^2)S_{AB})(t-c)\beta).\end{aligned}$$

Proof. We need to show the second part only. To see this observe that, by Section 2.1, the equation $u'' = A^2u$ with IVP's

$$u(c) = \gamma - B^2\alpha, \qquad u'(c) = \delta - B^2\beta$$

possesses the solution

$$u(t) = C_A(t-c)(\gamma - B^2\alpha) + S_A(t-c)(\delta - B^2\beta).$$

Similarly, a solution of $v'' = A^2v$ with conditions

$$v(c) = \gamma - A^2\alpha, \qquad v'(c) = \delta - A^2\beta$$

is given by

$$v(t) = C_B(t-c)(\gamma - A^2\alpha) + S_B(t-c)(\delta - A^2\beta).$$

Therefore, by Proposition 2.2 (c) we have

$$\begin{aligned}y_{AB}(t) &= (A^2-B^2)^{-1}(u(t)-v(t)) \\ &= (A^2-B^2)^{-1}(C_A(t-c)(\gamma - B^2\alpha) + S_A(t-c)(\delta - B^2\beta) \\ &\quad - C_B(t-c)(\gamma - A^2\alpha) - S_B(t-c)(\delta - A^2\beta)).\end{aligned}$$

By simplifying we have

$$\begin{aligned}y_{AB}(t) &= (C_{AB}(t-c)\gamma + S_{AB}(t-c)\delta \\ &+ (C''_{AB} - (A^2+B^2)C_{AB})(t-c)\alpha + (S''_{AB} - (A^2+B^2)S_{AB})(t-c)\beta). \quad \square\end{aligned}$$

Now we have the proper tools to obtain a required representation.

Theorem 2.4. *Let $A, B \in B(\mathbb{X})$ be commutative operators with $(A^2-B^2)^{-1}$ in $B(\mathbb{X})$. Then for each $f \in L^1(I,\mathbb{X})$ the equation* (2.6) *with the initial conditions* (2.7) *possesses the solution in the form*

$$y = y_{AB} + \mathcal{R}_{AB}f,$$

where $y_{AB}(t)$ is given by (2.12) *and $\mathcal{R}_{AB}f$ by* (2.11).

Moreover,

$$\begin{aligned}(\mathcal{R}_{AB}f)' &= (A^2-B^2)^{-1}((C_A - C_B) *_c f) = (S'_{AB} *_c f), \\ (\mathcal{R}_{AB}f)'' &= (A^2-B^2)^{-1}(A^2S_A - B^2S_B) *_c f = (S''_{AB} *_c f), \\ (\mathcal{R}_{AB}f)''' &= (A^2-B^2)^{-1}(A^2C_A - B^2C_B) *_c f = (S'''_{AB} *_c f).\end{aligned}$$

Remark 2.5. The function $S_{AB} = (A^2-B^2)^{-1}(S_A - S_B)$ plays the role of the Cauchy solution of $\mathcal{D}y = 0$.

EXAMPLE 2.6. Let $\mathbb{C}^2_\infty$ be the plane $\mathbb{C}^2$ consisting of the vectors $Y = [y_1, y_2]^T$, endowed with the norm $|Y| = \max(|y_1|, |y_2|)$. Let

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and take $B = 2A$. Then $A^2 + B^2 = -5J$ and $A^2B^2 = 4J$.

Consider the system

$$\mathcal{D}Y = Y'''' + 5JY'' + 4JY = F, \tag{2.13}$$

with conditions (2.8),

$$Y(0) = 0, \qquad Y'(0) = 0, \qquad Y''(0) = 0, \qquad Y'''(0) = 0, \tag{2.14}$$

where $F \in L^1([0,1], \mathbb{C}^2_\infty)$. The solution of the above problem is

$$Y = \frac{1}{3}((S_A - S_{2A}) * F) = \frac{1}{6}((\sinh(2x) - 2\sinh(x))J * F).$$

Indeed, applying the Jordan form

$$A = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1/2 & i/2 \\ 1/2 & -i/22 \end{bmatrix}$$

one can evaluate that

$$S_A - S_{2A} = \frac{1}{6}(\sinh(2x) - 2\sinh(x))J$$

Hence

$$Y = \frac{1}{6}\int_0^t ((\sinh 2x - 2\sinh x)F(t-x))\,dx.$$

In particular for $F = [18\sin t, 12]^T$ we have the solution

$$\overline{Y} = \overline{Y}(t) = [3t\cos t - \sin 2t - \sin t, 4\cos t - \cos 2t - 3]^T.$$

EXAMPLE 2.7. Denote by $\boldsymbol{m}$ the Banach space consisting of the bounded complex sequences $y = \{y_n\} = \{y_1, y_2, \ldots\}$ endowed with the norm $\|y\| = \sup|y_n|$.

Let $\boldsymbol{l}^2 \subset \boldsymbol{m}$ consists of such sequences that $\sum\limits_{n=1}^{\infty} |y_n|^2 < \infty$. $\boldsymbol{l}^2$ is a Hilbert space endowed with the scalar product $\langle y, z\rangle = \sum\limits_{n=1}^{\infty} y_n\overline{z_n}$.

Let $Ay = \{y_2, -y_1, \ldots, y_{2n}, -y_{2n-1}, \ldots\} \colon \boldsymbol{l}^2 \to \boldsymbol{l}^2$ and $B = 2A$. Then $A^2 = -J$, $A^2 + B^2 = -5J$, $A^2B^2 = 4J$, $(A^2 - B^2)^{-1} = J/3$, where $Jy = y$.

Consider in $\boldsymbol{l}^2$ the system

$$\mathcal{D}y = y'''' - (A^2 + B^2)y'' + A^2B^2y = y'''' + 5y'' + 4y = \left\{\frac{24x}{n}\right\} \in L^1([0,1], \boldsymbol{l}^2)$$

with conditions 2.8. The solution of the above problem is

$$\overline{y} = \frac{1}{3}\left((S_A - S_{2A}) * \left\{\frac{24x}{n}\right\}\right) = 4\left\{\frac{(2\sin(x) - \sin(2x)) * x}{n}\right\}.$$

Hence $\overline{y} = \overline{y}(x) = \{(6x + \sin 2x - 8\sin x)/n\}$. Indeed, we have

$$S_A(x) = A^{-1}\sinh(xA) = A^{-1}\sum_{n=0}^{\infty}\frac{(xA)^{2n+1}}{(2n+1)!} = \sin(x)J,$$

$$S_B(x) = S_{2A}(x) = (2A)^{-1}\sinh(2xA) = \frac{1}{2}\sin(2x)J,$$

$$(S_A - S_{2A})(x) = \frac{1}{6}(2\sin(x) - \sin(2x))J.$$

2.2.2. Real case. For $\mathbb{X} = \mathbb{R}$ we have a particular situation. We have a fourth order differential equation on I

$$\mathcal{D}_0 y = y'''' - (a^2 + b^2)y'' + a^2b^2y = f \tag{2.15}$$

with the initial conditions

$$y(c) = \alpha, \qquad y'(c) = \beta, \qquad y''(c) = \gamma, \qquad y'''(c) = \delta, \tag{2.16}$$

where $a, b, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $a^2 \neq b^2$, $a, b \neq 0$. In this case the cosine family $C_a(t) = \cosh(at)$, while $S_a(t) = (\sinh(at))/a$. Note that the function

$$S_{ab}(t) = \frac{b\sinh(at) - a\sinh(bt)}{ab(a^2 - b^2)}$$

is the Cauchy solution of $\mathcal{D}_0 y = 0$. Therefore the unknown function $y = y(t)$ is real and can be represented in the form $y = y_{ab} + \mathcal{R}_{ab}f$, where

$$\begin{aligned} y_{ab}(t) = \frac{1}{(a^2 - b^2)}&((\cosh(a(t-c)) - \cosh(b(t-c)))\gamma \\ &+ (\sinh(a(t-c)) - \sinh(b(t-c)))\delta \\ &+ (a^2\cosh(b(t-c)) - b^2\cosh a(t-c))\alpha \\ &+ (a^2\sinh(b(t-c)) - b^2\sinh a(t-c))\beta) \end{aligned}$$

and

$$(\mathcal{R}_{ab}f)(t) = (S_{ab} *_c f) = \int_c^t \frac{b\sinh(a(t-z)) - a\sinh(b(t-z))}{ab(a^2-b^2)} f(z)\,dz.$$

