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ON INFINITE DIMENSIONAL DISCRETE TIME PERIODICALLY CORRELATED PROCESSES

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ABSTRACT. Periodically correlated processes with values in Hilbert spaces are introduced and studied. The harmonizability of such a process is discussed. The covariance operator is characterized. Time-dependent spectra on Hilbert spaces are introduced and a time-dependent spectral density for a periodically correlated process is given.

1. Introduction. *Periodically correlated, PC* in short, sequences, introduced and studied first by Gladyshev in 1961, have recently received tremendous attention from different authors. This is due to a variety of applications of this class of nonstationary processes in different areas of sciences and engineering. The works of Hurd, Miamee and Salehi, among others (see the references), have elaborated the theory of periodically correlated processes. The book of Gardner [**3**] provides a good view on the applications of the PC processes in different branches of engineering and physics. Most of the works on the PC processes are confined to one-dimensional PC processes. The multi-dimensional processes, to the best of our knowledge, have not yet been treated in good detail. This article studies the PC sequences with values in a Hilbert space. The harmonizability, the structure of the covariance and the existence of a time-dependent spectral density are topics which are furnished in this article. The article brings the authors' works in [**18, 19**] to the contents of probability theory, together with a new perspective on the structure of the covariance function. The approach to spectral representation, presented in this article, is different from the one employed by Gladyshev, 1961, which was via forming a correlation matrix, and also from the one employed by Hurd, 1989, which was via using a certain root of a unitary operator. For recent works on PC processes, see [**10, 17**].

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The paper is organized as follows. Section 2 provides some preliminaries. Harmonizability is discussed in Section 3, where it is proved that a discrete time PC process is the Fourier transform of a spectral random measure, which indeed is an operator-valued measure. The main results are Theorems 3.5 and 3.9. Section 4 is for the characterization of the covariance operator. Necessary and sufficient conditions for an operator-valued function to be a covariance operator of an infinite-dimensional discrete time PC process are given. Theorem 4.13 is the main result. A time-dependent spectral density is derived in Section 5. The result indicates that, in the Gaussian case, the law of such a process can be determined by a positive measure and a time dependent kernel, which are fully specified. Theorems 5.7 and 5.14 give the details.

2. Notations and preliminaries. Let X be a Hilbert space, and let $L(X)$ stand for the bounded linear operators on X. The inner product on X is denoted by $(\ ,\ )$. The probability space is denoted by $(\Omega, \mathcal{B}, P)$. A random variable $\xi : \Omega \to X$ is called second order if $x^*(\xi(\omega)) \in L^2(\Omega, \mathcal{B}, P)$ for all $x^* \in X^*$, where X^* stands for the dual space of X. Since X is a Hilbert space, $x^*(\xi(\omega)) = (\xi(\omega), x)$ for some $x \in X$. This leads to a bounded linear transformation $T : X \to L^2(\Omega, \mathcal{B}, P)$, defined by $Tx = (\xi(\omega), x)$; indeed, it follows from the closed graph theorem that T is bounded. Thus, by letting $\xi_x \equiv Tx$, the mapping T can be identified with the collection of random variables $\{\xi_x,\ x \in X\}$. Therefore, in order to study a second order stochastic process with values in a Hilbert space X, one may consider a collection $\xi = \{\xi_x^n,\ n \in Z,\ x \in X\}$, Z is the set of integers of random variables for which

$$E|\xi_x^n|^2 < \infty \quad \text{and} \quad E\,\xi_x^n = 0 \quad n \in Z,\ x \in X,$$

and for each n, ξ_x^n is linear and continuous in x,

$$(E|\xi_x^n|^2)^{1/2} \leq C_n \|x\|, \quad x \in X.$$

We refer to the $\xi = \{\xi_x^n,\ n \in Z,\ x \in X\}$ as a second order X-valued stochastic process. Let $\mathcal{H} =$ span closure $\{\xi_x^n,\ n \in Z,\ x \in X\}$ denote the space generated by the process ξ. A process ξ is called PC if there exist an integer $T > 0$ such that for every $x, y \in X$ and $m, n \in Z$,

$$E\,\xi_x^n\,\overline{\xi_y^m} = E\,\xi_x^{n+T}\,\overline{\xi_y^{m+T}}. \tag{2.1}$$

If ξ is a PC process, then the dimension of $\mathcal{H}(n) =$ span closure $\{\xi_x^n, x \in X\}$, $n \in Z$ is periodic in n with period T, and the dimension of the process is specified by the dimensions of $\mathcal{H}(n)$, $n = 0, \dots, T-1$.

