Topological **M**ethods in **N**onlinear **A**nalysis
Volume 49, No. 2, 2017, 463–480
DOI: 10.12775/TMNA.2016.082

BOUNDEDNESS IN A TWO-SPECIES QUASI-LINEAR CHEMOTAXIS SYSTEM WITH TWO CHEMICALS

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Abstract. We consider the two-species quasi-linear chemotaxis system generalizing the prototype

$$\begin{cases} u_t = \nabla\cdot(D_1(u)\nabla u) - \chi_1\nabla\cdot(S_1(u)\nabla v), & x\in\Omega,\ t>0,\\ 0=\Delta v - v + w, & x\in\Omega,\ t>0,\\ w_t = \nabla\cdot(D_2(w)\nabla w) - \chi_2\nabla\cdot(S_2(w)\nabla z), & x\in\Omega,\ t>0,\\ 0=\Delta z - z + u, & x\in\Omega,\ t>0, \end{cases} \tag{0.1}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subseteq\mathbb{R}^N$ ($N\ge 1$). Here $D_i(u)=(u+1)^{m_i-1}$, $S_i(u)=u(u+1)^{q_i-1}$ ($i=1,2$), with parameters $m_i\ge 1$, $q_i>0$ and $\chi_1,\chi_2\in\mathbb{R}$. Hence, (0.1) allows the interaction of attraction-repulsion, with attraction-attraction and repulsion-repulsion type. It is proved that
(i) in the attraction-repulsion case $\chi_1<0$: if $q_1<m_1+2/N$ and $q_2<m_2+2/N-(N-2)^+/N$, then for any nonnegative smooth initial data, there exists a unique global classical solution which is bounded;
(ii) in the doubly repulsive case $\chi_1=\chi_2<0$: if $q_1<m_1+2/N-(N-2)^+/N$ and $q_2<m_2+2/N-(N-2)^+/N$, then for any nonnegative smooth initial data, there exists a unique global classical solution which is bounded;
(iii) in the attraction-attraction case $\chi_1=\chi_2>0$: if $q_1<2/N+m_1-1$ and $q_2<2/N+m_2-1$, then for any nonnegative smooth initial data, there exists a unique global classical solution which is bounded.
In particular, these results demonstrate that the circular chemotaxis mechanism underlying (0.1) goes along with essentially the same destabilizing features as known for the quasi-linear chemotaxis system in the doubly attractive case. These results generalize the results of Tao and Winkler (Discrete Contin. Dyn. Syst. Ser. B. 20 (9) (2015), 3165–3183) and also enlarge the parameter range $q>2/N-1$ (see Cieślak and Winkler (Nonlinearity 21 (2008), 1057–1076)).

2010 *Mathematics Subject Classification.* 35K55, 47J40, 92D25.
Key words and phrases. Boundedness; two-species; quasi-linear; chemotaxis.

1. Introduction

In this paper, we consider the initial-boundary value problem for the two-species quasi-linear chemotaxis system with two chemicals

$$
(1.1)\qquad \begin{cases} u_t = \nabla\cdot(D_1(u)\nabla u) - \chi_1\nabla\cdot(S_1(u)\nabla v), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + w, & x\in\Omega,\ t>0,\\ w_t = \nabla\cdot(D_2(w)\nabla w) - \chi_2\nabla\cdot(S_2(w)\nabla z), & x\in\Omega,\ t>0,\\ \tau z_t = \Delta z - z + u, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = \dfrac{\partial z}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x),\ w(x,0) = w_0(x), & x\in\Omega, \end{cases}
$$

where $\tau \in \{0,1\}$, Ω is a bounded domain in $\mathbb{R}^N$ $(N\geq 1)$ with smooth boundary $\partial\Omega$, $\Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2$, $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$, $\chi_i\in\mathbb{R}$ $(i=1,2)$ are parameters, which determine the attraction-repulsion case ($\chi_1 = 1$ and $\chi_2 = -1$), the repulsion-repulsion case ($\chi_1 = \chi_2 = -1$) and the attraction-attraction case ($\chi_1 = \chi_2 = 1$), respectively.

The first species, with density denoted by u, adapts its motion according to a chemical substance with concentration v, the latter being secreted by the second species, mathematically represented through its density w. The individuals of the second population themselves orient their movement along concentration gradients of a second signal with density z which in turn is produced by the first species. Moreover, we assume that

$$
(1.2)\qquad D_i, S_i \in C^2([0,\infty)) \quad\text{and}\quad S_i(u)\geq 0 \quad\text{for all } u\geq 0,
$$

satisfy

$$
(1.3)\qquad D_i(u) \geq C_{D_i}(u+1)^{m_i-1} \quad \text{for all } u\geq 0,
$$

$$
(1.4)\qquad S_i(u) \leq C_{S_i}u^{q_i} \quad \text{for all } u\geq 0,
$$

with $m_i\geq 1$, $q_i, C_{D_i}, C_{S_i} > 0$ $(i=1,2)$.

System (1.1) may be viewed as a simplified variant of a fully parabolic two-species chemotaxis model with two chemicals, involving slightly more general crossdiffusion mechanisms, as it has been proposed in [22] to describe chemotaxis-driven processes of cell sorting.

During the past decades, the chemotaxis models have become one of the best study models in numerous biological and ecological contexts, and one of the main issues is under what conditions the solutions of chemotaxis system blow up or exist globally. In order to better understand problem (1.1), let us mention some previous contributions in this direction. When $w\equiv u$ and $v\equiv z$, PDE system

(1.1) transforms into the classical chemotaxis system (one-species chemotaxis system with one chemical),

$$(1.5)\qquad \begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \chi\nabla\cdot(S(u)\nabla v), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v + u - v, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\Omega, \end{cases}$$

where $\chi>0$. This model has been studied extensively on blow-up and global existence (see e.g. Horstmann et al. [14], [15], Tao and Winkler [25], Ishida et al. [16], Winkler [35], Cieślak and Stinner [6], [7]). In particular, with $D(u)\equiv 1$ and $S(u)=u$, (1.5) turns into the classical Keller–Segel system, which has successfully been investigated up to now.

