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# THE EXISTENCE OF POSITIVE SOLUTIONS FOR THE SINGULAR TWO-POINT BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we consider the following boundary value problem:

$$\begin{cases} ((-u'(t))^n)' = nt^{n-1}f(u(t)) & \text{for } 0 < t < 1, \\ u'(0) = 0, \quad u(1) = 0, \end{cases}$$

where $n > 1$. Using the fixed point theory on a cone and approximation technique, we obtain the existence of positive solutions in which $f$ may be singular at $u = 0$ or $f$ may be sign-changing.

## 1. Introduction

In this paper, we consider the following problem:

$$\begin{cases} ((-u'(t))^n)' = nt^{n-1}f(u(t)) & \text{for } 0 < t < 1, \\ u'(0) = 0, \quad u(1) = 0, \end{cases} \tag{1.1}$$

where $n > 1$ and $f$ is not identically zero.

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Such a problem arises in the study of radially symmetric solutions to the following Dirichlet problem for the Monge–Ampère equations in $\mathbb{R}^n$:

$$\begin{cases} \det(D^2u) = \lambda f(-u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.2}$$

where $B = \{x \in \mathbb{R}^n : |x| < 1\}$ is the unit ball in $\mathbb{R}^n$ and $D^2u = (\partial^2 u/\partial x_i \partial x_j)$ is the Hessian of $u$ (see [8]).

The Monge–Ampère equation has attracted a growing attention in recent years because of its important role in several areas of applied mathematics. In [11], Lions considered the existence of a unique eigenvalue $\lambda_1$ to the boundary value problem (1.2) with $f(u) = u^n$ and showed that $\lambda_1$ acts like a bifurcation point for the boundary value problem (1.2). Kutev [9] obtained the existence of a unique nontrivial convex radially symmetric solution to the boundary value problem (1.2) with $f(u) = u^p$, for all $0 < p \neq n$, reducing (1.2) to (1.1). Hu and Wang [8] established sufficient conditions for the existence and multiplicity of positive solutions to problem (1.1), where the function $f$ is continuous on $[0, +\infty)$. In [3], Dai discussed unilateral global bifurcation results for the problem with $f(u) = u^n + g(u)$. In [17]–[18], Wang considered the existence, multiplicity and nonexistence of nontrivial radial convex solutions to systems of Monge–Ampère equations with superlinearity or sublinearity assumptions for an appropriately chosen parameter. In [16], using the Leggett–Williams fixed point theorem, Wang and An investigated the existence of at least three nontrivial radial convex solutions to systems of Monge–Ampère equations. We refer to [4], [7], [12], [20] and references therein for further discussions regarding solutions to the Monge–Ampère equations with continuous nonlinearities. For the case that $f(x)$ is singular at $x = 0$, there are some interesting results also. In [10], using the existing regularity theory and a subsolution-supersolution method, Lazer and McKennar discussed the existence and uniqueness of positive solutions to singular BVP (1.2). Using the sub-super solution technique, Mohammed [13]–[14] established the existence and uniqueness of negative convex solution also to BVP (1.2).

The goal of this paper is to consider the existence of positive solutions under the conditions that $n > 1$ and $f(x)$ is singular at $x = 0$ and sign-changing. Firstly, in order to overcome difficulties caused by singularity of $f$ we pose new conditions which are different from those in [8], [17]–[18], and establish the multiplicity of positive solutions to BVP (1.1) different from that in [10], [13]–[14] under the condition that $f(x)$ is suplinear at $x = +\infty$. Secondly, when $f$ is singular and sign-changing, we establish the existence of at least one positive solution to BVP (1.1) which is different from that in [6], [8], [13]–[14], [17]–[18] where $f$ is supposed to be positive on $(0, +\infty)$.

Our paper is organized as follows. In Section 2, we present some lemmas and preliminaries. Section 3 discusses the existence of multiple positive solutions to BVP (1.1) when $f$ is positive. In Section 4, we discuss the existence of at least one positive solution to BVP (1.1) when $f$ is singular at $u = 0$ and sign-changing. Some of our ideas come from [1]–[2], [15], [19].

## 2. Preliminaries

Here we state some auxiliary lemmas needed in the sequel.

Lemma 2.1 (see [6]). *Let $\Omega$ be a bounded open set in the real Banach space $E$, $P$ be a cone in $E$, $\theta \in \Omega$ and $A\colon \overline{\Omega} \cap P \to P$ be continuous and compact. Suppose $\lambda Ax \neq x$, for all $x \in \partial\Omega \cap P$, $\lambda \in (0,1]$. Then*

$$i(A, \Omega \cap P, P) = 1.$$

Lemma 2.2 (see [6]). *Let $\Omega$ be a bounded open set in the real Banach space $E$, $P$ be a cone in $E$, $\theta \in \Omega$ and $A\colon \overline{\Omega} \cap P \to P$ be continuous and compact. Suppose $Ax \not\leq x$, for all $x \in \partial\Omega \cap P$. Then*

$$i(A, \Omega \cap P, P) = 0.$$

Lemma 2.3 (see [6]). *Let $E$ be a Banach space, $R > 0$, $B_R = \{x \in E : \|x\| \leq R\}$, and $F\colon B_R \to E$ be a continuous compact operator. If $x \neq \lambda F(x)$, for any $x \in E$ with $\|x\| = R$ and $0 < \lambda < 1$, then $F$ has a fixed point in $B_R$.*

Let $C[0,1] = \{y\colon [0,1] \to \mathbb{R} : y(t)$ is continuous on $[0,1]\}$ with the norm $\|y\| = \max\limits_{t\in[0,1]} |y(t)|$. It is easy to see that $C[0,1]$ is a Banach space. Define

$$\begin{aligned} P = \{y \in C[0,1] : {} & y \text{ is decreasing on } [0,1] \\ & \text{with } y(t) \geq (1-t)\|y\|, \text{ for all } t \in [0,1] \text{ and } y(1) = 0\}. \end{aligned}$$

It is easy to prove $P$ is a cone in $C[0,1]$ (see [8]).

Lemma 2.4 (see [8]). *For any function $v \subseteq C[0,1]$ with $v(t) \geq 0$ and $v'(t)$ decreasing in $[0,1]$, $v(0) = \|v\|$, we have $v(t) \geq (1-t)\|v\|$.*

We shall pose the following conditions on the function $f$:

($C_1$) $f\colon (0,\infty) \to (-\infty,\infty)$ is continuous.

($C_2$) $\lim\limits_{x\to 0+} f(x) = +\infty$.

## 3. Multiplicity of positive solutions to the singular BVP (1.1)

In this section, we consider the existence of multiple positive solutions to BVP (1.1). For $y \in P$, we define the operator

$$(T_\varepsilon y)(t) = \int_t^1 \left( \int_0^s n\tau^{n-1} f(\max\{\varepsilon, y(\tau)\})\, d\tau \right)^{1/n} ds, \tag{3.1}$$

for $0 \le t \le 1$, $1 \ge \varepsilon > 0$.