2.3. Estimates in norm. Now we are looking for the relations between solutions in $\mathbb{X}$ and $\mathbb{R}$, especially for the estimates in norm of solutions of

$$\mathcal{D}y = y'''' - (A^2 + B^2)y'' + A^2B^2y = f,$$
$$\mathcal{D}_{ab}y = y'''' - (a^2 + b^2)y'' + a^2b^2y = |f|,$$

where $a = \|A\| > 0$, $b = \|B\| > 0$ and $a \neq b$. We start with the following pointwise inequalities

$$\|A^n\| \leq a^n, \qquad \|\sinh(At)\| \leq \sinh(at),$$

$$\|S_A(t)\| = \left\|\sum_{n=0}^{\infty}\frac{A^{2n}t^{2n+1}}{(2n+1)!}\right\| \leq \sum_{n=0}^{\infty}\frac{a^{2n}t^{2n+1}}{(2n+1)!} = S_a(t),$$

which hold for all $t \geq 0$. Moreover, for any $f \in L^1([c,d],\mathbb{R})$ we have for all $t \in I = [c,d]$ the inequality

$$|(S_A *_c u)(t)| \leq (S_a *_c |f|)(t). \tag{2.17}$$

Now assume that A^2 and B^2 are commutative and $(A^2 - B^2) \in GL(\mathbb{X})$. Let

$$S_{AB}(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}(A^2 - B^2)^{-1}(A^{2n} - B^{2n}) = (A^2 - B^2)^{-1}(S_A(t) - S_B(t)).$$

Then, for all $n = 1, 2, \ldots$, we have

$$\begin{aligned}\|(A^2 - B^2)^{-1}(A^{2n} - B^{2n})\| &= \bigg\| \sum_{i=0}^{n-1} A^{2i} B^{2(n-1-i)} \bigg\| \\ \leq \sum_{i=0}^{n-1} \|A^{2i} B^{2(n-1-i)}\| &\leq \sum_{i=0}^{n-1} a^{2i} b^{2(n-1-i)} = \frac{a^{2n} - b^{2n}}{a^2 - b^2}.\end{aligned}$$

Hence

$$\begin{aligned}
\|S_{AB}(t)\| &\leq \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \frac{a^{2n} - b^{2n}}{a^2 - b^2} = \frac{(S_a - S_b)(t)}{a^2 - b^2} = S_{ab}(t), \\
\|S'_{AB}(t)\| &= \bigg\| \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}(A^2 - B^2)^{-1}(A^{2n} - B^{2n}) \bigg\| \\
&\leq \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \frac{a^{2n} - b^{2n}}{a^2 - b^2} = \frac{(C_a - C_b)(t)}{a^2 - b^2} = S'_{ab}(t), \\
\|S''_{AB}(t)\| &= \bigg\| \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}(A^2 - B^2)^{-1}(A^{2n+2} - B^{2n+2}) \bigg\| \\
&\leq \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \frac{a^{2n+2} - b^{2n+2}}{a^2 - b^2} = \frac{a \sinh(at) - b \sinh(bt)}{a^2 - b^2} = S''_{ab}(t),
\end{aligned}$$

$$\begin{aligned}
\|S'''_{AB}(t)\| &= \bigg\| \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}(A^2 - B^2)^{-1}(A^{2n+2} - B^{2n+2}) \bigg\| \\
&\leq \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \frac{a^{2n+2} - b^{2n+2}}{a^2 - b^2} = \frac{a^2 C_a(t) - b^2 C_b(t)}{a^2 - b^2} = S'''_{ab}(t).
\end{aligned}$$

As a consequence of the above inequalites we have the following result:

THEOREM 2.8. *For each $f \in L^1([c,d,\mathbb{X})$ we have for all $t \in [c,d]$ the estimates*

$$\begin{aligned} |(\mathcal{R}_{AB} f)(t)| &\leq (\mathcal{R}_{ab}|f|)(t), & |(\mathcal{R}_{AB} f)'(t)| &\leq (\mathcal{R}_{ab}|f|)'(t), \\ |(\mathcal{R}_{AB} f)''(t)| &\leq (\mathcal{R}_{ab}|f|)''(t), & |(\mathcal{R}_{AB} f)'''(t)| &\leq (\mathcal{R}_{ab}|f|)'''(t).\end{aligned}$$

The proof follows from the fact that $\mathcal{R}_{AB} f = S_{AB} * f$; we leave the details for readers.

Remark 2.9. As a consequence of the latter estimates we get that for each $u, v \in L^1([c,d], \mathbb{X})$ and all $t \in [c,d]$

$$\begin{aligned}
|(\mathcal{R}_{AB}u)(t) - (\mathcal{R}_{AB}v)(t)| &\leq (\mathcal{R}_{ab}|u-v|)(t),\\
|(\mathcal{R}_{AB}u)'(t) - (\mathcal{R}_{AB}v)'(t)| &\leq (\mathcal{R}_{ab}|u-v|)'(t),\\
|(\mathcal{R}_{AB}u)''(t) - (\mathcal{R}_{AB}v)''(t)| &\leq (\mathcal{R}_{ab}|u-v|)''(t),\\
|(\mathcal{R}_{AB}u)'''(t) - (\mathcal{R}_{AB}v)'''(t)| &\leq (\mathcal{R}_{ab}|u-v|)'''(t).
\end{aligned}$$

3. A Gronwall type lemma

In the case $\mathbb{X} = \mathbb{R}$ take $p_0, l \in L^1([c,d], \mathbb{R}_+)$ and let $\varphi\colon [0, d-c] \to \mathbb{R}_+$ be an increasing continuous function. Consider sequences $(p_n)_{n\geq 0} \subset L^1([c,d], \mathbb{R}_+)$ and $(\varepsilon_n)_{n\geq 0} \subset \mathbb{R}_+$ satisfying for $n = 0, 1, \ldots$ the inequalities

$$0 \leq p_{n+1}(t) \leq l(t)(\varphi *_c (p_n + \varepsilon_n))(t) = l(t)\bigg(\int_c^t \varphi(t-s)(p_n(s) + \varepsilon_n)\,ds\bigg). \tag{3.1}$$

Denote

$$\sigma(t) = \sum_{n=1}^{\infty} p_n(t). \tag{3.2}$$

Then we have the following estimates:

Lemma 3.1. *Assume that the sequences $(p_n)_{n\geq 0} \subset L^1([c,d], \mathbb{R}_+)$ and $(\varepsilon_n)_{n\geq 0} \subset \mathbb{R}_+$ satisfy, for $n = 0, 1, \ldots$, the inequality* (3.1). *Then*

(a) *For $n \geq 0$ we have*

$$\begin{aligned}
p_{n+1}(t) \leq l(t)\bigg(&\int_c^t [\varphi(t-s)]^{n+1}\,\frac{[m(t)-m(s)]^n}{n!} p_0(s)\,ds\\
&+ \bigg(\sum_{i=0}^{n} \varepsilon_i \int_c^t \frac{[\varphi(t-s)]^{n+1-i}[m(t)-m(s)]^{n-i}}{(n-i)!}\,ds\bigg)\bigg),
\end{aligned} \tag{3.3}$$

where $m(t) = \int_0^t l(s)\,ds$.