Note that $E\,\xi_x^{n+\tau}\,\overline{\xi_y^n}$, $n, \tau \in Z$, is a bilinear function on $X \times X$ and bounded on the set $\{(x, y),\ \|x\| = \|y\| = 1\}$, as

$$\begin{aligned}|E\,\xi_x^{n+\tau}\,\overline{\xi_y^n}| &\le E|\xi_x^{n+\tau}\,\overline{\xi_y^n}| \le (E|\xi_x^{n+\tau}|^2)^{1/2}(E|\xi_y^n|^2)^{1/2}\\ &\le C_{n+\tau}\,\|x\| C_n\,\|y\|.\end{aligned}$$

Therefore it follows from [**14**, Theorem 12.8] that

$$E\,\xi_x^{n+\tau}\,\overline{\xi_y^n} = (x, S(n,\tau)y), \tag{2.2}$$

where $S(n,\tau) \in L(X)$ for each $n, \tau \in Z$. We refer to the collection $\gamma(.) = \{S(n, .), n \in Z\}$, or $S(.,.)$ if there is no ambiguity, as the *covariance* of the process. When $T = 1$ the process is called *stationary*. The theory of stationary processes is very well developed, see [**12, 15, 16**]. It is well known that for a stationary process $\{\zeta_x^n,\ x \in X,\ n \in Z\}$ the spectral representation

$$\zeta^n = \int_0^{2\pi} e^{-in\lambda}\Phi(d\lambda), \tag{2.3}$$

in the sense that

$$E\,\zeta_x^n\,\overline{\zeta_y^m} = \int_0^{2\pi} e^{-i(n-m)\lambda} E\Phi(d\lambda)x\,\overline{\Phi(d\lambda)y} \tag{2.4}$$

is fulfilled. In (2.3), $\Phi(d\lambda)$ is a spectral random measure with orthogonal increments on X, (RMOI), i.e., Φ is a finitely additive set function on the Borel sets of $[0, 2\pi)$ with values in the space of bounded linear transformations from X into $L^2(\Omega, \mathcal{B}, P)$. More precisely Φ satisfies the following.

i) $\Phi(\cup_{j=1}^n \Delta_j) = \sum_{j=1}^n \Phi(\Delta_j)$, for disjoint sets Δ_j, $j = 1, 2, \dots, n$,

ii) $E\Phi(\Delta)x\,\overline{\Phi(\Delta')x'} = 0$ if $\Delta \cap \Delta' = \varnothing$ $x, x' \in X$,

iii) $\Phi(\varnothing) = 0$,

iv) $E\Phi(.)x\bar{h}$ is a complex measure for each fixed $h \in \mathcal{H}$, $x \in X$,

v) $E|\Phi(\Delta)x| \leq M_\Delta \|x\|$, $x \in X$, where M_Δ is a constant which depends on Δ.

vi) $\Phi(.)x$ is countable additive in $\mathcal{H}$, for every $x \in X$.

The spectral distribution of a stationary process ζ is an $L(X)$-valued measure F for which

$$(x, F(d\lambda)y) = E\Phi(d\lambda)x\,\overline{\Phi(d\lambda)y},$$

giving that

$$E\,\zeta_x^n\,\overline{\zeta_y^m} = \int_0^{2\pi} e^{-i(n-m)\lambda}(x, F(d\lambda)y). \tag{2.5}$$

We say a second order process $\xi = \{\xi_x^n,\ x \in X,\ n \in Z\}$ admits a time-dependent spectral representation if

$$\xi^n = \int_0^{2\pi} e^{-isn}\Phi(ds)V_n(s), \tag{2.6}$$

in the sense that

$$E\,\xi_x^n\,\overline{\xi_y^m} = \int_0^{2\pi} e^{-is(n-m)}E\Phi(ds)V_n(s)x\,\overline{\Phi(ds)V_m(s)y},$$

where $V_n(.)$ are operator-valued functions, $V_n(.) : X \to X$, and Φ is an RMOI. The measure $\Upsilon_n(ds) = \Phi(ds)V_n(s)$ is called the *time-dependent random spectral*. The corresponding time-dependent spectral distribution, denoted by $G_n(.)$, is a measure with values in $L(X)$ given by

$$(x, G_n(ds)y) = E\Phi(ds)V_n(s)x\,\overline{\Phi(ds)V_n(s)y}. \tag{2.7}$$

We refer to a matrix which its entries are operators as an *operator-matrix*.