It is known that the model has only bounded solutions if $N=1$ ([21]); if $N=2$, there exists a threshold value for the initial mass that decides whether the solutions can blow up or exist globally in time ([11], [14], [33]); while in the case $N\geq 3$, there is no such threshold ([6], [33]–[35]). Especially, if $S(u)=u$, Horstmann and Winkler ([15]) showed that the solutions of (1.5) are global and bounded provided that $S(u)\leq c(u+1)^{2/N-\varepsilon}$ for all $u\geq 0$ with some $\varepsilon>0$ and $c>0$; while, if $S(u)\geq c(u+1)^{2/N+\varepsilon}$ for all $u\geq 0$ with $\varepsilon>0$ and $c>0$, $\Omega\subset\mathbb{R}^N$ ($N\geq 2$) is a ball, and some further technical conditions are satisfied, then the solutions become unbounded in finite or infinite time. In [25], Tao and Winkler proved that if $S(u)/D(u)\leq c(u+1)^{2/N+\varepsilon}$ for all $u\geq 0$ with some $\varepsilon>0$ and $c>0$, then the corresponding solutions are global and bounded provided that $D(u)$ satisfies some other technical conditions. We should point out that Winkler and Djie ([36]) discussed the following initial-boundary value problem:

$$(1.6)\qquad \begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \chi\nabla\cdot(S(u)\nabla v), & x\in\Omega,\ t>0,\\ 0 = \Delta v - M + u, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega,\\ \displaystyle\int_\Omega v(x,t)=0, & t>0, \end{cases}$$

where $M:=(1/|\Omega|)\int_\Omega u_0(x)\,dx$, and the functions $D(u)\cong u^{-p}$ and $S(u)\cong u^q$ as $u\cong\infty$ with some $p\geq 0$ and $q\in R$. They proved that if $p+q<2/N$, then all solutions of (1.6) are global in time and bounded. Conversely, if $p+q>2/N$ with $q>0$, and Ω is a ball, then the corresponding solutions of (1.6) will blow up in finite time.

However, to the best of our knowledge, few results are known for the two-species chemotaxis system with two chemicals (see Bellomo et al. [2], Murray [20], Cantrell et al. [5], Hibbing et al. [12], Kelly et al. [18], Painter and Sherratt [23], Biler et al. [3], Conca et al. [9], Espejo et al. [10], Lin et al. [19], Tello et al. [24], [28]). In particular, in [3], [24], [28], the authors showed that two species produce the same signal the gradient of which directs their movement. Let us remark that in the recent paper [27], Tao and Winkler proved the boundedness and blow-up to a two-species chemotaxis system with two chemicals

$$
\begin{cases}
u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\
0 = \Delta v - v + w, & x \in \Omega,\ t > 0, \\
w_t = \Delta u - \chi_2 \nabla \cdot (w \nabla z), & x \in \Omega,\ t > 0, \\
0 = \Delta z - z + u, & x \in \Omega,\ t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x),\ w(x,0) = w_0(x), & x \in \Omega.
\end{cases}
\tag{1.7}
$$

They found that the circular chemotaxis mechanism underlying (1.7) goes along with essentially the same destabilizing features as known for the classical Keller–Segel system in the doubly attractive case, but totally suppresses any blow-up phenomenon when only one, or both, taxis directions are repulsive.

Motivated by the above works, the aim of present paper is to study the chemotaxis system

$$
\begin{cases}
u_t = \nabla \cdot (D_1(u) \nabla u) - \chi_1 \nabla \cdot (S_1(u) \nabla v), & x \in \Omega,\ t > 0, \\
0 = \Delta v - v + w, & x \in \Omega,\ t > 0, \\
w_t = \nabla \cdot (D_2(w) \nabla w) - \chi_2 \nabla \cdot (S_2(w) \nabla z), & x \in \Omega,\ t > 0, \\
0 = \Delta z - z + u, & x \in \Omega,\ t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x),\ w(x,0) = w_0(x), & x \in \Omega,
\end{cases}
\tag{1.8}
$$

where D_i and S_i $(i = 1, 2)$ satisfy (1.2)–(1.4).

THEOREM 1.1. *Assume that the initial data u_0, w_0 are nonnegative functions with $(u_0, w_0) \in (C^0(\overline{\Omega}))^2$ and D_i, S_i $(i = 1, 2)$ satisfy (1.2)–(1.4). If one of the following cases holds:*

$$
\begin{aligned}
\chi_1 &< 0, & q_1 &< m_1 + \frac{2}{N} & &\text{and} & q_2 &< m_2 + \frac{2}{N} - \frac{(N-2)^+}{N}, \\
\chi_1 = \chi_2 &< 0, & q_1 &< m_1 + \frac{2}{N} - \frac{(N-2)^+}{N} & &\text{and} & q_2 &< m_2 + \frac{2}{N} - \frac{(N-2)^+}{N}, \\
\chi_1 = \chi_2 &> 0, & q_1 &< \frac{2}{N} + m_1 - 1 & &\text{and} & q_2 &< \frac{2}{N} + m_2 - 1,
\end{aligned}
$$

then problem (1.8) possesses a unique and uniformly bounded global classical solution $(u, v, w, z) \in (C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}((\overline{\Omega} \times (0, \infty)))^4$.

REMARK 1.2. (a) If $D_1(u) = S_1(u) = (u+1)^{m-1}$, $D_2(u) = S_2(u) = u(u+1)^{q-1}$, $\chi_1 = \chi_2 =: \chi > 0$ and $m = 1-p$, then Theorem 1.1 is consistent with Corollary 3.3 of Tello and Winkler ([36]). The result concerning a chemotactic collapse in the special case $D_1(u) \equiv D_2(u) = (u+1)^{-p}$ and $S_1(u) \equiv S_2(u) = u(u+1)^{-q}$ for problem is optimal. Namely, in [36], Winkler and Djie proved that if $p+q > 2/N$ with $p \geq 0$ and $q \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a ball, then the solutions will blow up.