Lemma 3.1. *Suppose* $(\mathrm{C}_1)$ *hold and* $f(x) > 0$ *for all* $x \in (0, +\infty)$. *Then* $T_\varepsilon \colon P \to P$ *is continuous and compact for all* $1 \ge \varepsilon > 0$.

Proof. It is easy to prove that $T_\varepsilon$ is well defined and $(T_\varepsilon y)(t) \ge 0$ for all $t \in P$. For $y \in P$, we have

$$(T_\varepsilon\, y)'(t) = -\bigg(\int_0^t ns^{n-1} f(\max\{\varepsilon, y(s)\})\, ds\bigg)^{1/n} < 0 \quad \text{on } (0,1),$$

which implies that $(T_\varepsilon y)'(t)$ is decreasing on $[0,1]$. Since $(T_\varepsilon y)'(0) = 0$, we have $(T_\varepsilon y)'(t) < 0$ for all $t \in (0,1)$, which together with $(T_\varepsilon y)(1) = 0$, implies that

$$\|T_\varepsilon y\| = (T_\varepsilon y)(0).$$

Hence, Lemma 2.4 guarantees that $T_\varepsilon P \subseteq P$. A standard argument shows that $T_\varepsilon \colon P \to P$ is continuous and compact (see [6]). $\square$

Define

$$\begin{aligned}\Phi_r = \big\{x \in P \cap C^2((0,1), R) : \|x\| \le r \text{ and } x \text{ satisfies}\\ ((-x'(t))^n)' = nt^{n-1} f(\max\{\varepsilon, x(t)\}) = 0,\\ 0 < t < 1,\ x'(0) = 0,\ x(1) = 0, \text{ for all } 1 \ge \varepsilon > 0\big\}.\end{aligned}$$

Lemma 3.2. *If* $\Phi_r \ne \emptyset$ *and* $(\mathrm{C}_2)$ *hold with* $f(x) > 0$ *for all* $x \in (0, +\infty)$, *then there exists* $\delta_r > 0$ *such that*

$$x(t) \ge \delta_r(1 - t), \quad \textit{for all } t \in [0,1],\ x \in \Phi_r. \tag{3.2}$$

Proof. Suppose $x \in \Phi_r$. By the proof of Lemma 3.1, we have $x \in P$. Condition $(\mathrm{C}_2)$ guarantees that there exist $1 > b > 0$ and $a > 0$ such that

$$f(x) \ge a, \quad \text{for all } 0 < x \le b.$$

Since $f > 0$ is continuous on $[b,1]$, we have $\min\limits_{x \in [b,1]} f(x) > 0$. Then

$$f(x) \ge \min\Big\{a, \min_{x \in [b,1]} f(x)\Big\} > 0, \quad \text{for all } x \in (0,1]. \tag{3.3}$$

There are two cases to consider. (I) $\|x\| > 1$. Lemma 2.4 implies that

$$x(t) \ge (1-t)\|x\| \ge (1-t), \quad \text{for all } t \in [0,1]. \tag{3.4}$$

(II) $0 < \|x\| \le 1$. (3.3) guarantees that

$$
\begin{aligned}
x(t) &= \int_t^1 \left( \int_0^s n\tau^{n-1} f(\max\{\varepsilon, x(\tau)\})\, d\tau \right)^{1/n} ds \qquad (3.5)\\
&\ge \int_t^1 \left( \int_0^s n\tau^{n-1} \min\left\{a, \min_{x\in[b,1]} f(x)\right\} d\tau \right)^{1/n} ds \\
&= \min\left\{a, \min_{x\in[b,1]} f(x)\right\}^{1/n} (1-t^2) \\
&= \min\left\{a, \min_{x\in[b,1]} f(x)\right\}^{1/n} (1+t)(1-t) \\
&\ge \min\left\{a, \min_{x\in[b,1]} f(x)\right\}^{1/n} (1-t), \quad \text{for all } t \in [0,1].
\end{aligned}
$$

Let $\delta_r = \min\left\{1, \min\left\{a, \min_{x\in[b,1]} f(x)\right\}^{1/n}\right\}$. From (3.4) and (3.5), one has

$$x(t) \ge \delta_r(1-t), \quad \text{for all } t \in [0,1]. \qquad \square$$

LEMMA 3.3. *Suppose that $f(x) > 0$ for all $x \in (0, +\infty)$ and*

$$\lim_{x\to+\infty} \frac{f(x)}{x^n} = +\infty. \tag{3.6}$$

*Then, there exists $R' > 1$ such that for all $R \ge R'$*

$$i(T_\varepsilon, \Omega_R \cap P, P) = 0, \quad \textit{for all } 0 < \varepsilon \le 1.$$

PROOF. From (3.6), there exists $R_1 > \max\{1, r\}$ such that

$$f(x) \ge N^* x^n, \quad \text{for all } x \ge R_1, \tag{3.7}$$

where $N^* > 2^{3n}$. Let $R' = 2R_1$ and $\Omega_R = \{x \in C[0,1] : \|x\| < R\}$, for all $R \ge R'$. Now we show that

$$T_\varepsilon y \not\le y \quad \text{for } y \in P \cap \partial\Omega_R \text{ and all } 0 < \varepsilon \le 1. \tag{3.8}$$

Suppose that there exists $y_0 \in P \cap \partial\Omega_R$ with $T_\varepsilon y_0 \le y_0$. Then, $\|y_0\| = R$. Since $y_0 \in P$ we have from Lemma 2.4 that $y_0(t) \ge (1-t)\|y_0\| \ge (1-t)R$ for $t \in [0,1]$. For $t \in [0, 1/2]$, one has

$$y_0(t) \ge \frac{1}{2} R \ge \frac{1}{2} R' = R_1, \quad \text{for all } t \in \left[0, \frac{1}{2}\right],$$

which together with (3.7) yields that

$$f(\max\{\varepsilon, y_0(t)\}) = f(y_0(t)) \ge N^*(y_0(t))^n \ge N^*\left(\frac{1}{2} R\right)^n, \tag{3.9}$$

for all $t \in [0,1/2]$. Then we have, using (3.9),

$$\begin{aligned}
y_0(0) \ge (T_\varepsilon\, y_0)(0) &= \int_0^1 \bigg( \int_0^s n\tau^{n-1} f(\max\{\varepsilon, y_0(\tau)\})\, d\tau \bigg)^{1/n} ds \\
&\ge \int_{1/2}^1 \bigg( \int_0^{1/2} n\tau^{n-1} f(\max\{\varepsilon, y_0(\tau)\})\, d\tau \bigg)^{1/n} ds \\
&\ge \int_{1/2}^1 \bigg( \int_0^{1/2} n\tau^{n-1} f(y_0(\tau))\, d\tau \bigg)^{1/n} ds \\
&\ge \int_{1/2}^1 \bigg( \int_0^{1/2} n\tau^{n-1} N^*(y_0(\tau))^n\, d\tau \bigg)^{1/n} ds \\
&\ge \int_{1/2}^1 \bigg( \int_0^{1/2} n\tau^{n-1} N^*(1/2R)^n\, d\tau \bigg)^{1/n} ds = \frac{1}{8}\,(N^*)^{1/n} R > R = \|y_0\|,
\end{aligned}$$

which is a contradiction. Hence (3.8) is true. Lemma 2.2 guarantees that

$$i(T_\varepsilon, \Omega_R \cap P, P) = 0, \quad \text{for all } 0 < \varepsilon \le 1. \qquad \square$$