(b) *Given $\varepsilon \geq 0$ and let $\varepsilon_n = \varepsilon/2^n$, $n = 0, 1, \ldots$ The function σ given by* (3.2) *is integrable and for any $t \in [c,d]$ we have*

$$\sigma(t) \leq l(t)\bigg(\int_c^t \varphi(t-s)K(t,s)p_0(s)\,ds + 2\varepsilon \int_c^t \varphi(t-s)K^2(t,s)p_0(s)\,ds\bigg),$$

where $K(t,s) = \exp[\varphi(t-s)(m(t)-m(s))]$.

(c) *If*

$$\varphi(t) = S_{ab}(t) = \frac{b\sinh(at) - a\sinh(bt)}{ab(a^2-b^2)}, \quad t \in \mathbb{R}$$

and for $n \geq 0$ *we have* $p_{n+1}(t) = l(t)(\varphi *_c (p_n + \varepsilon/2^n))(t)$*, then*

$$(\sigma *_c \varphi)(t) \leq \int_c^t \varphi(t-s)(K(t,s) + 2\varepsilon K^2(t,s) - 1)p_0(s)\,ds, \tag{3.4}$$

$$(\sigma *_c \varphi')(t) \leq \int\_c^t \varphi'(t-s)K(t,s)p_0(s)\,ds + 2\varepsilon \int_c^t \varphi'(t-s)K^2(t,s)\,ds. \tag{3.5}$$

Moreover, $\psi = \sigma/l$ *is the solution of the* IVP

$$\psi'''' - (a^2+b^2)\psi'' + (a^2b^2 - l)\psi = p_0 + \varepsilon, \quad \psi(c) = \psi'(c) = \psi''(c) = \psi'''(c) = 0.$$

Proof. (a) We apply the induction. It is obvious for $n = 0$, since

$$p_1(t) \leq l(t)\bigg(\int_c^t \varphi(t-s)p_0(s)\,ds + \varepsilon_0 \int_c^t \varphi(t-s)\,ds\bigg).$$

Assume that the required inequality holds for all $i \leq n+1$. Then for $n+2$ we have, by induction step, that

$$\begin{aligned}
0 \leq p_{n+2}(t) &\leq l(t)\bigg(\int_c^t \varphi(t-z)p_{n+1}(z)\,dz + \varepsilon_{n+1}\int_c^t \varphi(t-z)\,dz\bigg) \\
&\leq l(t)\bigg(\int_c^t \varphi(t-z)l(z)\int_c^z [\varphi(z-s)]^{n+1}\frac{[m(z)-m(s)]^n}{n!}p_0(s)\,ds\,dz \\
&\quad + \int_c^t \varphi(t-z)l(z)\bigg(\sum_{i=0}^n \varepsilon_i \int_c^z \frac{[\varphi(z-s)]^{n+1-i}[m(z)-m(s)]^{n-i}}{(n-i)!}\bigg)\,ds\,dz \\
&\quad + \varepsilon_{n+1}\int_c^t \varphi(t-z)\,dz\bigg)
\end{aligned}$$

Applying the Fubini Theorem we get

$$\begin{aligned}
p_{n+2}(t) \leq l(t)\bigg(&\int_c^t p_0(s)\int_s^t (\varphi(t-z)l(z)[\varphi(z-s)]^{n+1}\frac{[m(z)-m(s)]^n}{n!}\,dz\bigg)\,ds \\
&+ \int_c^t \bigg(\int_s^t \bigg(\sum_{i=0}^n \varepsilon_i\varphi(t-z)l(z)\frac{[\varphi(z-s)]^{n+1-i}[m(z)-m(s)]^{n-i}}{(n-i)!}\bigg)\,dz\bigg)\,ds \\
&\qquad + \varepsilon_{n+1}\int_c^t \varphi(t-s)\,ds\bigg).
\end{aligned}$$

But $\varphi(t)$ is increasing and, hence, for $s \leq z \leq t$ we have

$$\varphi(t-z) \leq \varphi(t-s) \quad \text{and} \quad \varphi(z-s) \leq \varphi(t-s).$$

Moreover, $d(m(z)-m(s))/dz = l(z)$ and thus

$$\begin{aligned}
p_{n+2}(t) \leq l(t)\bigg(&\int_c^t p_0(s)[\varphi(t-s)]^{n+2}\int_s^t \bigg(\bigg(\frac{[m(z)-m(s)]^{n+1}}{(n+1)!}\bigg)'\,dz\bigg)\,ds \\
&+ \int_c^t \bigg(\sum_{i=0}^n \varepsilon_i[\varphi(t-s)]^{n+2-i}\bigg)\bigg(\int_s^t \bigg(\frac{[m(z)-m(s)]^{n+1-i}}{(n+1-i)!}\bigg)'\,dz\bigg)\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ \varepsilon_{n+1} \int_c^t \varphi(t-s)\,ds\bigg) \\
&= l(t)\bigg(\int_c^t p_0(s)[\varphi(t-s)]^{n+2}\frac{[m(t)-m(s)]^{n+1}}{(n+1)!}\,ds \\
&\quad + \int_c^t \bigg(\sum_{i=0}^{n} \varepsilon_i[\varphi(t-s)]^{n+2-i}\bigg)\frac{[m(t)-m(s)]^{n+1-i}}{(n+1-i)!}\,ds \\
&\quad + \varepsilon_{n+1} \int_c^t \varphi(t-s)\,ds\bigg) \\
&= l(t)\bigg(\int_c^t p_0(s)[\varphi(t-s)]^{n+2}\frac{[m(t)-m(s)]^{n+1}}{(n+1)!}\,ds \\
&\quad + \int_c^t \bigg(\sum_{i=0}^{n+1} \varepsilon_i[\varphi(t-s)]^{n+2-i}\bigg)\frac{[m(t)-m(s)]^{n+1-i}}{(n+1-i)!}\,ds\bigg),
\end{aligned}
$$

what ends the induction step and the proof of (a).

(b) By (a) we have

$$
\begin{aligned}
\sigma(t) &= \sum_{n=0}^{\infty} p_{n+1}(t) \le l(t)\bigg(\sum_{n=0}^{\infty}\int_c^t [\varphi(t-s)]^{n+1}\frac{[m(t)-m(s)]^n}{n!}p_0(s)\,ds \\
&\quad + \sum_{n=0}^{\infty}\bigg(\sum_{i=0}^{n}\varepsilon_i\int_c^t \frac{[\varphi(t-s)]^{n+1-i}[m(t)-m(s)]^{n-i}}{(n-i)!}\,ds\bigg)\bigg) \\
&\le l(t)\bigg(\int_c^t \varphi(t-s)\exp[\varphi(t-s)(m(t)-m(s))]p_0(s)\,ds \\
&\quad + \sum_{n=0}^{\infty}\frac{\varepsilon}{2^n}\int_c^t \varphi(t-s)\exp[2\varphi(t-s)(m(t)-m(s))]p_0(s)\,ds\bigg) \\
&= l(t)\bigg(\int_c^t \varphi(t-s)K(t,s)p_0(s)\,ds + 2\varepsilon\int_c^t \varphi(t-s)K^2(t,s)p_0(s)\,ds\bigg).
\end{aligned}
$$

(c) To see the next inequalities we proceed as follows:

- for (3.4)

$$
\begin{aligned}
(\sigma *_c \varphi)(t) &\le \sum_{n=1}^{\infty}\bigg(\varphi *_c \bigg(p_n + \frac{\varepsilon}{2^n}\bigg)\bigg)(t) = \frac{1}{l(t)}\sum_{n=1}^{\infty} p_{n+1}(t) = \frac{\sigma(t)-p_1(t)}{l(t)} \\
&\le \int_c^t \varphi(t-s)K(t,s)p_0(s)\,ds \\
&\quad + 2\varepsilon\int_c^t \varphi(t-s)K^2(t,s)p_0(s)\,ds - \int_c^t \varphi(t-s)(p_0(s)+\varepsilon)(t) \\
&\le \int_c^t \varphi(t-s)(K(t,s)+2\varepsilon K^2(t,s)-1)p_0(s)\,ds.
\end{aligned}
$$

- for (3.5)

$$(\sigma *_c \varphi')(t) = \sum_{n=0}^{\infty} \int_c^t \varphi'(t-s) p_{n+1}(s)\, ds$$

$$\leq \sum_{n=0}^{\infty} \int_c^t \varphi'(t-s) l(s) \int_c^s (\varphi(s-x))^n \frac{(m(s)-m(x))^n}{n!} p_0(x)\, dx\, ds$$

$$+ \sum_{n=0}^{\infty} \int_c^t \varphi'(t-s) l(s) \biggl(\sum_{i=0}^{n} \frac{\varepsilon}{2^{n-i}} \int_c^s \frac{[\varphi(s-x)]^{n+1-i}[m(s)-m(x)]^{n-i}}{(n-i)!} \biggr) dx\, ds.$$

Now the same Fubini Theorem argument yields

$$(\sigma *_c \varphi')(t) \leq \sum_{n=0}^{\infty} \int_c^t \biggl(\int_x^t \varphi'(t-s)(\varphi(s-x))^n \biggl(\frac{(m(s)-m(x))^n}{n!} \biggr)' ds \biggr) p_0(x)\, dx$$

$$+ \sum_{n=0}^{\infty} \frac{\varepsilon}{2^n} \int_c^t \biggl(\int_x^t \varphi'(t-s) \biggl(\sum_{i=0}^{n+1} [\varphi(s-x)]^i \biggl(\frac{(2(m(s)-m(x)))^i}{i!} \biggr)' \biggr) \biggr) dx\, ds.$$

But for all $0 \leq x \leq s \leq t \leq T$ we have

$$\varphi(t-s) \leq \varphi(t-x) \quad \text{and} \quad \varphi'(s-x) \leq \varphi'(t-x).$$

Therefore

$$(\sigma *_c \varphi')(t) \leq \sum_{n=0}^{\infty} \int_c^t \biggl(\int_x^t \varphi'(t-x)(\varphi(t-x))^n \biggl(\frac{(m(s)-m(x))^n}{n!} \biggr)' ds \biggr) p_0(x)\, dx$$

$$+ \sum_{n=0}^{\infty} \frac{\varepsilon}{2^n} \int_c^t \biggl(\int_x^t \varphi'(t-x) \biggl(\sum_{i=0}^{n+1} [\varphi(t-x)]^i \biggl(\frac{(2(m(s)-m(x)))^i}{i!} \biggr)' \biggr) ds \biggr) dx$$

$$\leq \int_c^t \varphi'(t-x) \exp(\varphi(t-x)(m(t)-m(x))) p_0(x)\, dx$$

$$+ 2\varepsilon \int_c^t \varphi'(t-x) \exp(2\varphi(t-x)(m(t)-m(x))) dx$$

$$= \int_c^t \varphi'(t-x) K(t,x) p_0(x)\, dx + 2\varepsilon \int_c^t \varphi'(t-x) K^2(t,x)\, dx$$

what gives (3.5). Finally, observe that

$$\psi(t) = \frac{\sigma(t)}{l(t)} = \frac{1}{l(t)} \sum_{n=0}^{\infty} p_{n+1}(t) = \biggl(\varphi *_c \sum_{n=0}^{\infty} \Bigl(p_n + \frac{\varepsilon}{2^n} \Bigr) \biggr)(t)$$

$$= (\varphi *_c (\sigma + p_0 + \varepsilon))(t) = (\varphi *_c (l\psi + p_0 + \varepsilon))(t)$$

what means that

$$\psi'''' - (a^2+b^2)\psi'' + a^2b^2\psi = (l\psi + p_0 + \varepsilon).$$

In other words

$$\psi'''' - (a^2+b^2)\psi'' + (a^2b^2 - l)\psi = p_0 + \varepsilon, \qquad \square$$

4. A version of Filippov lemma

Let $\mathcal{D}y = y'''' - (A^2 + B^2)y'' + A^2B^2y$ and consider an IVP problem in the Banach space $(\mathbb{X}, |\cdot|)$ on $[c,d] \subset [0,T]$

$$\mathcal{D}y \in F(t,y), \tag{4.1}$$

$$y(c) = \alpha, \qquad y'(c) = \beta, \qquad y''(c) = \gamma, \qquad y'''(c) = \delta. \tag{4.2}$$

Let $V = \{y \in W^{4,1}([c,d],\mathbb{X}) : y(c) = y'(c) = y''(c) = y'''(c) = 0\}$.

By a solution of (4.1) with initial conditions (4.2) we mean a function y in $W = y_{AB} + V$ satisfying (4.1), where y_{AB} is given by (2.12).

We shall pose the following assumptions on $F\colon [0,T]\times\mathbb{X} \rightsquigarrow c(\mathbb{X})$, where $c(\mathbb{X})$ stands for the family of all nonempty compact subsets of $\mathbb{X}$:

Condition 4.1. *For every $y \in \mathbb{X}$ the multifunction $F(\,\cdot\,,y)$ is Lebesgue measurable in t.*

Condition 4.2. *The multifunction $F(t,\,\cdot\,)$ is Lipschitz continuous in y with a positive integrable function $l(\,\cdot\,)$, i.e. for every $y_1, y_2 \in \mathbb{X}$ the inequality*

$$\mathrm{d}_H(F(t,y_1),F(t,y_2)) \le l(t)|y_1 - y_2| \tag{4.3}$$

holds for almost all $t \in [0,T]$, where $\mathrm{d}_H(K,L)$ stands for the Hausdorff distance between sets $K, L \in c(\mathbb{X})$.

Condition 4.3. *The multivalued mapping $t \to F(t,y)$ is integrably bounded by an $\gamma \in L^1[0,T]$, i.e. for each y*

$$\sup\{|z| : z \in F(t,y)\} \le \gamma(t) \quad \textit{a.e. in } [0,T].$$

The main result of the paper is the following:

Theorem 4.4 (Filippov Lemma). *Assume that $F\colon [0,T]\times\mathbb{X} \rightsquigarrow c(\mathbb{X})$ satisfies Conditions 4.1–4.3. Let $y_0 \in W$ be an arbitrary function with (4.2) fulfilling an estimate*

$$\mathrm{dist}(\mathcal{D}y_0(t), F(t,y_0(t))) \le p_0(t) \quad \textit{a.e. in } [c,d] \subset [0,T],$$

where $p_0 \in L^1\,[0,T]$. Denote $a = \|A\| > 0$, $b = \|B\| > 0$ and assume that $a \ne b$. Take

$$\varphi(t) = S_{ab}(t) = \frac{b\sinh(at) - a\sinh(bt)}{ab(a^2 - b^2)}.$$

Then, for each $\varepsilon > 0$, there exists a solution $y \in W$ of (4.1) with (4.2) such that, for almost every t,

$$|\mathcal{D}y(t) - \mathcal{D}y_0(t)| \le \sigma_0(t), \qquad |y(t) - y_0(t)| \le (\varphi *_c \sigma_0)(t),$$
$$|y'(t) - y_0'(t)| \le (\varphi' *_c \sigma_0)(t), \qquad |y''(t) - y_0''(t)| \le (\varphi'' *_c \sigma_0)(t),$$
$$|y'''(t) - y_0'''(t)| \le (\varphi''' *_c \sigma_0)(t),$$

where

$$\sigma_0(t) = l(t)\int_c^t \Phi(t,s)p_0(s)\,ds + p_0(t) + \varepsilon(l(t)+2),$$
$$\Phi(t,s) = \varphi(t-s)(\exp(\varphi(t-s)(m(t)-m(s)))).$$