(b) If $D_1(u) = S_1(u) = (u+1)^{m-1}$, $D_2(u) = S_2(u) = u$, $\chi_1 = \chi_2 =: \chi > 0$ and $m = 1-p$, then problem (1.8) possesses a unique and uniformly bounded global classical solution, which is consistent with Theorem 2.4 of Cieślak and Winkler ([8]). The result concerning a chemotactic collapse in the special case $D_1(u) \equiv D_2(u) = (u+1)^{-p}$ and $S_1(u) \equiv S_2(u) = u$ for problem is optimal. Indeed, in [8], the boundedness result was obtained for $p < 2/N - 1$, whereas for each $p > 2/N - 1$ radially symmetric solutions were constructed that blow up in finite time.

This paper is organized as follows. In Section 2, we recall some preliminary results and prove the local existence of classical solution to (1.8). Section 3 is devoted to prove the main results of this paper. More precisely, in this section, we first give a suitable upper bound of the $L^k(\Omega)$ $(k \geq 1)$ norm of solutions to problem (1.8). Next, the main results are proved by the standard Alikakos–Moser iteration (see e.g. [1] and Lemma A.1 of [25]).

2. Preliminaries

Before proving our main results, we will give some preliminary lemmas, which play a crucial role in getting the main results. As for the proofs of these lemmas, here we will not repeat them again. Throughout this paper the Hilbert space $H = L^2(\Omega)$ is equipped with usual inner product $(\,\cdot\,,\,\cdot\,)$ and norm $|\cdot|_2$.

LEMMA 2.1 ([37]). *Let $\theta \in (0,p)$. There exists a positive constant C_{GN} such that for all $u \in W^{1,2}(\Omega) \cap L^\theta(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \leq C_{GN}(|\nabla u|_2^a \|u\|_{L^\theta(\Omega)}^{1-a} + \|u\|_{L^\theta(\Omega)}),$$

is valid with $a = (N/\theta - N/p)/(1 - N/2 + N/\theta) \in (0,1)$.

The following lemma plays an important role in the proof of Theorem 1.1.

LEMMA 2.2 ([29]). *Let $y(t) \geq 0$ be a solution of problem*

$$\begin{cases} y'(t) + Ay^p \leq B, & t > 0, \\ y(0) = y_0 \geq 0, \end{cases} \tag{2.1}$$

with $A > 0$, $p > 0$ and $B \geq 0$. Then we have

$$y(t) \leq \max\left\{y_0, \left(\frac{B}{A}\right)^{1/p}\right\}, \quad t > 0.$$

The following local existence result is rather standard, which is similar with the reasoning in [8], [30], [31], [32], [36], [37]. We omit it here.

Lemma 2.3. *Let the nonnegative pair of functions $(u_0, w_0) \in (W^{1,\infty}(\Omega))^2$. Then there exist a maximal existence time $T_{\max} \in (0, \infty]$ and a quadruple of nonnegative functions $(u, v, w, z) \in (C^0(\Omega \times [0, T_{\max})) \cap C^{2,1}(\Omega \times [0, T_{\max})))^4$ classically solving* (1.8) *in $\Omega \times [0, T_{\max})$. Moreover, if $T_{\max} < +\infty$, then*

$$\|u(\,\cdot\,, t)\|_{L^\infty(\Omega)} + \|w(\,\cdot\,, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}. \tag{2.2}$$

3. A priori estimates

In this section, we are going to establish an iteration step to develop the main ingredient of our results. Before proving the main results, we shall introduce some notations. We can assume that $C_{D_i} = C_{S_i} = 1$ $(i = 1, 2)$ without loss of generality. The iteration depends on a series of a priori estimates.

Lemma 3.1 ([4]). *Suppose $f \in L^1(\Omega)$. Let ψ be a solution of the following initial boundary value problem:*

$$\begin{cases} -\Delta\psi + \psi = f, & x \in \Omega, \\ \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases}$$

Then, for all $l \in (1, N/(N-1))$, there exists a constant $c > 0$ such that

$$\|\psi(\,\cdot\,, t)\|_{W^{1,l}} \leq c\|f\|_{L^1(\Omega)} + c\|\psi\|_{L^1(\Omega)} \quad \textit{for all } \psi \in C(\overline{\Omega}) \textit{ fulfilling } \frac{\partial\psi}{\partial\nu} = 0.$$

Firstly, let us derive the following a priori boundness for the solutions of model (1.8).

Lemma 3.2. *Assume that (u, v, w, z) is the solution of* (1.8). *Then*

$$\int_\Omega u(x, t)\,dx = \int_\Omega u_0(x)\,dx \quad \textit{for all } t \in (0, T_{\max})$$

and

$$\int_\Omega w(x, t)\,dx = \int_\Omega w_0(x)\,dx \quad \textit{for all } t \in (0, T_{\max}).$$

Applying Lemmas 3.1 and 3.2, we can get the following lemma:

Lemma 3.3. *For all $l \in (1, N/(N-1))$, there exists a constant $c > 0$ such that*

$$\|v(\,\cdot\,, t)\|_{W^{1,l}} \leq c \quad \textit{and} \quad \|z(\,\cdot\,, t)\|_{W^{1,l}} \leq c \quad \textit{for all } t \in (0, T_{\max}).$$

PROOF. By Lemma 3.1, for all $l \in (1, N/(N-1))$, there exists a constant $c > 0$ such that

$$\|\psi(\,\cdot\,,t)\|_{W^{1,l}} \le c\|f\|_{L^1(\Omega)} + c\|\psi\|_{L^1(\Omega)} \quad \text{for all } \psi \in C(\overline{\Omega}) \text{ fulfilling } \frac{\partial \psi}{\partial \nu} = 0,$$

where $f = -\Delta\psi + \psi$. Hence

$$\|v(\,\cdot\,,t)\|_{W^{1,l}} \le c\|{-\Delta v}+v\|_{L^1(\Omega)}+c\|v\|_{L^1(\Omega)} \quad \text{for all } v \in C(\overline{\Omega}) \text{ fulfilling } \frac{\partial v}{\partial \nu} = 0.$$

On the other hand, due to Lemma 3.2, we have

$$\|v(\,\cdot\,,t)\|_{W^{1,l}} \le c\| - \Delta v + v\|_{L^1(\Omega)} + c\|v\|_{L^1(\Omega)} = c\|w\|_{L^1(\Omega)} + c\|v\|_{L^1(\Omega)} \le c.$$

By the same arguments as in the above proof, we get $\|w(\cdot,t)\|_{W^{1,l}} \le c$. □

Next, we are in a position to improve the regularity of u in a higher L^p space. Firstly, we give the following lemma which plays an important role in obtaining the main results.