Theorem 3.4. *Suppose that* ($C_1$) *and* ($C_2$) *hold,* $0 < f(v) \le [g(v) + h(v)]^n$ *on* $(0,\infty)$ *with* $g > 0$ *continuous and nonincreasing on* $(0,\infty)$*,* $h \ge 0$ *continuous on* $[0,\infty)$ *and* $h/g$ *nondecreasing on* $(0,\infty)$*,*

$$\sup_{r\in(0,+\infty)} \frac{1}{1 + h(r)/g(r)} \int_0^r \frac{du}{g(u)} > \frac{1}{2} \tag{3.10}$$

*hold. Then* BVP (1.1) *has a solution* $v \in C[0,1] \cap C^2(0,1)$ *with* $v > 0$ *on* $(0,1)$ *and* $\|v\| < r$.

Proof. From (3.10), choose $0 < r$ with

$$\frac{1}{1 + h(r)/g(r)} \int_0^r \frac{du}{g(u)} > \frac{1}{2}. \tag{3.11}$$

Let $n_0 \in \{1,2,\ldots\}$ be chosen so that $1/n_0 < r$ and $\mathbb{N}_0 = \{n_0, n_0+1, \ldots\}$. Set $\Omega_1 = \{y \in C[0,1] : \|y\| < r\}$. For $m \in \mathbb{N}_0$, we define $T_{1/m}$ as that in (3.1). Lemma 3.1 guarantees that $T_{1/m}\colon P \to P$ is continuous and compact.

Now we show that

$$y \ne \lambda T_{1/m} y, \quad \text{for all } y \in \partial\Omega_1 \cap P,\ \lambda \in (0,1],\ m \in \mathbb{N}_0. \tag{3.12}$$

Suppose that there are $y_0 \in \partial\Omega_1 \cap P$ and $\lambda_0 \in (0,1]$ with $y_0 = \lambda_0 T_{1/m} y_0$, i.e. $y_0$ satisfies

$$\begin{cases} ((-y_0'(t))^n)' = \lambda n t^{n-1} f(\max\{1/m, y_0(t)\}) & \text{for } 0 < t < 1, \\ y_0'(0) = 0, \quad y_0(1) = 0 & \text{for } m \in \mathbb{N}_0,\ \lambda \in (0,1]. \end{cases}$$

Since $y_0$ is nonincreasing and nonnegative on $(0,1)$ with $y_0(0)=0$, we have $y_0(0)=\|y_0\|=r$. Then,

$$\begin{aligned}((-y_0'(t))^n)' &= nt^{n-1}f\bigg(\max\bigg\{\frac{1}{m},y_0(t)\bigg\}\bigg)\\ &\le nt^{n-1}g^n\bigg(\max\bigg\{\frac{1}{m},y_0(t)\bigg\}\bigg)\bigg\{1+\frac{h(\max\{1/m,y_0(t)\})}{g(\max\{1/m,y_0(t)\})}\bigg\}^n\\ &\le nt^{n-1}g^n(y_0(t))\bigg\{1+\frac{h(r)}{g(r)}\bigg\}^n.\end{aligned}$$

Integrate both sides from 0 to $t$ to obtain

$$(-y_0'(t))^n \le ng^n(y_0(t))\bigg\{1+\frac{h(r)}{g(r)}\bigg\}^n\int_0^t s^{n-1}\,ds = t^n g^n(y_0(t))\bigg\{1+\frac{h(r)}{g(r)}\bigg\}^n.$$

Then

$$-y_0'(t)\le tg(y_0(t))\bigg\{1+\frac{h(r)}{g(r)}\bigg\}. \tag{3.13}$$

Integrate both sides from $t$ to 1 to obtain

$$\int_{y_0(1)}^{y_0(t)}\frac{du}{g(u)}\le\frac{1}{2}(1-t^2)\bigg\{1+\frac{h(r)}{g(r)}\bigg\},$$

i.e.

$$\frac{1}{1+h(r)/g(r)}\int_0^{y_0(t)}\frac{du}{g(u)}\le\frac{1}{2}(1-t^2)\le\frac{1}{2},\quad\text{for all } t\in(0,1).$$

Consequently

$$\frac{1}{1+h(r)/g(r)}\int_0^r\frac{du}{g(u)}\le\frac{1}{2}.$$

This is a contradiction. Lemma 2.1 guarantees that

$$i(T_{1/m},P\cap\Omega_1,P)=1,\quad\text{for all } m\in\mathbb{N}_0,$$

which implies that there exists $v_m\in P\cap\Omega_1$ with $v_m=T_{1/m}v_m$, i.e. $v_m\in\Phi_r$. From Lemma 3.2, there exists a $\delta_r>0$ such that

$$v_m(t)\ge\delta_r(1-t),\quad\text{for all } t\in[0,1]. \tag{3.14}$$

Now we will show that

$$\{v_m(t)\}_{m\in N_0}\text{ is a bounded, equicontinuous family on }[0,1]. \tag{3.15}$$

Obviously, $\{v_m(t)\}_{m\in N_0}$ is uniformly bounded. Returning to (3.13) (with $y_0$ replaced by $v_m$) we have

$$\frac{-v_m'(t)}{g(v_m(t))}\le t\bigg\{1+\frac{h(v_m(0))}{g(v_m(0))}\bigg\},\quad\text{for all } t\in(0,1). \tag{3.16}$$