3. A spectral characterization. In this section we obtain a spectral representation for a PC process. It is easy to see that if $\xi = \{\xi_x^n,\ x \in X,\ n \in Z\}$ is a PC process, then the process $\zeta = \{\zeta_{\tilde{x}}^n,\ \tilde{x} \in \prod_{i=1}^T X_i,\ n \in Z\}$, defined by

$$\zeta_{\tilde{x}(p)}^n = \xi_x^{nT+p}, \tag{3.1}$$

is a stationary process, where $\prod_{i=1}^{T} X_i = \{(x_1, \dots, x_T),\ x_i \in X,\ i = 1, \dots, T\}$, and $\tilde{x}(p) = (0, \dots, x, \dots, 0) \in \prod_{i=1}^{T} X_i$ with all coordinates being zero except the $(p+1)$th which is x. Since ζ is a stationary process, it admits the representation

$$\zeta^n = \int_0^{2\pi} e^{-in\lambda} \Phi(d\lambda), \tag{3.2}$$

in the sense that

$$E\, \zeta_{\tilde{x}}^n\, \overline{\zeta_{\tilde{y}}^m} = \int_0^{2\pi} e^{-i(n-m)\lambda} E\Phi(d\lambda)\tilde{x}\, \overline{\Phi(d\lambda)\tilde{y}}, \tag{3.3}$$

where Φ is a random measure with orthogonal increments on $\prod_{i=1}^{T} X_i$. Corresponding to Φ define the column vector $\tilde{\Phi}$ whose entries are

$$\Phi_p(d\lambda)x = \Phi(d\lambda)(0, \dots, x, \dots, 0), \quad p = 0, \dots, T-1. \tag{3.4}$$

Note that for each $x \in X$, $\Phi_p(d\lambda)x$ is a random variable. Moreover each Φ_p is an RMOI on X.

Theorem 3.5. *Let $\xi = \{\xi_x^n,\ x \in X,\ n \in Z\}$ be a PC process. Then*

$$\xi_x^n = \sum_{j=0}^{T-1} e^{-i2\pi jn/T} \eta_{\tilde{x}(j)}^n, \tag{3.6}$$

where $\eta^n = \{\eta_{\tilde{x}}^n,\ \tilde{x} \in \prod_{i=1}^{T} X_i,\ n \in Z\}$ is the unique stationary process given by

$$\eta^n = \int_0^{2\pi/T} e^{-in\lambda} \Psi(d\lambda). \tag{3.7}$$

The Ψ, in (3.7), is related to Φ through

$$\tilde{\Psi}\left(\frac{d\lambda}{T}\right) = M^{-1} \rho\left(-\frac{\lambda}{T}\right) \tilde{\Phi}(d\lambda), \tag{3.8}$$

where Φ is given by (3.3), $\tilde{\Psi}$ corresponding to Ψ throughout (3.4), $M = [m_{pj}]$ and $\rho(\lambda/T) = [\delta_{pj}\rho_j]$ are $T \times T$ matrices with:

$$m_{pj} = e^{-i(p-1)(j-1)(2\pi/T)} \quad \text{and} \quad \rho_j = e^{-i(j-1)\lambda/T}, \quad p, j = 1, \dots, T.$$