LEMMA 3.4. *Assume that* $\chi_2 < 0$. *Let* (u,v,w,z) *be a solution to* (1.8) *on* $(0,T_{\max})$. *Then, for all* $k > 1$, *there exists a positive constant* C *such that*

$$\int_\Omega (w(x,t)+1)^k\,dx \le C \quad \textit{for all } t \in (0,T_{\max}). \tag{3.1}$$

PROOF. Without loss of generality, we may assume that $\chi_2 = -1$. Multiplying $(1.8)_3$ by $(w+1)^{k-1}$, integrating over Ω, we get

$$\begin{aligned} \frac{1}{k}\frac{d}{dt}\|w+1\|^k_{L^k(\Omega)} + (k-1)\int_\Omega D_2(w)(w+1)^{k-2}|\nabla w|^2\,dx \\ = -(k-1)\int_\Omega S_2(w)(w+1)^{k-2}\nabla w\cdot\nabla z\,dx. \end{aligned} \tag{3.2}$$

Integrating by parts the first term on the right-hand side of (3.2), we obtain from the second equation in (1.8)

$$\begin{aligned} -(k-1)\int_\Omega S_2(w)(w+1)^{k-2}\nabla w\cdot\nabla z\,dx &= -(k-1)\int_\Omega \nabla\Psi(w)\cdot\nabla z\,dx \\ &= (k-1)\int_\Omega \Psi(w)(z-w)\,dx \le (k-1)\int_\Omega \Psi(w)z\,dx \\ &\le (k-1)\int_\Omega (z+1)\int_0^w S_2(\tau)(\tau+1)^{k-2}\,d\tau dx \\ &\le \frac{(k-1)}{k+q_2-1}\int_\Omega (z+1)(w+1)^{k+q_2-1}\,dx \\ &\le \frac{(k-1)}{k+q_2-1}\bigg(\int_\Omega (w+1)^{(k+q_2-1)\gamma}\,dx\bigg)^{1/\gamma}\bigg(\int_\Omega (z+1)^{\gamma'}\,dx\bigg)^{1/\gamma'}, \end{aligned} \tag{3.3}$$

where

$$\Psi(w) = \int_0^w S_2(\tau)(\tau+1)^{k-2}\,d\tau \tag{3.4}$$

and $1/\gamma + 1/\gamma' = 1$.

Case 1. $N \leq 2$, due to Lemma 3.3 and the Sobolev embedding theorem, we have

$$\left(\int_\Omega (z+1)^{\gamma'}\, dx \right)^{1/\gamma'} \leq C_1 \|z+1\|_{W^{1,s}(\Omega)} \leq C_2, \tag{3.5}$$

where C_i $(i = 1,2)$ are independent of $T_{\max}$ and $\gamma' < +\infty$. As $q_2 < m_2 + 2/N$, choose $\gamma' = 1 + 1/(m_2 + 2/N - q_2) > 1$ in (3.5). Next, due to (3.3) and (3.5), we have

$$\begin{aligned} -(k-1) & \int_\Omega S_2(w)(w+1)^{k-2} \nabla u \cdot \nabla z \, dx \\ & \leq C_3 \left(\int_\Omega (w+1)^{(q_2+k-1)\gamma}\, dx \right)^{1/\gamma} \\ & = C_3 \big\| (w+1)^{(k+m_2-1)/2} \big\|_{2(q_2+k-1)\gamma/(k+m_2-1)}^{2(q_2+k-1)/(k+m_2-1)} \\ & \leq C_4 \Big(\big| \nabla (w+1)^{(k+m_2-1)/2} \big|_2^{\lambda} \big\| (w+1)^{(k+m_2-10/2} \big\|_{2/(k+m_2-1)}^{1-\lambda} \\ & \quad + \big\| (w+1)^{(k+m_2-1)/2} \big\|_{2/(k+m_2-1)} \Big)^{2(q_2+k-1)/(k+m_2-1)} \\ & \leq C_5 \Big(\big| \nabla (w+1)^{(k+m_2-1)/2} \big|_2^{2\lambda(q_2+k-1)/(k+m_2-1)} + 1 \Big) \end{aligned} \tag{3.6}$$

with

$$\lambda = \frac{\dfrac{N[k+m_2-1]}{2} - \dfrac{N[k+m_2-1]}{2(q_2+k-1)\gamma}}{1 - \dfrac{N}{2} + \dfrac{N[k+m_2-1]}{2}} = [k+m_2-1] \frac{\dfrac{N}{2} - \dfrac{N}{2(q_2+k-1)\gamma}}{1 - \dfrac{N}{2} + \dfrac{N[k+m_2-1]}{2}}$$

in $(0,1)$. As $q_2 < m_2 + 2/N$, we have

$$\begin{aligned} \frac{2\lambda(q_2+k-1)}{k+m_2-1} & = 2(q_2+k-1) \frac{\dfrac{N}{2} - \dfrac{N}{2(q_2+k-1)\gamma}}{1 - \dfrac{N}{2} + \dfrac{N[k+m_2-1]}{2}} \\ & = \frac{N\left(q_2 + k - 1 - \dfrac{1}{\gamma}\right)}{1 - \dfrac{N}{2} + \dfrac{N[k+m_2-1]}{2}} < 2. \end{aligned} \tag{3.7}$$

Case 2. $N \geq 3$, due to Lemma 3.3 and the Sobolev embedding theorem, we have

$$\left(\int_\Omega (z+1)^{\gamma'}\, dx \right)^{1/\gamma'} \leq C_1 \|z+1\|_{W^{1,s}(\Omega)} \leq C_2, \tag{3.8}$$

where C_i $(i = 1,2)$ are independent of $T_{\max}$ and $\gamma' < N/(N-2)$.

As $q_2 < m_2 + 4/N - 1$, choose $\gamma' = N/(N-2) - \varepsilon(m_2 + 4/N - 1 - q_2) > 1$ in (3.8), where $\varepsilon > 0$ is a small constant.