Proof. We begin with the observation that for any $y \in W \subset L^\infty$ the multivalued mapping $t \to F(t,y(t))$ is measurable with compact values and integrably bounded by $\gamma(t)$, i.e.

$$\sup\{|z| : z \in F(t,y(t))\} \le \gamma(t) \quad \text{a.e. in } [0,T]. \tag{4.4}$$

Denote by

$$\mathcal{K}(u) = \{f \in L^1([0,T],\mathbb{X}) : f(t) \in F(t, y_{AB}(t) + (\mathcal{R}_{AB}u)(t)) \text{ a.e. in } [a,b]\}.$$

Since $y_{AB} + \mathcal{R}_{AB}u \in W \subset L^\infty([c,d],\mathbb{X})$ then, by (4.4), each $\mathcal{K}(u)$ is nonempty. Moreover, for every $u,v \in L^1([c,d],\mathbb{X})$, any $f \in \mathcal{K}(u)$ and arbitrary $\delta > 0$ there is a $g \in \mathcal{K}(v)$ such that almost everywhere in $[c,d]$

$$\begin{aligned}
|f(t) - g(t)| &\le \operatorname{dist}(f(t), F(t, y_{AB}(t) + (\mathcal{R}_{AB}v)(t))) + \delta l(t) \\
&\le \mathrm{d}_H(F(t, y_{AB}(t) + (\mathcal{R}_{AB}u)(t)), F(t, y_{AB}(t) + (\mathcal{R}_{AB}v)(t))) + \delta l(t) \\
&\le l(t)(|\mathcal{R}_{AB}u - \mathcal{R}_{AB}v|)(t) + \delta l(t) \le l(t)((\mathcal{R}_{ab}|u-v|)(t) + \delta).
\end{aligned}$$

In what follows we shall adopt the Filippov technique with some necessary changes. Fix $\varepsilon > 0$. Let

$$M = 2\max_{t,s\in[c,d]}\left\{K^2(t,s), \int_c^t \varphi(t-s)K^2(t,s)p_0(s)\,ds\right\}$$

and take $\varepsilon_n = \varepsilon/(2^n M)$. Starting with $y_0 = y_{AB} + \mathcal{R}_{AB}u_0$ ($\mathcal{D}y_0 = u_0$), we may choose such $u_1 \in \mathcal{K}(u_0)$ that

$$|(\mathcal{D}y_1)(t) - (\mathcal{D}y_0)(t)| = |u_0(t) - u_1(t)| \le \operatorname{dist}(u_0(t), F(t, y_0(t))) + \varepsilon_0 \le p_0(t) + \varepsilon_0$$

almost everywhere in $[c,d]$, where $y_1 = y_{AB} + \mathcal{R}_{AB}u_1$. Hence, by Theorem 2.4, for all $t \in [c,d]$ we have

$$\begin{aligned}
|(y_1 - y_0)(t)| &= |(\mathcal{R}_{AB}(u_0 - u_1))(t)| \le |(\mathcal{R}_{ab}|u_0 - u_1|)(t)| \\
&\le |(\mathcal{R}_{ab}(p_0 + \varepsilon_0))(t)| = (\varphi *_c (p_0 + \varepsilon_0))(t).
\end{aligned}$$

Moreover,

$$\begin{aligned}
|(y_1 - y_0)'(t)| &\le (\varphi' *_c (p_0 + \varepsilon_0))(t), \\
|(y_1 - y_0)''(t)| &\le (\varphi'' *_c (p_0 + \varepsilon_0))(t), \\
|(y_1 - y_0)'''(t)| &\le (\varphi''' *_c (p_0 + \varepsilon_0))(t).
\end{aligned}$$

The relation (4.3) yields

$$\operatorname{dist}((\mathcal{D}y_1)(t), F(t, y_1(t))) \le l(t)((\varphi *_c (p_0 + \varepsilon_0))(t)) = p_1(t) \quad \text{a.e. in } [c,d].$$

We now may pick up $y_2 = y_{AB} + \mathcal{R}_{AB}u_2 \in W$ such that $u_2 = \mathcal{D}y_2 \in \mathcal{K}(u_1)$ and

$$|(\mathcal{D}y_2)(t) - (\mathcal{D}y_1)(t)| \le p_1(t) + \varepsilon_1 \quad \text{a.e. in } [c,d].$$

Observe that for all $t \in [c,d]$

$$\begin{gathered}
|y_2(t) - y_1(t)| = |(\mathcal{R}_{AB}(u_2 - u_1))(t)| \le (\varphi *_c (p_1 + \varepsilon_1))(t), \\
|y_2'(t) - y_1'(t)| \le (\varphi' *_c (p_1 + \varepsilon_1))(t), \\
|y_2''(t) - y_1''(t)| \le (\varphi'' *_c (p_1 + \varepsilon_1))(t), \\
|y_2'''(t) - y_1'''(t)| \le (\varphi''' *_c (p_1 + \varepsilon_1))(t).
\end{gathered}$$

The latter together with (4.3) gives

$$\operatorname{dist}((\mathcal{D}y_2)(t), F(t, y_2(t))) \le l(t)(\varphi *_c (p_1 + \varepsilon_1))(t) = p_2(t) \quad \text{a.e. in } [c,d].$$

Continuing this procedure by induction, we obtain sequences $(p_n) \subset L^1([c,d])$, $(u_n) \subset L^1([c,d], X)$ and $(y_n) \subset V$ with

$$\begin{gathered}
p_{n+1}(t) = l(t)(\varphi *_c (p_n + \varepsilon_n))(t), \\
u_{n+1} \in \mathcal{K}(u_n), \qquad y_n = y_{AB} + \mathcal{R}_{AB}u_n, \\
|(\mathcal{D}y_{n+1})(t) - (\mathcal{D}y_n)(t)| \le p_n(t) + \varepsilon_n \quad \text{a.e. in } [c,d], \\
|y_{n+1}(t) - y_n(t)| \le (\varphi *_c (p_n + \varepsilon_n))(t), \\
|y_{n+1}'(t) - y_n'(t)| \le (\varphi' *_c (p_n + \varepsilon_n))(t), \\
|y_{n+1}''(t) - y_n''(t)| \le (\varphi'' *_c (p_n + \varepsilon_n))(t), \\
|y_{n+1}'''(t) - y_n'''(t)| \le (\varphi''' *_c (p_n + \varepsilon_n))(t)
\end{gathered}$$

for $n = 0, 1, \ldots$ Thus

$$\operatorname{dist}((\mathcal{D}y_{n+1})(t), F(t, y_{n+1}(t))) \le l(t)((\varphi *_c (p_n + \varepsilon_n))(t)) = p_{n+1}(t)$$

almost everywhere in $[c,d]$. Denote $\sigma(t) = \sum_{n=1}^{\infty} p_n(t)$. By Lemma 3.1, with ε/M, we conclude that

$$\sigma(t) \le l(t)\bigg(\int_c^t \Phi(t,s)p_0(s)\,ds + \varepsilon\bigg),$$

where $\Phi(t,s) = \varphi(t-s)(\exp(\varphi(t-s)(m(t) - m(s))))$. Thus σ and σ/l are integrable. Moreover, for $n = 0, 1, \ldots$ and $m = 1, 2, \ldots$ we have, almost everywhere in $[c,d]$,

$$\begin{aligned}
(4.5)\quad |\mathcal{D}y_{n+m}(t) - \mathcal{D}y_n(t)| &\le \sum_{i=n}^{n+m} (p_i(t) + \varepsilon_i) \le \sum_{i=n}^{\infty} p_i(t) + \frac{2\varepsilon}{2^n} \\
&\le \sigma(t) + p_0(t) + 2\varepsilon \le l(t)\int_c^t \Phi(t,s)p_0(s)\,ds + p_0(t) + \varepsilon(l(t) + 2) = \sigma_0(t).
\end{aligned}$$