Proof. Let ξ be a PC process. It follows from (3.1), (3.2) and (3.8) that for every $x \in X$ and $h \in \mathcal{H}$,

$$\begin{aligned}
E\,\xi_x^{nT+p}\,\bar{h} &= E\,\zeta_{\tilde{x}(p)}^{n}\bar{h} \\
&= \int_0^{2\pi} e^{-in\lambda} E\Phi_p(d\lambda)x\bar{h} \\
&= \int_0^{2\pi} e^{-in\lambda} \sum_{j=0}^{T-1} e^{-i(2\pi jp)/T - i(p\lambda)/T}\, E\Psi_j\left(\frac{1}{T}\,d\lambda\right)x\bar{h} \\
&= \sum_{j=0}^{T-1} e^{-i(2\pi jp)/T} \int_0^{(2\pi)/T} e^{-i(nT+p)\lambda} E\Psi_j(d\lambda)x\bar{h} \\
&= \sum_{j=0}^{T-1} e^{-i(2\pi jp)/T} E\,\eta_{\tilde{x}(j)}^{nT+p}\,\bar{h}.
\end{aligned}$$

Therefore, (3.6) holds. For the uniqueness, let (3.6) hold; then

$$\begin{aligned}
E\,\xi_x^{nT+p}\,\bar{h} &= \sum_{j=0}^{T-1} e^{-i(2\pi j(nT+P))/T} E\eta_{\tilde{x}(j)}^{nT+p}\,\bar{h} \\
&= \sum_{j=0}^{T-1} e^{-i(2\pi jp)/T} \int_0^{(2\pi)/T} e^{-i(nT+p)\lambda}\, E\Psi_j(d\lambda)x\bar{h} \\
&= \int_0^{2\pi} e^{-in\lambda} \sum_{j=0}^{T-1} e^{-i(2\pi jp)/T} e^{-ip(\lambda/T)} E\Psi_j\left(\frac{1}{T}\,d\lambda\right)x\bar{h}.
\end{aligned}$$

On the other hand,

$$E\,\xi_x^{nT+p}\,\bar{h} = \int_0^{2\pi} e^{-in\lambda} E\Phi_p(d\lambda)x\bar{h}.$$

Therefore (3.8) follows from the uniqueness of the spectral representation for stationary processes.

The Theorem 3.5 given above provides a transparent proof for the harmonizability of a PC process with values in a Hilbert space. The following theorem gives the details.

Theorem 3.9. *Let ξ be a* PC *process. Then*

$$\xi^n = \int_0^{2\pi} e^{-in\lambda}\mathcal{Z}(d\lambda), \tag{3.10}$$

where the spectral random measure $\mathcal{Z}$ is given by

$$\mathcal{Z}(d\lambda) = \Psi_p\left(d\lambda - \frac{2\pi p}{T}\right), \quad \lambda \in \left[\frac{2\pi p}{T}, \frac{2\pi(p+1)}{T}\right). \tag{3.11}$$

Furthermore, the spectral distribution $F(.,.)$, introduced by

$$E\mathcal{Z}(ds)x\,\overline{\mathcal{Z}(dt)y} = (x, F(ds,dt)y), \tag{3.12}$$

is a measure on $[0,2\pi)\times[0,2\pi)$ which is supported by lines $d_k = \{(s,t) \in [0,2\pi)^2,\ s-t = (2\pi k)/T\}$, $k = 1-T,\dots,T-1$.

Proof. Let $\{\xi_x^n\}$ be a *PC* process. By (3.6) we will have

$$\begin{aligned}
\xi_x^n &= \sum_{p=0}^{T-1} e^{-i(2\pi pn)/T}\eta^n_{\tilde{x}(p)} \\
&= \sum_{p=0}^{T-1} e^{-i(2\pi pn)/T}\int_0^{(2\pi)/T} e^{-in\lambda}\Psi_p(d\lambda)x \\
&= \sum_{p=0}^{T-1} \int_{(2\pi p)/T}^{2\pi(p+1)/T} e^{-in\lambda}\Psi_p\left(d\lambda - \frac{2\pi p}{T}\right)x \\
&= \int_0^{2\pi} e^{-in\lambda}\mathcal{Z}(d\lambda)x.
\end{aligned}$$

The fact that the distribution $F(.,.)$ is supported on the cited lines follows from the point that $\{\Psi_p(d\lambda)\}$ is an independently scattered random measure, i.e.,

$$E\mathcal{Z}(ds)x\,\overline{\mathcal{Z}(dt)y} = \begin{cases} (x, F(ds,dt)y) & s - \dfrac{2\pi p}{T} = t - \dfrac{2\pi l}{T} \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

for $s \in [(2\pi p)/T, (2\pi(p+1))/T)$ and $t \in [(2\pi l)/T, (2\pi(l+1))/T)$.