Due to $q_2 < m_2 + 4/N - 1$, we have

$$\tag{3.9} \frac{2\lambda(q_2+k-1)}{k+m_2-1} = 2(q_2+k-1)\frac{\dfrac{N}{2}-\dfrac{N}{2(q_2+k-1)\gamma}}{1-\dfrac{N}{2}+\dfrac{N[k+m_2-1]}{2}} = \frac{N\left(q_2+k-1-\dfrac{1}{\gamma}\right)}{1-\dfrac{N}{2}+\dfrac{N[k+m_2-1]}{2}} < 2.$$

Next, due to (3.3) and (3.8), we have

$$\tag{3.10} \begin{aligned} -(k-1)&\int_\Omega S_2(w)(w+1)^{k-2}\nabla w\cdot\nabla z\,dx \\ &\le C_3\left(\int_\Omega (w+1)^{(q_2+k-1)\gamma}\,dx\right)^{1/\gamma} \\ &= C_3\big\|(w+1)^{(k+m_2-1)/2}\big\|_{2(q_2+k-1)\gamma/(k+m_2-1)}^{2(q_2+k-1)/(k+m_2-1)} \\ &\le C_4\big(\big|\nabla(w+1)^{(k+m_2-1)/2}\big|_2^{\lambda}\big\|(w+1)^{(k+m_2-1)/2}\big\|_{2/(k+m_2-1)}^{1-\lambda} \\ &\quad + \big\|(w+1)^{(k+m_2-1)/2}\big\|_{2/(k+m_2-1)}\big)^{2(q_2+k-1)/(k+m_2-1)} \\ &\le C_5\big(\big|\nabla(w+1)^{(k+m_2-1)/2}\big|_2^{2\lambda(q_2+k-1)/(k+m_2-1)}+1\big) \end{aligned}$$

with

$$\lambda = \frac{\dfrac{N[k+m_2-1]}{2}-\dfrac{N[k+m_2-1]}{2(q_2+k-1)\gamma}}{1-\dfrac{N}{2}+\dfrac{N[k+m_2-1]}{2}} = [k+m_2-1]\frac{\dfrac{N}{2}-\dfrac{N}{2(q_2+k-1)\gamma}}{1-\dfrac{N}{2}+\dfrac{N[k+m_2-1]}{2}}$$

in $(0,1)$. Thus, combining (3.6)–(3.9) and using the Young inequality, we have that there exists a positive constant C_6 such that

$$\tag{3.11} \begin{aligned} -(k-1)&\int_\Omega S_2(w)(w+1)^{k-2}\nabla w\cdot\nabla z\,dx \\ &\le \frac{(k-1)}{2}\int_\Omega D_2(w)(w+1)^{k-2}|\nabla w|^2\,dx + C_6, \end{aligned}$$

which together with (3.2) implies that

$$\tag{3.12} \frac{1}{k}\frac{d}{dt}\|w+1\|_{L^k(\Omega)}^k + \frac{(k-1)}{2}\int_\Omega D_2(w)(w+1)^{k-2}|\nabla w|^2\,dx \le C_7,$$

where C_7 is a positive constant. Employing the Hölder inequality to the second term on the left-hand side of (3.12) and using Lemma 2.2, we obtain the desired results. □

Lemma 3.5. *Let (u,v,w,z) be a solution to (1.8) on $(0,T_{\max})$. Then for any $k>1$, there exists a positive constant C such that*

$$\int_\Omega (u(x,t)+1)^k\,dx \le C \quad \textit{for all } t\in(0,T_{\max}). \tag{3.13}$$

Proof. Multiplying $(1.8)_1$ by $(u+1)^{k-1}$, integrating over Ω, we get

$$\begin{aligned}\frac{1}{k}\frac{d}{dt}\|u+1\|^k_{L^k(\Omega)} &+ (k-1)\int_\Omega D_1(u)(u+1)^{k-2}|\nabla u|^2\,dx \\ &= (k-1)\chi_1\int_\Omega S_1(u)(u+1)^{k-2}\nabla u\cdot\nabla v\,dx.\end{aligned} \tag{3.14}$$

Integrating by parts to the first term on the right-hand side of (3.14), we obtain from the second equation in (1.8)

$$\begin{aligned}(k-1)\chi_1&\int_\Omega S_1(u)(u+1)^{k-2}\nabla u\cdot\nabla v\,dx \\ &= (k-1)\chi_1\int_\Omega \nabla\widetilde{\Psi}(u)\cdot\nabla v\,dx \\ &= -(k-1)\chi_1\int_\Omega \widetilde{\Psi}(u)(v-w)\,dx \le (k-1)\chi_1\int_\Omega\widetilde{\Psi}(u)w\,dx \\ &\le (k-1)\chi_1\int_\Omega (w+1)\int_0^u S_1(\tau)(\tau+1)^{k-2}\,d\tau dx \\ &\le \frac{(k-1)}{k+q_1-1}\chi_1\int_\Omega (w+1)(u+1)^{k+q_1-1}\,dx \\ &\le \frac{(k-1)}{k+q_1-1}\chi_1\bigg(\int_\Omega (u+1)^{(k+q_1-1)\gamma}\,dx\bigg)^{1/\gamma}\bigg(\int_\Omega (w+1)^{\gamma'}\,dx\bigg)^{1/\gamma'},\end{aligned} \tag{3.15}$$

where

$$\widetilde{\Psi}(u) = \int_0^u S_1(\tau)(\tau+1)^{k-2}\,d\tau \tag{3.16}$$

and $1/\gamma + 1/\gamma' = 1$. On the other hand, due to Lemma 3.4, we have

$$\bigg(\int_\Omega (w+1)^{\gamma'}\,dx\bigg)^{1/\gamma'} \le C_1 \quad \text{for all } \gamma'>1. \tag{3.17}$$

