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EXISTENCE OF MONOTONE SOLUTIONS FOR A NONLINEAR QUADRATIC INTEGRAL EQUATION OF VOLTERRA TYPE

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ABSTRACT. In this paper a sufficient condition for the existence of monotone solutions of the following nonlinear quadratic integral equation of Volterra type

$$x(t) = a(t) + g(x(t)) \int_0^t v(t, s, x(s))\, ds, \quad \text{for all } t \in [0, T],$$

is established. Our approach is based on Darbo's fixed point theorem and the measure of noncompactness introduced by Banaś and Olszowy. As applications, some examples to demonstrate our result are given.

1. Introduction and preliminaries. We are interested in the existence of monotone solutions for the following nonlinear quadratic integral equation of Volterra type:

$$x(t) = a(t) + g(x(t)) \int_0^t v(t, s, x(s))\, ds, \quad \text{for all } t \in [0, T], \tag{1.1}$$

where the functions $a = a(t)$, $g = g(x)$ and $v = v(t, s, x)$ appearing in (1.1) are given while $x = x(t)$ is an unknown function.

It is known that the theory of integral equations has various applications in engineering, mathematical physics, economics and biology. For details, we refer to [**1, 2, 10, 16, 17**] and the references therein. Within the past 20 year or so, many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations,

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which are special cases of equation (1.1). In particular, Banaś and Martinon [**5**] established the existence of monotone solutions for equation (1.1) with $g(x) \equiv x$ for all $x \in (-\infty, \infty)$.

The aim of this paper is to establish simple criteria for the existence of monotone solutions of equation (1.1). The main tools used in this paper are Darbo's fixed point theorem and the measure of noncompactness introduced by Banaś and Olszowy [**6**]. In Section 3, we construct three concrete examples to illustrate our result.

Let us recall some necessary notions, concepts and results. Throughout this paper, let $R = (-\infty, \infty)$, $R_+ = [0, \infty)$ and $I = [0, T]$, where T is a positive constant. Let $(E, \|\cdot\|)$ be an infinite-dimensional Banach space with the zero element θ and $B(\theta, r)$ denote the closed ball centered at θ and with radius r. Assume that $B(E)$ stands for the family of all nonempty bounded subsets of E and $C(I)$ represents the classical Banach space of all continuous functions $x : I \to R$ endowed with the standard norm

$$\|x\| = \max\{|x(t)| : t \in I\}, \quad x \in C(I).$$

For any nonempty bounded subset X of $C(I)$, $x \in X$ and $\varepsilon > 0$, define

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in I \text{ with } |t - s| \le \varepsilon\},$$

$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\}, \quad \omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon),$$

$$d(x) = \sup\{|x(t) - x(s)| - [x(t) - x(s)] : t, s \in I \text{ with } s \le t\},$$

$$d(X) = \sup\{d(x) : x \in X\} \quad \text{and} \quad \mu(X) = \omega_0(X) + d(X).$$

It follows from Banaś and Olszowy [**6**] that the function μ is a measure of noncompactness in the space $C(I)$ in the sense of the definition accepted in [**3**] and the kernel $\ker \mu$ of this measure includes nonempty bounded sets X such that functions from X are equicontinuous and nondecreasing on the interval I.

Theorem 1.1 [**3, 15**]. *Let D be a nonempty bounded closed convex subset of the space $C(I)$, and let $f : D \to D$ be a continuous mapping such that $\mu(fA) \le k\mu(A)$ for each nonempty subset A of D, where*

$k \in [0,1)$ *is a constant and* μ *is a measure of noncompactness on* $B(E)$. *Then* f *has at least one fixed point in* D.

2. Monotone solutions for the nonlinear quadratic integral equation of Volterra type (1.1). The following assumptions are adopted throughout this section:

(a) $a \in C(I)$ and the function a is nondecreasing and nonnegative on the interval I;

(b) $v : I \times I \times R \to R$ is a continuous function such that $v : I \times I \times R_+ \to R_+$ and for each $s \in I$ and $x \in R_+, v(\cdot, s, x)$ is nondecreasing on I;

(c) there exists a nondecreasing function $f : R_+ \to R_+$ satisfying

$$|v(t,s,x)| \leq f(|x|), \quad \text{for all} \quad t, s \in I \quad \text{and} \quad x \in R;$$

(d) $g : R \to R$ is continuous and $|g(t)| \leq g(|t|)$ for any $t \in R$, $g : R_+ \to R_+$ is differentiable and g' is nondecreasing on R_+ with $g'(0) \geq 0$;

(e) there exists a positive constant r satisfying

$$\|a\| + Tg(r)f(r) \leq r \quad \text{and} \quad Tg'(r)f(r) < 1.$$

Theorem 2.1. *Under assumptions* (a)–(e), *the equation* (1.1) *possesses at least one solution* $x = x(t)$ *which belongs to the space* $C(I)$ *and is nondecreasing on the interval* I.