Remark 3.14. It is well known that, if a distribution F is given, then there is a unique $\mathcal{Z}$ for which (3.12) is satisfied, see [**13**].

4. The covariance operator. In this section we characterize the covariance operator of a PC process, introduced in Section 2. Let $F_k(s)$ be the restriction of $F(s,t)$ to the line, d_k, $k = -T + 1, \dots, T-1$. Now, for the point (s,t) on the diagonal of the square, $[(2\pi p)/T, (2\pi(p+1))/T) \times [(2\pi l)/T, (2\pi(l+1))/T)$, $p, l = 0, \dots, T-1$, i.e., $s - t = ((2\pi)/T)(p-l)$, $s \in [(2\pi p)/T, (2\pi(P+1))/T)$, $t \in [(2\pi l)/T, (2\pi(l+1))/T)$, we have

$$(x,\, F(s,t)y) = (x, F_{p-l}(s)y). \tag{4.1}$$

Thus, in particular for $u \in [0, (2\pi)/T)$, we obtain that

$$E\Psi_p(du)x\,\overline{\Psi_l(du)y} = \left(x,\, F_{p-l}\left(du + \frac{2\pi p}{T}\right)y\right). \tag{4.2}$$

Now define a square matrix $\mathcal{F}$ by

$$\mathcal{F}(ds) = \left[F_{p-l}\left(ds + \frac{2\pi p}{T}\right)\right]_{p,l=0,\dots,T-1}, \quad s \in \left[0, \frac{2\pi}{T}\right), \tag{4.3}$$

Lemma 4.4. *The matrix defined by* (4.3) *is positive definite.*

Proof. Recall that an operator-matrix $T = [T_{i,j}]$, $i,j = 0, \dots, T-1$, is called positive definite if and only if, for each $x_{k_0}, \dots, x_{k_{T-1}}$, the matrix

$$[(x_{kj}, T_{j,l}\, x_{k_l})]_{j,l=0,\dots,T-1}, \tag{4.5}$$

is positive definite. It follows from (4.2) that, for $\mathcal{F}(ds)$, the matrix given by (4.5) is the covariance matrix of the random vector $(\Psi_0(du)x_{k_0}, \dots, \Psi_{T-1}(du)x_{k_{T-1}})$ and therefore is positive definite. The proof is complete.

Lemma 4.6. *If* $\{\xi_x^n\}$ *is a* PC *process, then*

$$E\,\xi_x^{n+\tau}\,\overline{\xi_y^n} = (x, S(n,\tau)y), \quad x,y \in X,\ n,\tau \in Z, \tag{4.7}$$

where the operator $S(n,\tau)$ *is given by*

$$S(n,\tau) = \sum_{k=0}^{T-1} e^{-i(2\pi kn)/T} R_k(\tau), \tag{4.8}$$

and $R_k(\tau)$, $k = 0, \dots, T-1$ *are certain operators in* $L(X)$.

Proof. Equation (4.7) follows from the fact that $E\,\xi_x^{n+\tau}\,\overline{\xi_y^n}$ is a bilinear and bounded function on $X \times X$, see [**14**, Theorem 12.8]. For fixed x, y in X, $E\,\xi_x^{n+\tau}\,\overline{\xi_y^n}$ is periodic in n with period T and so is $(x, S(n,\tau)y)$.

Thus,

$$(x, S(n,\tau)y) = \sum_{k=0}^{T-1} e^{i(2\pi kn)/T} B_{k,\tau,x,y}, \tag{4.9}$$

where

$$B_{k,\tau,x,y} = \frac{1}{T}\sum_{n=0}^{T-1} (x, S(n,\tau)y) e^{-i(2\pi kn)/T}. \tag{4.10}$$

It is easy to see that $B_{k,\tau,\cdot,\cdot}$ is a bilinear form and a bounded function on $X \times X$. Therefore, there is an operator-valued function $R_k(\tau) \in L(X)$ for which

$$B_{k,\tau,x,y} = (x, R_k(\tau)y). \tag{4.11}$$

It follows from (4.9) and (4.11) that

$$\begin{aligned}(x, S(n,\,\tau)y) &= \sum_{k=0}^{T-1} e^{i(2\pi kn)/T}(x, R_k(\tau)y) \\ &= \left(x, \sum_{k=0}^{T-1} e^{-i(2\pi kn)/T} R_k(\tau)y\right). \qquad \square\end{aligned} \tag{4.12}$$

The following theorem is the main theorem in this section.