By $q_1 < m_1 + 2/N$, choosing $\gamma' = 1 + 1/(m_1+2/N-q_1) > 1$ in (3.17) and from (3.15), we have

$$\begin{aligned}(k-1)\chi_1&\int_\Omega S_1(u)(u+1)^{k-2}\nabla u\cdot\nabla v\,dx \\ &\le C_2\bigg(\int_\Omega (u+1)^{(k+q_1-1)\gamma}\,dx\bigg)^{1/\gamma} \\ &= C_2\big\|(u+1)^{(k+m_1-1)/2}\big\|^{2(q_1+k-1)/(k+m_1-1)}_{2(q_1+k-1)\gamma/(k+m_1-1)} \\ &\le C_3\big(\big|\nabla(u+1)^{(k+m_1-1)/2}\big|^{\lambda}_2\big\|(u+1)^{(k+m_1-1)/2}\big\|^{1-\lambda}_{2/(k+m_1-1)}\end{aligned} \tag{3.18}$$

$$
\begin{aligned}
&+ \left\|(u+1)^{(k+m_1-1)/2}\right\|_{2/k+m_1-1}\Big)^{2(q_1+k-1)/(k+m_1-1)} \\
\le & C_4\big(|\nabla(u+1)^{(k+m_1-1)/2}|_2^{2\lambda(q_1+k-1)/(k+m_1-1)}+1\big)
\end{aligned}
$$

with

$$
\lambda = \frac{\dfrac{N[k+m_1-1]}{2} - \dfrac{N[k+m_1-1]}{2(q_1+k-1)\gamma}}{1-\dfrac{N}{2}+\dfrac{N[k+m_1-1]}{2}} = [k+m_1-1]\,\frac{\dfrac{N}{2}-\dfrac{N}{2(q_1+k-1)\gamma}}{1-\dfrac{N}{2}+\dfrac{N[k+m_1-1]}{2}}
$$

in $(0,1)$. Since $q_1 < m_1 + 2/N$, we have

$$
\begin{aligned}
\frac{2\lambda(q_1+k-1)}{k+m_1-1} &= 2(q_1+k-1)\,\frac{\dfrac{N}{2}-\dfrac{N}{2(q_1+k-1)\gamma}}{1-\dfrac{N}{2}+\dfrac{N[k+m_1-1]}{2}} \\
&= \frac{N\left(q_1+k-1-\dfrac{1}{\gamma}\right)}{1-\dfrac{N}{2}+\dfrac{N[k+m_1-1]}{2}} < 2.
\end{aligned} \tag{3.19}
$$

Thus, combining (3.18) with (3.19) and using the Young inequality, we have that there exists a positive constant C_5 such that

$$
\begin{aligned}
(k-1)\chi_1 &\int_\Omega S_1(u)(u+1)^{k-2}\nabla u\cdot\nabla v\,dx \\
&\le \frac{(k-1)}{2}\int_\Omega D_1(u)(u+1)^{k-2}|\nabla u|^2\,dx + C_5,
\end{aligned} \tag{3.20}
$$

which together with (3.2) implies that

$$
\frac{1}{k}\frac{d}{dt}\|u+1\|_{L^k(\Omega)}^k + \frac{(k-1)}{2}\int_\Omega D_1(u)(u+1)^{k-2}|\nabla u|^2\,dx \le C_6, \tag{3.21}
$$

where C_6 is a positive constant. Employing the Hölder inequality to the second term on the left-hand side of (3.21) and using Lemma 2.2, we obtain the desired results. □

LEMMA 3.6. *Assume that* $\chi_1, \chi_2 > 0$,

$$
q_1 < \frac{2}{N} + m_1 - 1 \quad \text{and} \quad q_2 < \frac{2}{N} + m_2 - 1. \tag{3.22}
$$

Let (u,v,w,z) *be a solution to* (1.8) *on* $(0,T_{\max})$. *Then there exist positive constants* C *and* γ_0 *such that*

$$
\int_\Omega (u(x,t)+1)^k\,dx + \int_\Omega (w(x,t)+1)^k\,dx \le C \tag{3.23}
$$

for all $t \in (0,T_{\max})$ *and* $k \in (1,\gamma_0]$.

PROOF. Firstly, due to (3.22) there exists

$$\gamma_1 \in \left(1, 1 + \min\left\{\frac{2/N + m_1 - 1 - q_1}{q_1}, \frac{2/N + m_2 - 1 - q_2}{q_2}, \frac{2}{(N-2)^+}, \frac{1}{N}\right\}\right),$$

such that

$$\gamma_1 q_1 < \frac{2}{N} + m_1 - 1 \quad \text{and} \quad \gamma_1 q_2 < \frac{2}{N} + m_2 - 1. \tag{3.24}$$

Hence, it then follows from $q_i > 0$ $(i = 1, 2)$ that there exists $\gamma_0 > \max\{N/2, \gamma_1/(\gamma_1 - 1) - q_1, \gamma_1/(\gamma_1 - 1) - q_2\}$ such that

$$\max\left\{\gamma_1(\gamma_0 + q_1 - 1), \frac{\gamma_1}{\gamma_1 - 1}\right\} = \gamma_1(\gamma_0 + q_1 - 1) < \gamma_0 + \frac{2}{N} + m_1 - 1, \tag{3.25}$$

$$\max\left\{\gamma_1(\gamma_0 + q_2 - 1), \frac{\gamma_1}{\gamma_1 - 1}\right\} = \gamma_1(\gamma_0 + q_2 - 1) < \gamma_0 + \frac{2}{N} + m_2 - 1. \tag{3.26}$$

Multiplying $(1.8)_1$ and $(1.8)_3$ by $(u+1)^{k-1}$ and $(w+1)^{k-1}$, respectively, and integrating over Ω, we get

$$\frac{1}{k}\frac{d}{dt}\|u+1\|^k_{L^k(\Omega)} + (k-1)\int_\Omega D_1(u)(u+1)^{k-2}|\nabla u|^2\,dx = (k-1)\chi_1\int_\Omega S_1(u)(u+1)^{k-2}\nabla u\cdot\nabla v\,dx, \tag{3.27}$$

$$\frac{1}{k}\frac{d}{dt}\|w+1\|^k_{L^k(\Omega)} + (k-1)\int_\Omega D_2(w)(w+1)^{k-2}|\nabla w|^2\,dx = (k-1)\chi_2\int_\Omega S_2(w)(w+1)^{k-2}\nabla w\cdot\nabla z\,dx. \tag{3.28}$$