Proof. First of all, we define an operator G as follows:

$$(Gx)(t) = a(t) + g(x(t)) \int_0^t v(t,s,x(s))\, ds, \tag{2.1}$$
$$\text{for all} \quad t \in I \quad \text{and} \quad x \in C(I).$$

It follows from assumptions (a), (b), (d) and (2.1) that Gx is continuous on I for each $x \in C(I)$. That is, G maps the space $C(I)$ into itself.

In light of assumption (c) and (2.1), we see that for any $x \in C(I)$ and $t \in R$,

$$\begin{aligned}|(Gx)(t)| &\le |a(t)| + |g(x(t))| \left| \int_0^t v(t,s,x(x))\,ds \right| \\ &\le \|a\| + g(|x(t)|) \int_0^t f(|x(s)|)\,ds \\ &\le ||a|| + Tg(\|x\|)f(\|x\|),\end{aligned}$$

which yields that

$$||Gx|| \le ||a|| + Tg(||x||)f(||x||), \quad \text{for all } x \in C(I).$$

Combining the above inequality and assumption (e), we get that the operator G transforms the ball $B(\theta, r)$ into itself. Let

$$B_r^+ = \{x \in B(\theta, r) : x(t) \ge 0, \text{ for all } t \in I\}.$$

It is easy to see that the set B_r^+ is nonempty, bounded, closed and convex. By virtue of assumptions (a), (b), (d) and (2.1), we conclude immediately that G transforms the set B_r^+ into itself. For any $\varepsilon > 0$, put

$$\begin{aligned}\alpha(\varepsilon, r) &= \sup\{|v(t,s,x) - v(t,s,y)| : \\ &\qquad \forall t, s \in I \text{ and } x, y \in [0, r] \text{ with } |x - y| \le \varepsilon\}, \\ \beta(\varepsilon, r) &= \sup\{|v(t,s,x) - v(p,s,x)| : \\ &\qquad \forall x \in [0, r] \text{ and } t, p, s \in I \text{ with } |t - p| \le \varepsilon\}.\end{aligned} \tag{2.2}$$

Now we assert that G is continuous on the set B_r^+. For each $\varepsilon > 0$, $t \in I$ and $x, y \in B_r^+$ with $||x - y|| \le \varepsilon$, by (2.1), (2.2), the mean value theorem and assumptions (b)–(d), we deduce that

$$\begin{aligned}&|(Gx)(t) - (Gy)(t)| \\ &\quad = \left| g(x(t)) \int_0^t v(t,s,x(s))\,ds - g(y(t)) \int_0^t v(t,s,y(t))\,dt \right| \\ &\quad \le |g(x(t)) - g(y(t))| \int_0^t v(t,s,x(s))\,ds + g(y(t)) \\ &\qquad \times \int_0^t |v(t,s,x(s)) - v(t,s,y(t))|\,ds \\ &\quad = g'(\zeta)|x(t) - y(t)| \int_0^t f(x(s))\,ds + g(r) \int_0^t \alpha(\varepsilon, r)\,ds \\ &\quad \le \varepsilon T g'(r) f(r) + T g(r) \alpha(\varepsilon, r),\end{aligned}$$

for some ζ, which implies that

$$||Gx - Gy|| \leq \varepsilon T g'(r) f(r) + T g(r)\alpha(\varepsilon, r), \quad \forall x, y \in B_r^+. \tag{2.3}$$

Notice that $\lim_{\varepsilon\to 0} \alpha(\varepsilon, r) = 0$ by the uniform continuity of the function v on the set $I \times I \times [0, r]$. Thus, (2.3) ensures that the operator G is continuous on the set B_r^+.

Next we claim that

$$\mu(GX) \leq T g'(r) f(r)\mu(X), \quad \text{for all } X \subseteq B_r^+. \tag{2.4}$$

To show this, let us fix X as an arbitrary nonempty subset of B_r^+. For any $\varepsilon > 0$, $x \in X$ and $t, p \in I$ with $|t-p| \leq \varepsilon$, without loss of generality we may assume that $t \leq p$. In view of assumptions (b)–(d) and (2.1), we arrive at the following claim of estimates:

$$\begin{aligned}
|(Gx)(p) - (Gx)(t)| &\leq |a(p) - a(t)| \\
&\quad + \left| g(x(p)) \int_0^p v(p, s, x(s))\, ds - g(x(t)) \int_0^t v(t, s, x(s))\, ds \right| \\
&\leq \omega(a, \varepsilon) + |g(x(p)) - g(x(t))| \int_0^p v(p, s, x(s))\, ds + g(x(t)) \\
&\quad \times \int_0^p |v(p, s, x(s)) - v(t, s, x(s))|\, ds + g(x(t)) \\
&\quad \times \int_t^p v(t, s, x(s))\, ds \\
&\leq \omega(a, \varepsilon) + g'(r)|x(p) - x(t)| \int_0^p f(x(s))\, ds \\
&\quad + g(r) \int_0^p \beta(\varepsilon, r)\, ds + g(r) \int_t^p f(x(s))\, ds \\
&\leq \omega(a, \varepsilon) + T g'(r) f(r)\omega(x, \varepsilon) + T g(r)\beta(\varepsilon, r) + \varepsilon g(r) f(r),
\end{aligned}$$

which implies that

$$\omega(GX, \varepsilon) \leq \omega(a, \varepsilon) + T g'(r) f(r)\omega(X, \varepsilon) + T g(r)\beta(\varepsilon, r) + \varepsilon g(r) f(r). \tag{2.5}$$

Since the function v on the set $I \times I \times [0, r]$ is uniformly continuous, it follows that $\lim_{\varepsilon\to 0} \beta(\varepsilon, r) = 0$. Letting $\varepsilon \to 0$ in (2.5), we have

$$\omega_0(GX) \leq T g'(r) f(r)\omega_0(X). \tag{2.6}$$

On the other hand, for any $x \in X$ and $t, p \in I$ with $t \le p$, in view of assumptions (a)–(d) and (2.1), we deduce that

$$\begin{aligned}
&|(Gx)(p) - (Gx)(t)| - [(Gx)(p) - (Gx)(t)] \\
&\quad = \left| a(p) + g(x(p)) \int_0^p v(p,s,x(s))\,ds - a(t) - g(x(t)) \right. \\
&\qquad\qquad \left. \times \int_0^t v(t,s,x(s))\,ds \right| \\
&\qquad - \left[a(p) + g(x(p)) \int_0^p v(p,s,x(s))\,ds \right. \\
&\qquad\qquad \left. - a(t) - g(x(t)) \int_0^t v(t,s,x(s))\,ds \right] \\
&\quad \le \{|a(p) - a(t)| - [a(p) - a(t)]\} \\
&\qquad + \left| g(x(p)) \int_0^p v(p,s,x(s))\,ds - g(x(t)) \int_0^t v(t,s,x(s))\,ds \right| \\
&\qquad - \left[g(x(p)) \int_0^p v(p,s,x(s))\,ds - g(x(t)) \int_0^t v(t,s,x(s))\,ds \right] \\
&\quad \le \left| g(x(p)) \int_0^p v(p,s,x(s))\,ds - g(x(t)) \int_0^p v(p,s,x(s))\,ds \right| \\
&\qquad + \left| g(x(t)) \int_0^p v(p,s,x(s))\,ds - g(x(t)) \int_0^t v(t,s,x(s))\,ds \right| \\
&\qquad - \left[g(x(p)) \int_0^p v(p,s,x(s))\,ds - g(x(t)) \int_0^p v(p,s,x(s))\,ds \right] \\
&\qquad - \left[g(x(t)) \int_0^p v(p,s,x(s))\,ds - g(x(t)) \int_0^t v(t,s,x(s))\,ds \right] \\
&\quad \le |g(x(p)) - g(x(t))| \int_0^p v(p,s,x(s))\,ds \\
&\qquad + g(x(t)) \left| \int_0^p v(p,s,x(s))\,ds - \int_0^t v(t,s,x(s))\,ds \right| \\
&\qquad - [g(x(p)) - g(x(t))] \int_0^p v(p,s,x(s))\,ds \\
&\qquad - g(x(t)) \left[\int_0^p v(p,s,x(s))\,ds - \int_0^t v(t,s,x(s))\,ds \right] \\
&\quad \le \{|g(x(p)) - g(x(t))| - [g(x(p)) - g(x(t))]\}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^p v(p,s,x(s))\,ds + g(x(t))\bigg| \int_t^p v(p,s,x(s))\,ds \\
&+ \int_0^t [v(p,s,x(s)) - v(t,s,x(s))]\,ds\bigg| \\
&- g(x(t))\bigg[\int_t^p v(p,s,x(s))\,ds \\
&+ \int_0^t [v(p,s,x(s)) - v(t,s,x(s))]\,ds\bigg] \\
\leq{}& g'(r)\{|x(p)-x(t)| - [x(p)-x(t)]\}\int_0^p f(x(s))\,ds \\
\leq{}& Tg'(r)f(r)\,d(x),
\end{aligned}
$$

which gives that

$$d(Gx) \leq Tg'(r)f(r)\,d(x), \quad \text{for all } x \in X.$$

From the above estimate we get

$$d(GX) \leq Tg'(r)f(r)\,d(X). \tag{2.7}$$

Thus (2.6) and (2.7) ensure that (2.4) holds. Consequently, Theorem 2.1 follows from (2.4), Theorem 1.1 and assumption (e). This completes the proof. □

Remark 2.1. If $a(0) > 0$, then each solution obtained by Theorem 2.1 is positive.