Let $I\colon[0,\infty)\to[0,\infty)$ be defined by

$$I(z)=\int_0^z\frac{du}{g(u)}.$$

Note that $I$ is an increasing map from $[0,\infty)$ onto $[0,\infty)$ (notice that $I(\infty)=\infty$ since $g>0$ is nonincreasing on $(0,\infty)$) with $I$ continuous on $[0,A]$ for any $A>0$. For $t,s\in[0,1]$ we have

$$\begin{aligned}|I(v_m(t))-I(v_m(s))| &= \bigg|\int_s^t \frac{v_m'(\tau)}{g(v_m(\tau))}\,d\tau\bigg| \\ &\le \bigg\{1+\frac{h(r)}{g(r)}\bigg\}\bigg|\int_s^t \tau\,d\tau\bigg| = \bigg\{1+\frac{h(r)}{g(r)}\bigg\}\frac{1}{2}\,|t^2-s^2|,\end{aligned}$$

which implies that

$$\{I(v_m(t))\}_{m\in N_0} \text{ is equicontinuous on } [0,1]. \tag{3.17}$$

Condition (3.17) and the uniform continuity of $I^{-1}$ on $[0,I(r)]$ together with

$$|v_m(t)-v_m(s)| = |I^{-1}(I(v_m(t)))-I^{-1}(I(v_m(s)))|$$

guarantees that (3.15) holds. Moreover, from (3.14), we have $\delta_r/2\le v_m(t)<r$, for all $t\in[0,1/2]$. Hence,

$$((-v_m'(s))^n)' = ns^{n-1}f\bigg(\max\bigg\{\frac{1}{m},y_m(s)\bigg\}\bigg) \le ns^{n-1}g^n\bigg(\delta_r\frac{1}{2}\bigg)\bigg\{1+\frac{h(r)}{g(r)}\bigg\}^n,$$

for all $s\in(0,1/2]$, which guarantees that

$$\begin{gathered}\text{the functions belonging to } \{(-v_m'(t))^n\} \\ \text{are equicontinuous and uniformly bounded on } [0,1/2],\end{gathered}$$

and so

$$\begin{gathered}\text{the functions belonging to } \{-v_m'(t)\} \\ \text{are equicontinuous and uniformly bounded on } [0,1/2].\end{gathered} \tag{3.18}$$

The Arzela–Ascoli Theorem guarantees that $\{v_m(t)\}$ has a uniformly convergent subsequence $\{v_{m_i}\}$ on $[0,1]$ and $\{v'_{m_i}(t)\}$ has a uniformly convergent subsequence $\{v'_{m_{i_j}}(t)\}$ on $[0,1/2]$. Without loss of generality, we may assume that there is a function $v\in C[0,1]\cap C^1[0,1/2]$ with $\lim\limits_{m\to\infty} v_m(t)=v(t)$ uniformly on $[0,1]$ and $\lim\limits_{m\to\infty} v_m'(t)=v'(t)$ uniformly on $[0,1/2]$. Obviously, $v'(0)=0$ and $v(1)=0$, $\|v\|\le r$. In particular, (3.14) implies that $v(t)\ge(1-t)\delta_r$ on $(0,1)$. Fixing $t\in(0,1)$, we have that $v_m$, $m\in\mathbb{N}_0$, satisfies the integral equation

$$v_m(t)=v_m(0)-\int_0^t\bigg(\int_0^s n\tau^{n-1}f\bigg(\max\bigg\{\frac{1}{m},v_m(\tau)\bigg\}\bigg)\,d\tau\bigg)^{1/n} ds,\quad t\in(0,1).$$

Let $m\to\infty$ through $\mathbb{N}_0$ (we note here that $f$ is uniformly continuous on compact subsets of $(0,r]$) to obtain

$$v(t)=v(0)-\int_0^t\bigg(\int_0^s n\tau^{n-1}f(v(\tau))\,dx\bigg)^{1/n} ds,\quad \text{for all } t\in(0,1).$$

We can do this argument for each $t \in (0,1)$ and so $((-v'(t))^n)' = nt^{n-1}f(v(t))$, for $0 < t < 1$. Finally it is easy to see that $\|v\| < r$. $\square$

THEOREM 3.5. *Suppose the conditions of Theorem* 3.4 *hold and*

$$\lim_{x\to+\infty} \frac{f(x)}{x^n} = +\infty. \tag{3.19}$$

*Then* BVP (1.1) *has at least two positive solutions.*

PROOF. From (3.10) and (3.19), choose $r > 0$ as in (3.11), $n_0 > 0$ with $1/n_0 < r$, and $R > \max\{r, R'\}$ in Lemma 3.3. Set $\mathbb{N}_0 = \{n_0, n_0+1, \dots\}$, and

$$\Omega_1 = \{y \in C[0,1] : \|y\| < r\}, \qquad \Omega_2 = \{y \in C[0,1] : \|y\| < R\}.$$

From the proofs of Theorem 3.4 and Lemma 3.3, we have

$$i(T_{1/m}, \Omega_1 \cap P, P) = 1 \quad \text{and} \quad i(T_{1/m}, \Omega_2 \cap P, P) = 0,$$

which imply that

$$i(T_{1/m}, (\Omega_2 - \overline{\Omega}_1) \cap P, P) = -1.$$

Then, there exist $x_{1,m} \in \Omega_1 \cap P$ and $x_{2,m} \in (\Omega_2 - \overline{\Omega}_1) \cap P$ such that

$$T_{1/m}x_{1,m} = x_{1,m}, \qquad T_{1/m}x_{2,m} = x_{2,m}.$$

From the proof of Theorem 3.4, there exist a subsequence $\{x_{1,m_i}\}$ of $\{x_{1,m}\}$ and $x_1 \in P \cap \Omega_1$ such that

$$\lim_{m_i\to+\infty} x_{1,m_i}(t) = x_1(t), \quad \text{for all } t \in [0,1],$$

and moreover, $x_1(t)$ is a positive solution to BVP (1.1) with $r > x_1(t) \geq \delta_r(1-t)$, for all $t \in [0,1]$.

A similar argument shows that there exist a subsequence $\{x_{2,m_j}\}$ of $\{x_{2,m}\}$ and $x_2 \in P \cap (\Omega_2 - \overline{\Omega}_1)$ such that

$$\lim_{m_j\to+\infty} x_{1,m_j}(t) = x_2(t), \quad \text{for all } t \in [0,1],$$

and $x_2(t)$ is a positive solution to BVP (1.1); while (3.11) guarantees $\|x_2\| > r$. Hence, $x_1$ and $x_2$ are two positive solutions to BVP (1.1). $\square$

THEOREM 3.6. *Suppose that all conditions of Theorem* 3.5 *hold. Then* BVP (1.1) *has a minimal positive solution and a maximal positive solution in* $C[0,1] \cap C^2(0,1)$.

PROOF. Let $\Omega = \{x(t) : x(t)$ is a $C[0,1] \cap C^2(0,1)$ positive solution to BVP (1.1)$\}$. From Theorem 3.4, we know that $\Omega$ is nonempty.