So, for almost all $t \in [c,d]$, is

$$|y_{n+m}(t) - y_n(t)| \le \sum_{i=n+1}^{\infty} (\varphi *_c (p_i + \varepsilon_i)) \le \varphi *_c \sigma_0. \tag{4.6}$$

Moreover,

$$\begin{aligned} |y'_{n+m}(t) - y'_n(t)| &\le \sum_{i=n+1}^{\infty} (\varphi' *_c p_i) \le \varphi' *_c \sigma_0, \\ |y''_{n+m}(t) - y''_n(t)| &\le \sum_{i=n+1}^{\infty} (\varphi'' *_c (p_i + \varepsilon_i)) \le \varphi'' *_c \sigma_0, \\ |y'''_{n+m}(t) - y'''_n(t)| &\le \sum_{i=n+1}^{\infty} (\varphi''' *_c (p_i + \varepsilon_i)) \le \varphi''' *_c \sigma_0. \end{aligned} \tag{4.7}$$

Therefore the sequences $\{u_n\} \subset L^1([c,d],X)$, $\{y_n\} = \{\mathcal{R}_{AB}u_n + y_{AB}\} \subset W$, $\{y'_n\} = \{(\mathcal{R}_{AB}u_n)' + y'_{AB}\}$, $\{y''_n\} = \{(\mathcal{R}_{AB}u_n)'' + y''_{AB}\}$ and $\{y'''_n\} = \{(\mathcal{R}_{AB}u_n)''' + y'''_{AB}\}$ are pointwise convergent and, by the Lebesgue Dominated Convergence Theorem, they are strongly convergent. Starting with $\lim u_n = u$ we conclude that

$$\lim y_n = y, \quad \lim y'_n = y', \quad \lim y''_n = y'', \quad \lim y'''_n = y''' \quad \text{and} \quad \lim \mathcal{D}y_n = \mathcal{D}y.$$

Since, for each $n = 0, 1 \ldots$, $(\mathcal{D}y_{n+1})(t) \in F(t, y_n(t))$ almost everywhere in $[c,d]$ and each $F(t,\,\cdot\,)$ is Lipschitz continuous then y is a solution of (4.1) with (4.2). We shall check that this is the required one. Indeed, taking $n = 0$ in (4.5), (4.6) and passing to the limit with $m \to \infty$ we obtain, almost everywhere in $[c,d]$,

$$\begin{aligned} &|\mathcal{D}y(t) - \mathcal{D}y_0(t)| \le \sigma_0(t), && |y(t) - y_0(t)| \le (\varphi *_c \sigma_0)(t), \\ &|y'(t) - y'_0(t)| \le (\varphi' *_c \sigma_0)(t), && |y''(t) - y''_0(t)| \le (\varphi'' *_c \sigma_0)(t), \\ &|y'''(t) - y'''_0(t)| \le (\varphi''' *_c \sigma_0)(t). \end{aligned}$$

This ends the proof. □

Example 4.5. Let $F\colon [0,1] \times \mathbb{R}^2_\infty \rightsquigarrow c(\mathbb{R}^2_\infty)$ be given by

$$F(t,y) = 6\sqrt{(y_1 - \sin t + 3t\cos t)^2 + (y_2 - 3 + 4\cos t)^2}\left\{\begin{bmatrix} 3\sin t \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}.$$

Since the function $y \to \sqrt{(y_1 - \sin t + 3t\cos t)^2 + (y_2 - 3 + 4\cos t)^2}$ satisfies the Lipschitz condition with constant 1 then $F(t,y)$ is a Lipschitz continuous mapping with $l = l(t) = 18(1 - \sin t)$.

Consider the differential inclusion

$$\mathcal{D}y = y'''' + 5y'' + 4y \in F(t,y) \tag{4.8}$$

with the IC's (2.8). Observe that the function

$$\overline{y} = \begin{bmatrix} \overline{y}_1(t) \\ \overline{y}_2(t) \end{bmatrix} = \begin{bmatrix} \sin t - 3t\cos t + \sin 2t \\ \cos 2t - 4\cos t + 3 \end{bmatrix}$$

is a particular solution of (4.8). Indeed, by Example 2.6, we have

$$\mathcal{D}\overline{y} = \begin{bmatrix} 18\sin t \\ 12 \end{bmatrix} \in \left\{ \begin{bmatrix} 18\sin t \\ 12 \end{bmatrix}, \begin{bmatrix} 18 \\ 12 \end{bmatrix} \right\} = F(t, \overline{y}),$$

since $\sqrt{(\overline{y}_1 - \sin t + 3t\cos t)^2 + (\overline{y}_2 - 3 + 4\cos t)^2} = 1$.

Let $y_0(t) = 0 = [0,\ 0]^T$. Then

$$d(\mathcal{D}0, F(t,0)) = 18\sin t\sqrt{(-\sin t + 3t\cos t)^2 + (-3 + 4\cos t)^2} \le 24\sin t = p_0(t).$$

In this case we have

$$a = \|A\| = 1, \qquad b = \|B\| = 2,$$
$$\varphi(t) = \frac{1}{6}(\sinh(2t) - 2\sinh t), \qquad m(t) = 18(t + \cos t - 1)$$

and

$$\begin{aligned} \sigma_0(t) &= 24\sin t + \varepsilon(20 - 18\sin t) \\ &\quad + 3(1 - \sin t)\int_0^t [(\sinh(2(t-s)) - 2\sinh(t-s)) \times (8\sin s - \sin 2s) \\ &\qquad \times \exp(54(\sinh(2(t-s)) - 2\sinh(t-s))(t - s + \cos t - \cos s))]\,ds. \end{aligned}$$

Checking that the solution

$$\overline{y} = \begin{bmatrix} \sin t - 3t\cos t + \sin 2t \\ \cos 2t - 4\cos t + 3 \end{bmatrix}$$

fulfills all statements of our version of the Filippov Lemma is quite labourious.

Example 4.6. Let $F\colon [0,1] \times \boldsymbol{l}^2 \rightsquigarrow c(\boldsymbol{l}^2)$ be given by

$$F(t,y) = \frac{4\sqrt{6}}{\pi}\sqrt{\sum_{n=1}^{\infty}\left(y_n - \frac{8\sin t - \sin 2t}{n}\right)^2} \cdot \left(\left\{\frac{1}{2n}\right\}, \left\{\frac{1}{n}\right\}\right).$$

Since the function

$$y = \{y_n\} \to \sqrt{\sum_{n=1}^{\infty}\left(y_n - \frac{8\sin t - \sin 2t}{n}\right)^2}$$

satisfies the Lipschitz condition with constant 1 and

$$\operatorname{diam}\left(\left\{\frac{1}{2n}\right\}, \left\{\frac{1}{n}\right\}\right) = \frac{\pi}{2\sqrt{6}},$$

then $F(t,y)$ is a Lipschitz continuous mapping with the constant $l = 2$.