Integrating by parts the first term on the right-hand side of (3.27) and (3.28), we obtain from the second and the fourth equation in (1.8)

$$(k-1)\chi_1\int_\Omega S_1(u)(u+1)^{k-2}\nabla u\cdot\nabla v\,dx \le \frac{(k-1)}{k+q_1-1}\,\chi_1\int_\Omega (w+1)(u+1)^{k+q_1-1}\,dx, \tag{3.29}$$

$$(k-1)\chi_2\int_\Omega S_2(w)(w+1)^{k-2}\nabla w\cdot\nabla z\,dx \le \frac{(k-1)}{k+q_2-1}\,\chi_2\int_\Omega (u+1)(w+1)^{k+q_2-1}\,dx. \tag{3.30}$$

Hence, by the Young inequality and (3.25)–(3.26), we have

$$(k-1)\chi_1\int_\Omega S_1(u)(u+1)^{k-2}\nabla u\cdot\nabla v\,dx + (k-1)\chi_2\int_\Omega S_2(w)(w+1)^{k-2}\nabla w\cdot\nabla z\,dx \tag{3.31}$$

$$
\begin{aligned}
&\le C_1 \int_\Omega \big((u+1)^{k+q_1-1}(w+1) + (u+1)(w+1)^{k+q_2-1}\big)\,dx \\
&\le C_2 \Big(\int_\Omega (u+1)^{\gamma_1(k+q_1-1)} + (w+1)^{\gamma_1'}\,dx \\
&\qquad + \int_\Omega (w+1)^{\gamma_1(k+q_2-1)} + (u+1)^{\gamma_1'}\,dx \Big) \\
&\le C_3 (\int_\Omega (u+1)^{\alpha_1}\,dx + \int_\Omega (w+1)^{\alpha_2}\,dx),
\end{aligned}
$$

where

$$
\alpha_1 = \gamma_1(k+q_1-1), \quad \alpha_2 = \gamma_1(k+q_2-1), \quad \frac{1}{\gamma_1} + \frac{1}{\gamma_1'} = 1. \tag{3.32}
$$

and C_i $(i = 1, 2, 3)$ are independent of $T_{\max}$ and k. On the other hand, due to Lemma 2.1, we have

$$
\begin{aligned}
&(k-1)\chi_1 \int_\Omega S_1(u)(u+1)^{k-2}\nabla u \cdot \nabla v\,dx \\
&\quad + (k-1)\chi_2 \int_\Omega S_2(w)(w+1)^{k-2}\nabla w \cdot \nabla z\,dx \\
&\le C_3 \int_\Omega (u+1)^{\alpha_1}\,dx + C_3 \int_\Omega (w+1)^{\alpha_2}\,dx \\
&= C_3 \|u+1\|_{\alpha_1}^{\alpha_1} + C_3 \|w+1\|_{\alpha_2}^{\alpha_2} \\
&= C_3 \big\|(u+1)^{(k+m_1-1)/2}\big\|_{2\alpha_1/(k+m_1-1)}^{2\alpha_1/(k+m_1-1)} \\
&\quad + C_3 \big\|(w+1)^{(k+m_2-1)/2}\big\|_{(2\alpha_2)/(k+m_2-1)}^{2\alpha_2/(k+m_2-1)} \\
&\le C_4 \Big(\big|\nabla (u+1)^{(k+m_1-1)/2}\big|_2^{\lambda} \big\|(u+1)^{(k+m_1-1)/2}\big\|_{2/(k+m_1-1)}^{1-\lambda} \\
&\quad + \big\|(u+1)^{(k+m_1-1)/2}\big\|_{2/(k+m_1-1)} \Big)^{2\alpha_1/(k+m_1-1)} \\
&\quad + C_5 \Big(\big|\nabla (w+1)^{k+m_2-1/2}\big|_2^{\mu} \big\|(w+1)^{(k+m_2-1)/2}\big\|_{2/(k+m_2-1)}^{1-\mu} \\
&\quad + \big\|(w+1)^{(k+m_2-1)/2}\big\|_{2/(k+m_2-1)} \Big)^{2\alpha_2/(k+m_2-1)} \\
&\le C_6 \Big(\big|\nabla (u+1)^{(k+m_1-1)/2}\big|_2^{2\lambda\alpha_1/(k+m_1-1)} + 1 \Big) \\
&\quad + C_7 \Big(\big|\nabla (w+1)^{(k+m_2-1)/2}\big|_2^{2\mu\alpha_2/(k+m_2-1)} + 1 \Big)
\end{aligned} \tag{3.33}
$$

with

$$
\lambda = \frac{\dfrac{N[k+m_1-1]}{2} - \dfrac{N[k+m_1-1]}{2\alpha_1}}{1 - \dfrac{N}{2} + \dfrac{N[k+m_1-1]}{2}} = [k+m_1-1]\,\frac{\dfrac{N}{2} - \dfrac{N}{2\alpha_1}}{1 - \dfrac{N}{2} + \dfrac{N[k+m_1-1]}{2}}
$$

in $(0,1)$ and

$$\mu = \frac{\frac{N[k+m_2-1]}{2} - \frac{N[k+m_2-1]}{2\alpha_2}}{1 - \frac{N}{2} + \frac{N[k+m_2-1]}{2}} = [k+m_2-1]\,\frac{\frac{N}{2} - \frac{N}{2\alpha_2}}{1 - \frac{N}{2} + \frac{N[k+m_2-1]}{2}}$$

in $(0,1)$. By (3.25) and (3.26) and $k \le \gamma_0$, we have

$$\begin{aligned}\frac{2\lambda\alpha_1}{k+m_1-1} &= 2\alpha_1\,\frac{\frac{N}{2} - \frac{N}{2\alpha_1}}{1 - \frac{N}{2} + \frac{N[k+m_1-1]}{2}} \\ &= \frac{N(\alpha_1-1)}{1 - \frac{N}{2} + \frac{N[k+m_1-1]}{2}} < 2\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}\frac{2\lambda\alpha_2}{k+m_2-1} &= 2\alpha_2\,\frac{\frac{N}{2} - \frac{N}{2\alpha_2}}{1 - \frac{N}{2} + \frac{N[k+m_2-1]}{2}} \\ &= \frac{N(\alpha_2-1)}{1 - \frac{N}{2} + \frac{N[k+m_2-1]}{2}} < 2.\end{aligned} \tag{3.35}$$