First, we show that $\Omega$ is bounded. From (3.19), there exists $R_1 > 1$ such that

$$f(x) \geq N^* x^n, \quad \text{for all } x \geq R_1, \tag{3.20}$$

where $N^* > 2^{3n}$. Let $R' = 2R_1$. We have

$$\|x\| \le R', \quad \text{for all } x \in \Omega. \tag{3.21}$$

Indeed, suppose that there exists $x_0 \in \Omega$ with $\|x_0\| > R'$. Lemma 2.4 guarantees that

$$x_0(t) \ge (1-t)\|x_0\| \ge (1-t)R', \quad \text{for all } t \in [0,1].$$

Then

$$x_0(t) \ge \frac{1}{2}\,\|x_0\| \ge \frac{1}{2}\,R' = R_1, \quad \text{for all } t \in \left[0, \frac{1}{2}\right],$$

which together (3.20) implies that

$$\begin{aligned} x_0(0) &= \int_0^1 \bigg( \int_0^s n\tau^{n-1} f(x_0(\tau))\,d\tau \bigg)^{1/n} ds \\ &\ge \int_{1/2}^1 \bigg( \int_0^{1/2} n\tau^{n-1} f(x_0(\tau))\,d\tau \bigg)^{1/n} ds \\ &\ge \int_{1/2}^1 \bigg( \int_0^{1/2} n\tau^{n-1} N^* (x_0(\tau))^n\,d\tau \bigg)^{1/n} ds = \frac{1}{8}\,{N^*}^{1/n}\|x_0\| > \|x_0\|, \end{aligned}$$

a contradiction. Hence (3.21) is true. Now, Lemma 3.2 implies that there exists $\delta_{R'} > 0$ such that

$$x(t) \ge (1-t)\delta_{R'}, \quad \text{for all } t \in [0,1].$$

Define a partial order "$\le$" in $\Omega$: $x \le y$ if and only if $x(t) \le y(t)$ for any $t \in [0,1]$. We prove only that any chain in $\langle \Omega, \le \rangle$ has lower and upper bounds in $\Omega$. The rest is obtained from Zorn's Lemma.

Let $\{x_\alpha(t)\}$ be a chain in $\langle \Omega, \le \rangle$. Since $C[0,1]$ is a separable Banach space, there exists an at most denumerable set $\{x_m(t)\}$, which is dense in $\{x_\alpha(t)\}$. Without loss of generality, we may assume that $\{x_m(t)\} \subseteq \{x_\alpha(t)\}$.

Set $z_m(t) = \min\{x_1(t), \ldots, x_m(t)\}$, $y_m(t) = \max\{x_1(t), \ldots, x_m(t)\}$. Since $\{x_\alpha(t)\}$ is a chain, $z_m(t)$, $y_m(t) \in \Omega$ for any $m \in \mathbb{N}_0$ and $\delta_{R'}(1-t) \le z_{m+1}(t) \le z_m(t)$, $R' \ge y_{m+1}(t) \ge y_m(t)$ for any $m \in \mathbb{N}_0$.

From the proofs of (3.15) and (3.18), we get that uniformly in $t$

$$\begin{aligned} &\lim_{m\to\infty} z_m(t) = z(t), \quad t \in [0,1], &&\lim_{m\to\infty} z'_m(t) = z'(t), \quad t \in [0,1/2], \\ &\lim_{m\to\infty} y_m(t) = y(t), \quad t \in [0,1], &&\lim_{m\to\infty} y'_m(t) = y'(t), \quad t \in [0,1/2]. \end{aligned}$$

We prove that $y, z \in \Omega$. From Theorem 3.4, we know that $y_m$ and $z_m$, $m \in \mathbb{N}_0$, satisfy the integral equations

$$y_m(t) = y_m(0) - \int_0^t \bigg( \int_0^s n\tau^{n-1} f(y_m(\tau))\,d\tau \bigg)^{1/n} ds, \quad \text{for all } t \in (0,1),$$

and

$$z_m(t) = z_m(0) - \int_0^t \bigg( \int_0^s n\tau^{n-1} f(z_m(\tau))\,d\tau \bigg)^{1/n} ds, \quad \text{for all } t \in (0,1).$$

Let $m \to \infty$ through $\mathbb{N}_0$ (we note here that $f$ is uniformly continuous on compact subsets of $(0,r]$) to obtain

$$y(t) = y(0) - \int_0^t \left( \int_0^s n\tau^{n-1} f(y(\tau))\, d\tau \right)^{1/n} ds, \quad \text{for all } t \in (0,1),$$

and

$$z(t) = z(0) - \int_0^t \left( \int_0^s n\tau^{n-1} f(z(\tau))\, d\tau \right)^{1/n} ds, \quad \text{for all } t \in (0,1),$$

and so $z, y \in \Omega$.

For any $x(t) \in \{x_\alpha(t)\}$, there exists $\{x_{m_k}(t)\} \subseteq \{x_m(t)\}$ such that $\|x_{m_k} - x\| \to 0$. Noticing that $y(t) \ge y_{m_k}(t) \ge x_{m_k}(t) \ge z_{m_k}(t) \ge z(t)$, $t \in [0,1]$, and letting $m_k \to \infty$, we have $y(t) \ge x(t) \ge z(t)$, $t \in [0,1]$, i.e. $\{x_\alpha(t)\}$ has lower and upper bounds in $\Omega$.

Zorn's Lemma shows that BVP (1.1) has a minimal $C[0,1] \cap C^2(0,1)$ positive solution and a maximal $C[0,1] \cap C^2(0,1)$ positive solution. □

Example 3.7. Consider

$$\begin{cases} ((-u'(t))^n)' = nt^{n-1}(u^{-\alpha} + u^{\beta} + 1 - \sin u^2)^n, & 0 < t < 1, \\ u'(0) = 0, \quad y(1) = 0, \end{cases} \tag{3.22}$$

where $\alpha > 0$, $\beta > 1$ and

$$\sup_{r \in (0,+\infty)} \frac{1}{1+\alpha} \frac{r^{\alpha+1}}{1 + r^{\alpha} + r^{\alpha+\beta}} > \frac{1}{2}.$$

Then BVP (3.22) has at least two positive solutions, a minimal positive solution and a maximal positive solution in $C[0,1] \cap C^2(0,1)$.

It is easy to prove that all conditions of Theorem 3.6 hold and our conclusion is true.

## 4. Positive solutions for singular boundary value problems with sign-changing nonlinearities

We shall consider the following conditions:

($H_1$) There exists a decreasing function $F(y) \in C((0,+\infty),(0,+\infty))$ and a function $G(y) \in C([0,+\infty),[0,+\infty))$ such that $f(y) \le (F(y) + G(y))^n$, and there exists $R > 1$ such that

$$\int_0^R \frac{dy}{F(y)} \cdot \left(1 + \frac{\overline{G}(R)}{F(R)}\right)^{-1} > \frac{1}{2},$$

where $\overline{G}(R) = \max_{s \in [0,R]} G(s)$.

($H_2$) $n > 1$ is a even number.