Consider the differential inclusion

$$\mathcal{D}y = y'''' + 5y'' + 4y \in F(t, y) \tag{4.9}$$

with the IC's (2.8). Observe that the function

$$\overline{y}(t) = \left\{ \frac{8 \sin t - 6t - \sin 2t}{n} \right\}$$

is a particular solution of 4.9. Indeed, by Example 2.7, we have

$$\mathcal{D}\overline{y} = \left\{ \frac{24t}{n} \right\} \in \left(\left\{ \frac{12t}{n} \right\}, \left\{ \frac{24t}{n} \right\} \right) = F(t, \overline{y}).$$

Let $y_0(t) = \{0\}$. Then

$$F(t, \{0\}) = 4(8 \sin t - \sin 2t) \cdot \left(\left\{ \frac{1}{2n} \right\}, \left\{ \frac{1}{n} \right\} \right)$$

and therefore

$$d(\mathcal{D}0, F(t, 0)) = \frac{\pi(8 \sin t - \sin 2t)}{\sqrt{6}} = p_0(t).$$

In this case we have

$$a = \|A\| = 1, \qquad b = \|B\| = 2, \qquad \varphi(t) = \frac{1}{6}(\sinh(2t) - 2 \sinh t)$$

and

$$\begin{aligned}\sigma_0(t) = {} & \frac{\pi}{\sqrt{6}}(8 \sin t - \sin 2t) + 4\varepsilon \\ & + \frac{2\pi}{6\sqrt{6}} \int_0^t \bigg[(\sinh(2(t-s)) - 2 \sinh(t-s))(8 \sin s - \sin 2s) \\ & \qquad \times \exp\left(\frac{(\sinh(2(t-s)) - 2 \sinh(t-s))(t-s)}{3} \right) \bigg] \, ds.\end{aligned}$$

One can check, after long calculation, that the solution

$$\overline{y}(t) = \left\{ \frac{8 \sin t - 6t - \sin 2t}{n} \right\}$$

fulfills the statements of our version of the Filippov Lemma.

5. Filippov–Ważewski Theorem on $[0, T]$

The celebrated Filippov–Ważewski result states that the solution set of differential inclusion

$$y' \in G(t, y), \qquad y(0) = \alpha,$$

where G is Lipschitz continuous multifunction satisfying Conditions 4.1–4.3, is dense in the solution set of

$$y' \in \operatorname{clco} G(t, y), \qquad y(0) = \alpha. \tag{5.1}$$

As a conclusion we can deduce that the similar result holds for higher order differential inclusions

$$\mathcal{D}y \in F(t,y), \tag{5.2}$$

$$\mathcal{D}y \in \operatorname{clco} F(t,y), \tag{5.3}$$

with the same IC

$$y(0)=\alpha, \qquad y'(0)=\beta, \qquad y''(0)=\gamma, \qquad y'''(0)=\delta, \tag{5.4}$$

where $\mathcal{D}y = y'''' - (A^2+B^2)y'' + A^2B^2y$ and $F\colon [0,T]\times\mathbb{X} \rightsquigarrow c(\mathbb{X})$ satisfies Conditions 4.1–4.3. Namely, the inclusion (5.2) can be transformed to (5.1) by introducing new unknown $Y=[y_1,y_2,y_3,y_4]^T$ and taking

$$G(t,Y) = \big\{[y_2,y_3,y_4,w]^T : w \in (A^2+B^2)y_3 - A^2B^2y_1 + F(t,y_1)\big\}.$$

This transformation preserves Conditions 4.1–4.3 but with larger parameters and therefore leads to less precise estimates. In this section we propose "an alternative proof" based on our version of the Filippov Lemma. Namely, we have the following:

Theorem 5.1. *Let r be a solution of* (5.3) *with* (5.4). *Then, for each $\varepsilon>0$, there exists a solution y of* (5.2) *with* (5.4) *such that*

$$\begin{aligned}
|y(t)-r(t)| &\le 2\varepsilon(\varphi *_c (l+1))(t)+\varepsilon,\\
|y'(t)-r'(t)| &\le 2\varepsilon(\varphi' *_c (l+1))(t)+\varepsilon,\\
|y''(t)-r''(t)| &\le 2\varepsilon(\varphi'' *_c (l+1))(t)+\varepsilon,\\
|y'''(t)-r'''(t)| &\le 2\varepsilon(\varphi''' *_c (l+1))(t)+\varepsilon,
\end{aligned}$$

where

$$\varphi(t) = \frac{\|A\|\sinh(\|B\|\,t) - \|B\|\sinh(\|A\|\,t)}{\|A\|\,\|B\|\,(\|A\|^2-\|B\|^2)}.$$

Proof. Fix $\varepsilon>0$ and denote

$$M = 1 + \sup_{t\in[0,T]}\bigg(\int_0^t \Phi(t,z)l(z)\,dz\bigg),$$

where $\Phi(t,s)=\varphi(t-s)(\exp(\varphi(t-s)(m(t)-m(s))))$.

Take a partition $0=t_0<t_1<\ldots<t_{N+1}=T$ such that for each $k=0,\ldots,N$ is

$$\int_{[t_k,t_{k+1}]} \gamma(t)\,dt < \frac{\varepsilon}{2M},$$

$$\bigg\|\int_{[t_k,t_{k+1}]} [\varphi(t_{k+1}-x),\ \varphi'(t_{k+1}-x),\ \varphi''(t_{k+1}-x),\ \varphi''(t_{k+1}-x)]^T\gamma(x)\,dx\bigg\| < \frac{\varepsilon}{2M}.$$

Let $\mathcal{D}r = v$, where $v(t) \in \operatorname{clco} F(t, r(t))$. Observe that, by (2.12), the function

$$\begin{aligned} z_k = z_k(t) = &(A^2 - B^2)^{-1}((C_A(t-t_k) - C_B(t-t_k))r''(t_k) \\ &+ (S_A(t-t_k) - S_B(t-t_k))r'''(t_k) \\ &+ (C_B(t-t_k)A^2 - C_A(t-t_k)B^2)r(t_k) \\ &+ (S_B(t-t_k)A^2 - S_A(t-t_k)B^2)r'(t_k)) \end{aligned}$$

is for $t \in [t_k, t_{k+1}]$ the unique solution of $\mathcal{D}z = 0$ with the IC's

$$z(t_k) = r(t_k), \qquad z'(t_k) = r'(t_k), \qquad z''(t_k) = r''(t_k) \quad \text{and} \quad z'''(t_k) = r'''(t_k).$$

Hence, by Theorem 2.4, for all $t \in [t_k, t_{k+1}]$ we have

$$\begin{bmatrix} r(t) - z_k(t) \\ r'(t) - z_k'(t) \\ r''(t) - z_k''(t) \\ r'''(t) - z_k'''(t) \end{bmatrix} = \begin{bmatrix} (\mathcal{R}_{AB}v)(t) \\ (\mathcal{R}_{AB}v)'(t) \\ (\mathcal{R}_{AB}v)''(t) \\ (\mathcal{R}_{AB}v)'''(t) \end{bmatrix} = \int_{t_k}^{t} \begin{bmatrix} S_{AB}(t-x) \\ S_{AB}'(t-x) \\ S_{AB}''(t-x) \\ S_{AB}'''(t-x) \end{bmatrix} v(x)\,dx.$$

Therefore

$$\begin{aligned} \left\| \begin{bmatrix} r(t) - r(t_k) \\ r'(t) - r'(t_k) \\ r''(t) - r''(t_k) \\ r'''(t) - r''(t_k) \end{bmatrix} \right\| &= \left\| \begin{bmatrix} r(t) - z_k(t) \\ r'(t) - z_k'(t) \\ r''(t) - z_k''(t) \\ r'''(t) - z_k'''(t) \end{bmatrix} \right\| = \left\| \begin{bmatrix} (\mathcal{R}_{AB}v)(t) \\ (\mathcal{R}_{AB}v)'(t) \\ (\mathcal{R}_{AB}v)''(t) \\ (\mathcal{R}_{AB}v)'''(t) \end{bmatrix} \right\| \\ &\le \left\| \begin{bmatrix} (\mathcal{R}_{ab}\,|v|)(t) \\ (\mathcal{R}_{ab}\,|v|)'(t) \\ (\mathcal{R}_{ab}\,|v|)''(t) \\ (\mathcal{R}_{ab}\,|v|)'''(t) \end{bmatrix} \right\| \le \left\| \int_{t_k}^{t} \begin{bmatrix} \varphi(t-x) \\ \varphi'(t-x) \\ \varphi''(t-x) \\ \varphi'''(t-x) \end{bmatrix} \gamma(x)dx \right\| \le \frac{\varepsilon}{2M}. \end{aligned}$$