Theorem 4.13. *In order for the operator $S(n,\tau)$ in (4.8) to be the covariance operator of a* PC *process, it is necessary and sufficient that the operator-valued functions $R_k(\tau)$, $k=0,\dots,T-1$ can be represented as*

$$R_k(s)=\int_0^{2\pi} e^{-i\tau s}dG_k(s), \tag{4.14}$$

where each $G_k(s)$ is defined in terms of the $F_k(s)$ by

$$G_k(s)=\begin{cases} F_k(s) & \dfrac{2\pi k}{T}<s \\ F_{-T+k}(s) & s\le \dfrac{2\pi k}{T}\end{cases}, \quad k=0,\dots,T-1.$$

Proof. Let ξ be a PC process and $S(n,\tau)$ its covariance. It follows from Theorem 3.9 that

$$\begin{aligned}
(x,S(n,\tau)y) &= E\,\xi_x^{n+\tau}\,\overline{\xi_y^n}\\
&=\int_0^{2\pi}\int_0^{2\pi} e^{-i(n+\tau)s+int}E\mathcal{Z}(ds)x\,\overline{\mathcal{Z}(dt)y}\\
&=\int_0^{2\pi}\int_0^{2\pi} e^{-i(n+\tau)s+int}(x,F(ds,dt)y)\\
&=\sum_{p=0}^{T-1}\sum_{l=0}^{T-1}\int_{(2\pi p)/T}^{(2\pi(p+1))/T}\int_{(2\pi l)/T}^{(2\pi(l+1))/T}\\
&\qquad\qquad e^{-i(n+\tau)s+int}(x,F(ds,dt)y)\\
&=\sum_{p=0}^{T-1}\sum_{l=0}^{T-1}\int_{(2\pi p)/T}^{(2\pi(p+1))/T} e^{-i\tau s-i(2\pi(p-l)n)/T}(x,F_{p-l}(ds)y)\\
&=\sum_{k=-T+1}^{-1}\sum_{p=0}^{T+k-1}\int_{(2\pi p)/T}^{(2\pi(p+1))/T} e^{-i\tau s-i(2\pi kn)/T}(x,F_k(ds)y)\\
&\quad+\sum_{p=0}^{T-1}\int_{(2\pi p)/T}^{(2\pi(p+1))/T} e^{-i\tau s}(x,F_0(ds)y)\\
&\quad+\sum_{k=1}^{T-1}\sum_{p=k}^{T-l}\int_{(2\pi p)/T}^{(2\pi(p+1))/T} e^{-i\tau s-i(2\pi kn)/T}(x,\,F_k(ds)y)
\end{aligned} \tag{4.15}$$

$$= \sum_{k=-T+1}^{-1} e^{-i(2\pi nk)/T} \int_0^{(2\pi(T+k))/T} e^{-i\tau s}(x,\, F_k(ds)y)$$
$$+ \int_0^{2\pi} e^{-i\tau s}(x,\, F_0(ds)y)$$
$$+ \sum_{k=1}^{T-1} e^{-i(2\pi nk)/T} \int_{(2\pi k)/T}^{2\pi} e^{-i\tau s}(x,\, F_k(ds)y)$$
$$= \sum_{k=0}^{T-1} e^{-i(2\pi nk)/T} \int_0^{2\pi} e^{-i\tau s}(x,\, G_k(ds)y).$$

Thus, (4.14) follows from (4.8) and the observation given above. Conversely, let

$$S(n,\tau) = \sum_{k=0}^{T-1} e^{-i(2\pi nk)/T} R_k(\tau),$$

where $R_k(\tau)$ are defined by (4.14). Since $\mathcal{F}(ds)$ is a positive-definite matrix, there exists a unique random measure Φ such that (4.2) holds.