For $y \in C[0,1]$, we define the operator $T_m$ as

$$(4.1)\qquad (T_m y)(t) = \frac{1}{m} + \int_t^1 \bigg( \int_0^s n\tau^{n-1} f\bigg( \max\bigg\{ \frac{1}{m}, y(\tau) \bigg\} \bigg) \, d\tau \bigg)^{1/n} ds,$$

for $0 \le t \le 1$, $m \in \{1, 2, \ldots\}$. From a standard argument (see [6]), we have the following result.

Lemma 4.1. *Suppose* $(\mathrm{C}_1)$–$(\mathrm{C}_2)$ *hold. Then the operator* $T_m$ *is continuous and compact from* $C[0,1]$ *to* $C[0,1]$.

Lemma 4.2. *Suppose* $(\mathrm{C}_1)$–$(\mathrm{C}_2)$ *and* $(\mathrm{H}_1)$–$(\mathrm{H}_2)$ *hold. Then, for* $m$ *big enough, there exists* $x_m \in C[0,1]$ *with* $1/m \le x_m(t) \le R$ *such that*

$$(4.2)\qquad x_m(t) = \frac{1}{m} + \int_t^1 \bigg( \int_0^s n\tau^{n-1} f(x_m(\tau)) \, d\tau \bigg)^{1/n} ds, \quad 0 \le t \le 1.$$

Proof. From $(\mathrm{C}_2)$, there exist two positive constants $a > 0$ and $b > 0$ such that $f(y) \ge a$, for all $y \in (0, b]$. $(\mathrm{H}_1)$ guarantees that there exists $\varepsilon_0 > 0$ such that

$$(4.3)\qquad \int_{\varepsilon_0}^R \frac{dy}{F(y)} \cdot \bigg( 1 + \frac{\overline{G}(R)}{F(R)} \bigg)^{-1} > \frac{1}{2}.$$

Choose $n_0 > 3$ with $1/n_0 < \min\{\varepsilon_0, b\}$ and let $\mathbb{N}_0 = \{n_0, n_0+1, \ldots\}$. Lemma 4.1 implies that the operator $T_m$ is continuous and compact from $C[0,1]$ to $C[0,1]$, for $m \in \mathbb{N}_0$.

Let $\Omega = \{y \in C : \|y\| < R\}$. For $y \in \partial\Omega$, we now prove that

$$(4.4)\qquad \begin{aligned} y(t) &\ne \lambda (T_m y)(t) \\ &= \lambda \frac{1}{m} + \lambda \int_t^1 \bigg( \int_0^s n\tau^{n-1} f\bigg( \max\bigg\{ \frac{1}{m}, y(\tau) \bigg\} \bigg) \, d\tau \bigg)^{1/n} ds, \end{aligned}$$

for $0 \le t \le 1$, $n \in \mathbb{N}_0$ and any $\lambda \in (0,1]$.

Suppose that (4.4) is not true. Then there exist $y \in C[0,1]$, with $\|y\| = R$, and $0 < \lambda \le 1$ such that

$$(4.5)\qquad y(t) = \lambda (T_m y)(t) = \lambda \frac{1}{m} + \lambda \int_t^1 \bigg( \int_0^s n\tau^{n-1} f\bigg( \max\bigg\{ \frac{1}{m}, y(\tau) \bigg\} \bigg) \, d\tau \bigg)^{1/n} ds,$$

for $0 \le t \le 1$, $n \in \mathbb{N}_0$. We first claim that

$$(4.6)\qquad y(t) \ge \lambda \frac{1}{m}, \quad \text{for any } t \in [0,1].$$

Suppose that there exists $\eta \in (0,1)$ with $y(\eta) < \lambda 1/m$. Let $\gamma_0 = \inf\{t_1 : y(s) < \lambda/m, \text{ for all } s \in [t_1, \eta]\}$ and $\gamma_1 = \sup\{t_1 : y(s) < \lambda/m, \text{ for all } s \in [\eta, t_1]\}$. Since $y(1) = \lambda/m$, we have $\gamma_1 \le 1$ and $y(\gamma_1) = \lambda/m$.

If $\gamma_0 > 0$ we have $y(t) < \lambda/m$, for all $t \in (\gamma_0, \gamma_1)$ and $y(\gamma_0) = y(\gamma_1) = \lambda/m$, which implies that there exists $t_0 \in (\gamma_0, \gamma_1)$ such that $y'(t_0) = 0$. Differentiating (4.5), we have

$$0 = n(-y'(t_0))^{n-1}y''(t_0) = ((-y'(t_0))^n)' = \lambda n t_0^{n-1} f(1/m) > 0,$$

a contradiction.

If $\gamma_0 = 0$, there two cases to consider.

(I) $y(\gamma_0) = \lambda/m$. By the same argument as for $\gamma_0 > 0$, we get a contradiction.

(II) $y(\gamma_0) < \lambda/m$. If there exists $t_0 \in (0, \gamma_1)$ with $y'(t_0) = 0$, we also get a contradiction. If $y'(t) \neq 0$, for all $t \in (\gamma_0, \gamma_1) = (0, \gamma_1)$, we have $y'(t) > 0$, for all $t \in (0, \gamma_1)$. Differentiating (4.5), from ($\mathrm{H}_2$), we have

$$n(-y'(t))^{n-1}y''(t) = ((-y'(t))^n)' = \lambda n t_0^{n-1} f(1/m) > 0,$$

which implies that $y''(t) < 0$, for all $t \in (0, \gamma_1)$. Since $y'(0) = 0$, we have $y'(t) < 0$. This is a contradiction. Consequently, (4.6) holds.

Let $t^* = \sup\{t : y(t) = R,\ y'(t) = 0\}$. Obviously, $0 \leq t^* < 1$, $y'(t^*) = 0$, $y(t^*) = R$, $y(t) < R$, for all $t \in (t^*, 1]$. Let $t_1 = \inf\{t^* < t \leq 1 : y(t) = \lambda y(1)\}$. It is easy to see that $t^* < t_1 \leq 1$, $y(t) > y(t_1)$ for all $t \in (t^*, t_1)$.

Now we consider the properties of $y$ on $(t^*, t_1)$. We get a countable set $\{t_i\}$ in $(t^*, t_1]$ such that

- $t^* > \ldots \geq t_{2m} > t_{2m-1} > \ldots > t_5 \geq t_4 > t_3 \geq t_2 > t_1$, $t_{2m} \to t^*$,
- $y(t_{2i}) = y(t_{2i+1})$, $y'(t_{2i}) = 0$, $i = 1, 2, \ldots$,
- $y(t)$ is strictly decreasing in $[t_{2i}, t_{2i-1}]$, $i = 1, 2, \ldots$ (if $y(t)$ is strictly decreasing in $[t^*, t_1]$, put $m = 1$; i.e. $[t_2, t_1] = [t^*, t_1]$).