Remark 2.2. In case $g(t) \equiv t$ for all $t \in R$, then Theorem 2.1 reduces to Theorem 3.1 of Banaś and Martinon [**5**]. Examples 3.1 and 3.2 in Section 3 reveal that Theorem 2.1 extends proper Theorem 3.1 of Banaś and Martinon [**5**].

3. Some examples. In this section, in order to illustrate our result, we consider a few examples.

Example 3.1. Consider the following nonlinear quadratic integral equation

$$x(t) = t^2 + (bx(t) + cx^2(t)) \int_0^t \frac{(t+s)x(s)}{1+x^2(s)}\, ds, \quad \text{for all } t \in [0,T] = I, \tag{3.1}$$

where b and c are constants with $b \geq 0$ and $c > 0$. Put

$$0 < T < \frac{1}{\sqrt{b+2\sqrt{c}}}, \quad \frac{1 - bT^2 - \sqrt{(1-bT^2)^2 - 4cT^4}}{2cT^2} \leq r < \frac{1-bT^2}{2cT^2},$$

$$\begin{aligned} a(t) &= t^2, && \text{for all } t \in I, \\ g(t) &= bt + ct^2, && \text{for all } t \in R, \\ f(t) &= T, && \text{for all } t \in R_+, \\ v(t,s,x) &= \frac{(t+s)x}{1+x^2}, && \text{for all } t, s \in I \quad \text{and } x \in R. \end{aligned}$$

It is easy to verify that assumptions (a)–(e) are satisfied. Thus Theorem 2.1 ensures that equation (3.1) has at least one solution $x = x(t)$ which belongs to the space $C(I)$ and is nondecreasing on the interval I. But Theorem 3.1 of Banaś and Martinon [**5**] is not applicable because $c > 0$.

Example 3.2. Consider the following nonlinear quadratic integral equation

$$x(t) = a(t) + \frac{x^2(t)}{1+|x(t)|} \int_0^t v(t,s,x(x))\, ds, \quad \text{for all } t \in [0,T] = I, \tag{3.2}$$

where a and v satisfy assumptions (a), (b) and $|v(t,s,x)| \leq T = f(|x|)$ for all $t, s \in I$ and $x \in R$. Set

$$g(x) = \frac{x^2}{1+|x|}, \quad \text{for all } x \in R.$$

Obviously,

$$g'(x) = \frac{x(2+x)}{(1+x)^2} \geq 0 \quad \text{and } g''(x) = \frac{2}{(1+x)^3} > 0, \quad \text{for all } x \in R_+.$$

It is easy to prove that assumptions (d) and (e) hold if one of the conditions below is satisfied.

(a1) $||a|| < 1$, $T = 1$ and $r > ||a||/(1 - ||a||)$;

(a2) $||a|| < 1$, $1 < T \leq \sqrt{1 + (1/4||a||)(1 - ||a||)^2}$, $r > T - 1$, and

$$\frac{2||a||}{1 - ||a|| + \sqrt{(1 - ||a||)^2 - 4||a||(T^2 - 1)}} \leq r \leq \frac{1 - ||a|| + \sqrt{(1 - ||a||)^2 - 4||a||(T^2 - 1)}}{2(T^2 - 1)}.$$

Hence, by Theorem 2.1, equation (3.2) has at least one solution $x = x(t)$ which belongs to the space $C(I)$ and is nondecreasing on the interval I. However, we cannot invoke Theorem 3.1 of Banaś and Martinon [**5**] to prove the existence of monotone solutions for equation (3.2) since g is not the identity function on R.

Example 3.3. Consider the following nonlinear quadratic integral equation

(3.3)
$$x(t) = \frac{25}{52}t + \frac{x^2(t)}{1 + |x(t)|} \int_0^t \frac{tx(x)}{1 + s + |x(s)|}\, ds, \quad \text{for all } t \in [0, T] = I.$$

Set $T = 1.04$, $a(t) = (25/52)t$ for all $t \in I$, $v(t, s, x) = tx/(1 + s + |x|)$ for all $t, s \in I$ and $x \in R$ and

$$\frac{1}{0.5+\sqrt{0.0868}} < 1.25846570311 \leq r \leq 4.86898527726 < \frac{0.5+\sqrt{0.0868}}{1.632}.$$

Clearly, assumptions (a)–(e) and (a2) hold. Consequently, Theorem 2.1 guarantees that equation (3.3) has at least one solution $x = x(t)$ which belongs to $C(I)$ and is nondecreasing on the internal I.

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