Now observe that, for $t = t_{k+1}$, we have

$$\begin{bmatrix} r(t_{k+1}) - z_k(t_{k+1}) \\ r'(t_{k+1}) - z_k'(t_{k+1}) \\ r''(t_{k+1}) - z_k''(t_{k+1}) \\ r'''(t_{k+1}) - z_k'''(t_{k+1}) \end{bmatrix} \in \int_{t_k}^{t_{k+1}} \begin{bmatrix} S_{AB}(t_{k+1}-x) \\ S_{AB}'(t_{k+1}-x) \\ S_{AB}''(t_{k+1}-x) \\ S_{AB}'''(t_{k+1}-x) \end{bmatrix} \operatorname{clco} F(x, r(x)).$$

But, by the properties of the Aumann integral (see [16]), we have

$$\operatorname{cl} \int_{[t_k, t_{k+1}]} \Psi(x) \operatorname{clco} F(x, r(x)) = \operatorname{cl} \int_{[t_k, t_{k+1}]} \Psi(x) F(x, r(x)),$$

where $\Psi(x) \in L^\infty([0,T], \mathbb{X}^n)$. Thus

$$\begin{bmatrix} r(t_{k+1}) - z_k(t_{k+1}) \\ r'(t_{k+1}) - z_k'(t_{k+1}) \\ r''(t_{k+1}) - z_k''(t_{k+1}) \\ r'''(t_{k+1}) - z_k'''(t_{k+1}) \end{bmatrix} \in \operatorname{cl} \int_{t_k}^{t_{k+1}} \begin{bmatrix} S_{AB}(t_{k+1}-x) \\ S_{AB}(t_{k+1}-x) \\ S_{AB}(t_{k+1}-x) \\ S_{AB}(t_{k+1}-x) \end{bmatrix} F(x, r(x)),$$

and therefore, for each $k = 0, \dots, N$, there exists an integrable selection $u_k(t) \in F(t, r(t))$almost everywhere in $[t_k, t_{k+1}]$ such that

$$\left\| \begin{bmatrix} r(t_{k+1}) - z_k(t_{k+1}) \\ r'(t_{k+1}) - z_k'(t_{k+1}) \\ r''(t_{k+1}) - z_k''(t_{k+1}) \\ r'''(t_{k+1}) - z_k'''(t_{k+1}) \end{bmatrix} - \int_{[t_k, t_{k+1}]} \begin{bmatrix} S_{AB}(t_{k+1} - x) \\ S_{AB}(t_{k+1} - x) \\ S_{AB}(t_{k+1} - x) \\ S_{AB}(t_{k+1} - x) \end{bmatrix} u_k(x)\, dx \right\| < \frac{\varepsilon}{2MN}.$$

Take

$$u = \sum_{k=0}^{N} u_k \cdot \chi_{[t_k, t_{k+1}]}$$

and let y_0 be a solution of $\mathcal{D}y = u$ with (5.4). Then, for $k = 0, \dots, N-1$, we have

$$\left\| \begin{bmatrix} r(t_{k+1}) - z_k(t_{k+1}) \\ r'(t_{k+1}) - z_k'(t_{k+1}) \\ r''(t_{k+1}) - z_k''(t_{k+1}) \\ r'''(t_{k+1}) - z_k'''(t_{k+1}) \end{bmatrix} - \begin{bmatrix} y_0(t_{k+1}) - z_k(t_{k+1}) \\ y_0'(t_{k+1}) - z_k'(t_{k+1}) \\ y_0''(t_{k+1}) - z_k''(t_{k+1}) \\ y_0'''(t_{k+1}) - z_k'''(t_{k+1}) \end{bmatrix} \right\| < \frac{\varepsilon}{2MN}.$$

Equivalently, for $k = 1, \dots, N$,

$$\left\| \left[r(t_k) - y_0(t_k),\ r'(t_k) - y_0'(t_k),\ r''(t_k) - y_0''(t_k),\ r'''(t_k) - y_0'''(t_k) \right]^T \right\| < \frac{\varepsilon}{2MN}.$$

Therefore, for all $t \in [t_k, t_{k+1}]$ is

$$y(a) = \alpha, \qquad y'(a) = \beta, \qquad y''(a) = \gamma, \qquad y'''(a) = \delta, \tag{5.5}$$

$$\begin{aligned} &\left\| \left[r(t) - y_0(t),\ r'(t) - y_0'(t),\ r''(t) - y_0''(t),\ r'''(t) - y_0'''(t) \right]^T \right\| \\ &\leq \left\| \begin{bmatrix} r(t) - r(t_k) \\ r'(t) - r'(t_k) \\ r''(t) - r''(t_k) \\ r'''(t) - r''(t_k) \end{bmatrix} \right\| + \left\| \begin{bmatrix} r(t_k) - y_0(t_k) \\ r'(t_k) - y_0'(t_k) \\ r''(t_k) - y_0''(t_k) \\ r'''(t_k) - y_0'''(t_k) \end{bmatrix} \right\| + \left\| \begin{bmatrix} y_0(t_k) - y_0(t) \\ y_0'(t_k) - y_0'(t) \\ y_0''(t_k) - y_0''(t) \\ y_0'''(t_k) - y_0'''(t) \end{bmatrix} \right\| < \frac{\varepsilon}{M}. \end{aligned}$$

Hence

$$\operatorname{dist}((\mathcal{D}y_0)(t), F(t, y_0(t))) = \operatorname{dist}(u(t), F(t, y_0(t))) \leq l(t)\, |r(t) - y_0(t)| \leq \frac{\varepsilon l(t)}{M}.$$

By the Filippov Lemma there exists a solution $y \in W$ of (5.2) with (5.4) such that almost everywhere in t,

$$|\mathcal{D}y(t) - \mathcal{D}y_0(t)| \leq \varepsilon l(t) \bigg(\bigg(\int_c^t \Phi(t, s) \frac{l(s)}{M}\, ds + \frac{1}{M} + 1 \bigg) + 2 \bigg) \leq 2\varepsilon (l(t) + 1),$$

$$|y(t) - y_0(t)| \leq 2\varepsilon (\varphi *_c (l + 1))(t), \qquad |y'(t) - y_0'(t)| \leq 2\varepsilon (\varphi' *_c (l + 1))(t),$$

$$|y''(t) - y_0''(t)| \leq 2\varepsilon (\varphi'' *_c (l + 1))(t), \qquad |y'''(t) - y_0'''(t)| \leq 2\varepsilon (\varphi''' *_c (l + 1))(t),$$

where

$$\Phi(t, s) = \varphi(t - s)(\exp(\varphi(t - s)(m(t) - m(s)))).$$

Thus

$$\begin{aligned}|y(t)-r(t)| &\le 2\varepsilon(\varphi *_c (l+1))(t)+\varepsilon,\\ |y'(t)-r'(t)| &\le 2\varepsilon(\varphi' *_c (l+1))(t)+\varepsilon,\\ |y''(t)-r''(t)| &\le 2\varepsilon(\varphi'' *_c (l+1))(t)+\varepsilon,\\ |y'''(t)-r'''(t)| &\le 2\varepsilon(\varphi''' *_c (l+1))(t)+\varepsilon.\end{aligned}$$

This ends the proof. □

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