Differentiating (4.5) and using assumption ($\mathrm{H}_1$), we obtain

$$\begin{aligned}
(4.7)\quad ((-y'(t))^n)' &= \lambda n t^{n-1} f\bigg(\max\bigg\{\frac{1}{m}, y(t)\bigg\}\bigg) \\
&\leq \lambda n t^{n-1}\bigg(F\bigg(\max\bigg\{\frac{1}{m}, y(t)\bigg\}\bigg) + G\bigg(\max\bigg\{\frac{1}{m}, y(t)\bigg\}\bigg)\bigg)^n \\
&= \lambda n t^{n-1} F^n\bigg(\max\bigg\{\frac{1}{m}, y(t)\bigg\}\bigg)\bigg(1 + \frac{G(\max\{1/m, y(t)\})}{F(\max\{1/m, y(t)\})}\bigg)^n \\
&< n t^{n-1} F^n\bigg(\max\bigg\{\frac{1}{m}, y(t)\bigg\}\bigg)\bigg(1 + \frac{\overline{G}(R)}{F(R)}\bigg)^n \\
&\leq n t^{n-1} F^n(y(t))\bigg(1 + \frac{\overline{G}(R)}{F(R)}\bigg)^n,
\end{aligned}$$

for $t \in [t_{2i}, t_{2i-1})$, $i = 1, 2, \ldots$

Integrating (4.7) from $t_{2i}$ to $t$, we have due to the decreasing property of $F$,

$$\int_{t_{2i}}^{t} ((-y'(s))^n)'\,ds \le \bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)^n \int_{t_{2i}}^{t} ns^{n-1}F^n(y(s))\,ds \le F^n(y(t))\bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)^n (t^n - t_{2i}^n),$$

for $t \in [t_{2i}, t_{2i-1})$, $i = 1, 2, \ldots$; that is to say

$$(-y'(t))^n \le F^n(y(t))\bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)^n (t^n - t_{2i}^n), \tag{4.8}$$

for $t \in [t_{2i}, t_{2i-1})$, $i = 1, 2, \ldots$ It follows from (4.8) that

$$-\frac{y'(t)}{F(y(t))} \le \bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)t, \tag{4.9}$$

for $t \in [t_{2i}, t_{2i-1})$, $i = 1, 2, \ldots$

On the other hand, for any $z \in (0,1)$ with $y(z) > \lambda 1/m$, we can choose $i_0$ and $z' \in (t^*, t_1)$ such that $z' \in [t_{2i_0}, t_{2i_0-1})$, $y(z') = y(z)$ and $z \le z'$. Integrating (4.9) from $t_{2i}$ to $t_{2i-1}$, $i = 1, \ldots, i_0 - 1$, and from $t_{2i_0}$ to $z'$, we have

$$\int_{y(t_{2i-1})}^{y(t_{2i})} \frac{dy}{F(y)} \le \bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)\int_{t_{2i}}^{t_{2i-1}} t\,dt = \bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)\frac{1}{2}(t_{2i-1}^2 - t_{2i}^2) \tag{4.10}$$

for $i = 1, \ldots, i_0 - 1$, and

$$\int_{y(t_{2i_0-1})}^{y(z')} \frac{dy}{F(y)} \le \bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)\int_{z'}^{t_{2i_0-1}} t\,dt = \bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)\frac{1}{2}({t_{2i_0-1}}^2 - {z'}^2). \tag{4.11}$$

Summing (4.10) from 1 to $i_0 - 1$, we have by (4.11) and $y(t_{2i}) = y(t_{2i+1})$, that

$$\int_{y(t_1)}^{y(z')} \frac{dy}{F(y)} \le \bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)\frac{1}{2}(t_1^2 - {z'}^2).$$

Since $y(z) = y(z')$,

$$\int_{y(t_1)}^{y(z)} \frac{dy}{F(y)} \le \bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)\frac{1}{2}(t_1^2 - z^2). \tag{4.12}$$

Letting $z \to t^*$ in (4.12), we have

$$\int_{\varepsilon_0}^{R} \frac{dy}{F(y)} \le \int_{y(t_1)}^{R} \frac{dy}{F(y)} \le \bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)\frac{1}{2}(t_1^2 - {t^*}^2) \le \bigg(1+\frac{\overline{G}(R)}{F(R)}\bigg)\frac{1}{2},$$

which contradicts (4.3). Hence (4.4) holds.

It follows from Lemma 2.3 that $T_m$ has a fixed point $x_m$ in $C[0,1]$. Using $x_m$ and 1 in place of $y$ and $\lambda$ in (4.5), we obtain easily that $1/m \le x_m(t) \le R$, $t \in [0,1]$. Since $x_m$ satisfies

$$x_m(t) = \frac{1}{m} + \int_t^1 \bigg(\int_0^s n\tau^{n-1} f\bigg(\max\bigg\{\frac{1}{m}, x_m(\tau)\bigg\}\bigg)\,d\tau\bigg)^{1/n} ds,$$

for $t \in [0,1]$, we have that (4.2) holds. $\square$

LEMMA 4.3. *Suppose that all conditions of Lemma* 4.2 *hold and* $x_m$ *satisfies* (4.2). *For a fixed* $h \in (0,1)$, *let* $M_{m,h} = \min\{x_m(t) : t \in [0,h]\}$. *Then*

$$M_h = \inf\{M_{m,h}\} > 0.$$

PROOF. Since $x_m(t) \geq 1/m > 0$, we get $M_h \geq 0$. For any fixed natural numbers $m$ ($m > n_0$ defined in Lemma 4.2), let $t_m \in [0,h]$ be such that $x_m(t_m) = \min\{x_m(t) : t \in [0,h]\}$. If $M_h = 0$, then there exists a countable set $\{m_i\}$ such that

$$\lim_{m_i \to +\infty} x_{m_i}(t_{m_i}) = 0.$$

So there exists $N_0$ such that $x_{m_i}(t_{m_i}) < b$ (defined in Lemma 4.2), $m_i > N_0$. Let $\overline{\mathbb{N}}_0 = \{m_i > N_0 : m_i \in \mathbb{N}_0$ with $\lim_{m_i \to +\infty} x_{m_i}(t_{m_i}) = 0\}$. Then we have two cases.

*Case 1.* There exist $m_k \in \overline{\mathbb{N}}_0$ and $t^*_{m_k} \in (0,1)$ such that $x'_{m_k}(t^*_{m_k}) = 0$. By the same argument as in Lemma 4.2, we have

$$0 = ((-x'_{m_k}(t^*_{m_k}))^n)' = n{t^*}^{n-1}_{m_k} f(x_{m_k}(t^*_{m_k})) > 0, \tag{4.13}$$

a contradiction.