Thus, if we produce the stationary process $\eta = \{\eta^n_{\tilde{x}},\ \tilde{x} \in \prod_{i=1}^{T-1} X_i,\ n \in Z\}$ by

$$\eta^n_{\tilde{x}(p)} = \int_0^{2\pi} e^{-in\lambda}\Phi_p(d\lambda)x, \quad p = 0, \dots, T-1$$

and ξ by (3.6), then it is easy to see that ξ is a PC process with the covariance $\gamma(.)$.

5. Time-dependent spectra. In this section we present a time-dependent spectral distribution for a PC process. The significance of such observation lies on the fact that the law of the process is governed by a positive measure μ and a deterministic kernel, see (5.17). We proceed with the following lemma concerning the Cholesky decomposition for a positive definite operator-matrix.

Lemma 5.1. *Let M_n be an $n \times n$ positive operator-matrix, then*

$$M_n = U_n^* U_n, \tag{5.2}$$

where U_n is an $n \times n$ upper triangular matrix.

Proof. See [**1**]. □

For any operator valued-measure F_k, there exists a positive measure μ_k such that, see [**2**],

$$\|F_k(\Delta)\| \leq \mu_k(\Delta).$$

Thus,

$$|F_k(\Delta)x| \leq \|F_k(\Delta)\| \, \|x\|_X \leq \mu_k(\Delta)\|x\|_X.$$

It is straightforward to show that there exists a positive measure μ such that

$$|F_k(\Delta)x| \leq \mu(\Delta) \, \|x\|_X, \quad k = -T+1, \dots, T-1,$$

and

$$\mu_{(2\pi k)/T}(dx) = \mu(dx) \quad \text{on} \quad \left[0, \frac{2\pi}{T}\right), \quad k = 1-T, \dots, T-1, \tag{5.4}$$

where, for a measure ν on $[0, 2\pi)$,

$$\nu_{(2\pi k)/T}(E) = \nu\left(\left(E + \frac{2\pi k}{T}\right) \cap [0, 2\pi)\right). \tag{5.5}$$

Now let f_k, $k = 1-T, \dots, T-1$, be an $L(X)$-valued function on $[0, 2\pi)$ such that

$$(x, F_k(\Delta)y) = \int_\Delta (x, f_k(s)y)\mu(ds), \tag{5.6}$$

[**7**].

Theorem 5.7. *A second order X-valued stochastic process $\{\xi_x^n$, $n \in Z$, $x \in X\}$ is* PC *if and only if*

$$\xi_x^n = \sum_{k=0}^{T-1} \int_0^{2\pi} e^{-ins} \Phi_{(-2\pi k)/T}(ds) a_k(s) x, \tag{5.8}$$

where i) Φ *is an independently scattered random measure satisfying*

$$E\Phi(d\lambda)x \, \overline{\Phi(d\lambda)y} = (x, y)_X \mu(d\lambda), \tag{5.9}$$

μ is a finite measure satisfying (5.4).

ii) $a_k(s)$ *are operators with* $a_k(s) = 0$ *for* $s \in [0, (2\pi k)/T)$. *Furthermore, the matrix*

$$A(x) = \left[\overline{a_{j-k}\left(x + \frac{2\pi j}{T}\right)}\right]_{k \le j}, \quad k, j = 0, \dots, T-1 \tag{5.10}$$

satisfies

$$f(x) = A^*(x)A(x) \quad x \in \left[0, \frac{2\pi}{T}\right), \tag{5.11}$$

where $f(x)$ *is given by*

$$f(x) = \left[f_{p-l}\left(dx + \frac{2\pi p}{T}\right)\right]_{l,p=0,\dots,T-1}. \tag{5.12}$$