Thus, combining (3.33)–(3.35) and using the Young inequality, we have that there exists a positive constant C_7 such that

$$\begin{aligned}&(k-1)\chi_1 \int_\Omega S_1(u)(u+1)^{k-2}\nabla u \cdot \nabla v\,dx \\ &\quad + (k-1)\chi_2 \int_\Omega S_2(w)(w+1)^{k-2}\nabla w \cdot \nabla z\,dx \\ &\le \frac{(k-1)}{2}\int_\Omega D_1(u)(u+1)^{k-2}|\nabla u|^2\,dx \\ &\quad + \frac{(k-1)}{2}\int_\Omega D_2(w)(w+1)^{k-2}|\nabla w|^2\,dx + C_7,\end{aligned} \tag{3.36}$$

which together with (3.27) and (3.28) implies that

$$\begin{aligned}&\frac{1}{k}\frac{d}{dt}\left(\|u+1\|^k_{L^k(\Omega)} + \|w+1\|^k_{L^k(\Omega)}\right) \\ &\qquad + \frac{(k-1)}{2}\int_\Omega D_1(u)(u+1)^{k-2}|\nabla u|^2\,dx \\ &\qquad + \frac{(k-1)}{2}\int_\Omega D_2(w)(w+1)^{k-2}|\nabla w|^2\,dx \le C_8,\end{aligned} \tag{3.37}$$

where C_8 is a positive constant.

Since

$$\begin{aligned}\int_\Omega D_1(u)(u+1)^{k-2}|\nabla u|^2\,dx &\geq \int_\Omega \left|\nabla(u+1)^{(m_1+k-1)/2}\right|^2 dx \\ &\geq C_9 \int_\Omega \left|(u+1)^{(m_1+k-1)/2}\right|^2 dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega D_2(w)(w+1)^{k-2}|\nabla w|^2\,dx &\geq \int_\Omega \left|\nabla(w+1)^{(m_2+k-1)/2}\right|^2 dx \\ &\geq C_{10} \int_\Omega \left|(w+1)^{(m_2+k-1)/2}\right|^2 dx,\end{aligned}$$

letting

$$y := \int_\Omega (u+1)^k\,dx + \int_\Omega (w+1)^k\,dx$$

in (3.37), we get

$$\frac{d}{dt}\,y(t) + C_{11}y^h(t) \leq C_{12} \quad \text{for all } t \in (0, T_{\max})$$

with some positive constant h. Thus a standard ODE comparison argument implies boundedness of $y(t)$ for all $t \in (0, T_{\max})$. The proof of Lemma 3.6 is complete. □

A straightforward adaptation of the well-established Moser-type iteration procedure ([1] or Lemma A.1 of [25]) allows us to formulate a general condition which is sufficient for the boundedness of u and w.

Lemma 3.7. *Assume that* $\chi_1, \chi_2 > 0$ *and* (3.25)–(3.26) *hold. Moreover, suppose that the solutions of* (1.8) *fulfill*

$$\sup_{t\in(0,T_{\max})} \|1+u(\,\cdot\,,t)\|_{L^k(\Omega)} + \sup_{t\in(0,T_{\max})} \|1+w(\,\cdot\,,t)\|_{L^k(\Omega)} < \infty \tag{3.38}$$

with some $k > 1$ *satisfying* $k > \max\{N/2, \gamma_1/(\gamma_1-1)-q_1, \gamma_1/(\gamma_1-1)-q_2\}$, *where*

$$\gamma_1 = 1 + \frac{1}{2^k}\min\left\{\frac{2/N+m_1-1-q_1}{q_1}, \frac{2/N+m_2-1-q_1}{q_2}, \frac{2}{(N-2)^+}, \frac{1}{N}\right\}.$$

Then there exists $C > 0$ *such that*

$$\|u(\,\cdot\,,t)\|_{L^\infty(\Omega)} + \|w(\,\cdot\,,t)\|_{L^\infty(\Omega)} \leq C \quad \textit{for all } t \in (0, T_{\max}). \tag{3.39}$$

Proof. Firstly, according to (3.38) we can pick $k_0 \geq \max\{N/2, \gamma_1/(\gamma_1-1) -q_1, \gamma_1/(\gamma_1-1)-q_2\}$ such that

$$\int_\Omega (u+1)^{k_0}(x,t)\,dx + \int_\Omega (w+1)^{k_0}(x,t)\,dx \leq C_0 \quad \text{for all } t \in (0, T_{\max}) \tag{3.40}$$

with some $C_0 > 0$. Now, we may invoke Lemma A.1 in [25] which by means of a Moser-type iteration applied to the first and the third equation in (1.8) establishes

$$\|u(\,\cdot\,,t)\|_{L^\infty(\Omega)} + \|w(\,\cdot\,,t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0,\infty). \tag{3.41}$$

The proof of Lemma 3.7 is complete. □

Collecting Lemmas 3.4–3.7, we can prove Theorem 1.1.

Proof of Theorem 1.1. Theorem 1.1 will be proved if we can show that $T_{\max} = \infty$. Suppose on the contrary that $T_{\max} < \infty$. In view of Lemmas 3.4–3.7, $\|u(\,\cdot\,,t)\|_{L^\infty(\Omega)} \leq C$ and $\|w(\,\cdot\,,t)\|_{L^\infty(\Omega)} \leq C$ for all $t \in (0, T_{\max})$, where the constant C is independent of $T_{\max}$. This contradicts with Lemma 2.3. Hence the classical solution (u,v,w,z) of (1.8) is global in time and bounded. □

Acknowledgements. The authors are very grateful to the anonymous reviewers for their carefully reading and valuable suggestions which greatly improved this work. This work is partially supported by the National Natural Science Foundation Shandong Province of China (No. ZR2016AQ17) and the Doctor Start-up Funding of Ludong University (No. LA2016006).

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Manuscript received December 2, 2015

accepted April 17, 2016

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