*Case 2.* $x'_{m_i}(t) < 0$ for all $t \in (0,1)$, $m_i \in \overline{\mathbb{N}}_0$. From $\lim_{m_i \to +\infty} x_{m_i}(t_{m_i}) = 0$, we have

$$\lim_{m_i \to +\infty} x_{m_i}(t) = 0 \text{ uniformly on } [h,1] \tag{4.14}$$

and $0 < x_{m_i}(t) < b$, for all $t \in [h,1]$, $m_i \in \overline{\mathbb{N}}_0$, which yields that $f(x_{m_i}(t)) \geq a$, for all $t \in [h,1]$, $m_i \in \overline{\mathbb{N}}_0$. Then, for any $t \in [h,(h+1)/2]$, we have

$$\begin{aligned}
x_{m_i}(t) &= \frac{1}{m_i} + \int_t^1 \left( \int_0^s n\tau^{n-1} f(x_{m_i}(\tau))\, d\tau \right)^{1/n} ds \\
&\geq \int_{(h+1)/2}^1 \left( \int_0^s n\tau^{n-1} f(x_{m_i}(\tau))\, d\tau \right)^{1/n} ds \\
&\geq \int_{(h+1)/2}^1 \left( \int_h^{(h+1)/2} n\tau^{n-1} f(x_{m_i}(\tau))\, d\tau \right)^{1/n} ds \\
&\geq \int_{(h+1)/2}^1 \left( \int_h^{(h+1)/2} n\tau^{n-1} a\, d\tau \right)^{1/n} ds \\
&= a^{1/n} \left( \left( \frac{h+1}{2} \right)^n - h^n \right)^{1/n} \frac{1-h}{2} > 0,
\end{aligned}$$

which contradicts (4.14). Hence, $M_h > 0$. □

THEOREM 4.4. *If* ($C_1$)–($C_2$) *and* ($H_1$)–($H_2$) *hold, then* BVP (1.1) *has at least one positive solution.*

Proof. For any natural numbers $n \in \mathbb{N}_0$ (defined in Lemma 4.2), it follows from Lemma 4.2 that there exist $x_m \in C$, $1/m \le x_m(t) \le R$ for all $t \in [0,1]$, satisfying (4.2). Now we divide the proof into two steps.

*Step 1.* There exists a convergent subsequence of $\{x_m\}$ in $[0,1)$. For a natural number $k \ge 3$, it follows from Lemma 4.3 that $0 < m_{1-1/k} \le x_m(t) \le R$, $t \in [0,1-1/k]$, for any natural numbers $m \in \mathbb{N}_0$; i.e. $\{x_m\}$ is uniformly bounded in $[0,1-1/k]$. Since $x_m$ also satisfies

$$|((-x_m'(t))^n)'| \le nt^{n-1}|f(x_m(t))| \le n \max_{x \in [m_{1-1/k},R]} f(r), \tag{4.15}$$

for $t \in [0,1-1/k]$, it follows from inequality (4.15) that $\{x_m\}$ and $\{x_m'\}$ are equicontinuous in $[0,1-1/k]$. The Ascoli–Arzela Theorem guarantees that there exists a subsequence of $\{x_n'(t)\}$ which converges uniformly on $[0,1-1/k]$. We may choose the diagonal sequence $\{{x_k^{(k)}}'(t)\}$ which converges everywhere in $[0,1)$ and it is easy to verify that $\{{x_k^{(k)}}'(t)\}$ converges uniformly on any interval $[0,d] \subseteq [0,1)$. Without loss of generality, let $\{{x_k^{(k)}}'(t)\}$ be $\{x_n'(t)\}$ in what follows. Putting $x(t) = \lim_{n\to+\infty} x_n(t)$ and $x'(t) = \lim_{n\to+\infty} x_n'(t)$, $t \in [0,1)$, we have that $x'(t)$ is continuous in $[0,1)$ and $x(t) \ge m_h > 0$, $t \in [0,h]$, for any $h \in (0,1)$ by Lemma 4.3.

*Step 2.* Fix $t \in (0,1)$, we have

$$x_m(t) = x_m(0) - \int_0^t \bigg(\int_0^s n\tau^{n-1} f(x_m(\tau))\,d\tau\bigg)^{1/n} ds.$$

Letting $m \to +\infty$ in the above equation, we have

$$x(t) = x(0) - \int_0^t \bigg(\int_0^s n\tau^{n-1} f(x(\tau))\,d\tau\bigg)^{1/n} ds. \tag{4.16}$$

Differentiating (4.16), we get

$$((-x'(t))^n)' = nt^{n-1} f(x(t)), \quad \text{for all } t \in (0,1). \tag{4.17}$$

Since $x_m'(0) = 0$ and $\{x_m'(t)\}$ is uniformly continuous on $[0,h]$ for any $1 > h > 0$, we have

$$x'(0) = 0. \tag{4.18}$$

Let $t_m = \sup\{t : x_m(t) = \|x_m\|,\ x_m'(t) = 0,\ t \in [0,1)\}$. Then $t_m \in [0,1)$, $x_m(t_m) = \|x_m\|$ and $x_m'(t_m) = 0$. Using $x_m(t)$, 1, $t_m$ in place of $y(t)$, $\lambda$ and $t^*$ in Lemma 4.2, from (4.12), we obtain easily by

$$\int_{1/m}^{\|x_m\|} \frac{dx}{F(x)} \le \bigg(1 + \frac{\overline{G}(R)}{F(R)}\bigg)\frac{1}{2}(1 - t_m^2).$$

It follows from above inequalities that $b_1 = \sup\{t_m\} < 1$. Fixed $z \in (b_1, 1)$, we get $1/m \le x_m(z) < \|x_m\| \le R$. From (4.12) and the proof of Lemma 4.2, one easily has

$$\int_{1/m}^{x_m(z)} \frac{dx}{F(x)} \le \left(1 + \frac{\overline{G}(R)}{F(R)}\right)\frac{1}{2}\,(1 - z^2), \quad \text{for all } z \in (b, 1).$$

Letting $m \to +\infty$ in the above inequality, we have

$$\int_{0}^{x(z)} \frac{dx}{F(x)} \le \left(1 + \frac{\overline{G}(R)}{F(R)}\right)\frac{1}{2}\,(1 - z^2), \quad \text{for all } z \in (b, 1). \tag{4.19}$$

It follows from (4.19) that

$$x(1) = \lim_{z \to 1^-} x(z) = 0. \tag{4.20}$$

Combining (4.17), (4.18) and (4.20), $x$ is a positive solution to BVP (1.1). □

EXAMPLE 4.5. Consider

$$\begin{cases} ((-u'(t))^8)' = 8t^7\left(\dfrac{1}{12}\,u^3(t) + \dfrac{1}{12}\,u^{-2}(t) - 100\right) & \text{for } 0 < t < 1, \\ u'(0) = 0, \quad u(1) = 0. \end{cases}$$

It is easy to prove that all conditions of Theorem 4.4 hold hence this problem has at least one positive solution.

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