Proof. Let $\{\xi_x^n\}$ be a process for which (5.8) holds. Thus

$$\begin{aligned}(x, S(n,\tau)y) &= E\,\xi_x^{n+\tau}\,\overline{\xi_y^n} \\ &= \sum_{k=0}^{T-1}\sum_{l=0}^{T-1}\int_0^{2\pi}\int_0^{2\pi} e^{-i(n+\tau)s+int} E\Phi_{(-2\pi k)/T}(ds) \\ &\qquad \times a_k(s)x\,\overline{\Phi_{(-2\pi l)/T}(dt)a_l(t)y};\end{aligned}$$

it also follows from (5.9) that

$$\begin{aligned}&E\Phi_{(-2\pi k)/T}(ds)a_k(s)x\,\overline{\Phi_{(-2\pi l)/T}(dt)a_l(t)y} \\ &= \begin{cases} (a_k(s)x,\, a_{p+k}\left(s + \left(\frac{2\pi p}{T}\right) y\right)\mu_{(-2\pi p)/T}(ds) & s - \frac{2\pi k}{T} = t - \frac{2\pi l}{T} \\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

where $p = l - k$. Since $\mu_{(-2\pi p)/T}(ds) = \mu(ds)$, $p = 1-T, \dots, T-1$, we obtain that

(5.13)

$$
\begin{aligned}
(x, S(n,\tau)y) = &\sum_{p=-(T-1)}^{-1} \int_0^{2\pi} e^{-i(n+\tau)s+in(s+((2\pi p)/T))} \\
&\qquad \times \sum_{k=-p}^{T-1} \left(a_k(s)x, a_{k+p}\left(s+\frac{2\pi p}{T}\right)y\right)\mu(ds) \\
&+ \int_0^{2\pi} e^{-i(n+\tau)s+ins} \sum_{k=0}^{T-1} (a_k(s)x,\, a_k(s)y)\mu(ds) \\
&+ \sum_{p=1}^{T-1} \int_0^{2\pi} e^{-i(n+\tau)s+in(s+((2\pi p)/T))} \\
&\qquad \times \sum_{k=0}^{T-1-p} \left(a_k(s)x, a_{k+p}\left(s+\frac{2\pi p}{T}\right)y\right)\mu(ds).
\end{aligned}
$$

It is clear from (5.13) that $\{\xi_x^n\}$ is a PC process. Conversely, let $\{\xi_x^n\}$ be a PC process with covariance $S(.,.)$. As demonstrated in the proof of Theorem 4.13, $E\,\xi_x^{n+\tau}\,\overline{\xi_y^n}$ can be expressed by (4.15). By using (5.6) and comparing with (5.13), we obtain that

(5.14)

$$
\begin{aligned}
&\sum_{k=-p}^{T-1} \left(a_k(s)x, a_{p+k}\left(s+\frac{2\pi p}{T}\right)y\right) = (x, f_{-p}(s)y), \\
&\qquad p = -T+1, \dots, -1, \\
&\sum_{k=0}^{T-1} (a_k(s)x, a_k(s)y) = (x, f_0(s)y) \\
&\sum_{k=0}^{T-1-p} \left(a_k(s)x, a_{k+p}\left(s+\frac{2\pi p}{T}\right)y\right) = (x, f_{-p}(s)y), \\
&\qquad p = 1, \dots, T-1.
\end{aligned}
$$

It is easy to verify that (5.14) and (5.11) are identical. On the other hand, it follows from Lemma 5.1 that (5.11) is satisfied, as it is a Cholesky decomposition for $f(.)$, for details see [**9**].

Theorem 5.15. *Let* $\{\xi_x^n,\ x \in X,\ n \in Z\}$ *be a* PC *process with period* T. *Then the process* $\{\xi_x^n\}$ *admits the following time-dependent spectral representation*

$$\xi_x^n = \int_0^{2\pi} e^{ins}\Phi(ds)V_n(s)x, \tag{5.16}$$

in the sense that

$$E\,\xi_x^n\,\overline{\xi_x^m} = \int_0^{2\pi} e^{i(n-m)s}(V_n(s)x, V_m(s)y)\mu(ds), \tag{5.17}$$

where

$$V_n(s) = \sum_{k=0}^{T-1} e^{-i(2\pi kn)/T} a_k\left(s + \frac{2\pi k}{T}\right), \quad s \in [0, 2\pi), \quad n \in Z. \tag{5.18}$$

Proof. It follows from Theorem 5.7 that

$$\begin{aligned}
\xi_x^n &= \sum_{k=0}^{T-1} \int_{(2\pi k)/T}^{2\pi} e^{-ins}\Phi_{(-2\pi k)/T}(ds)a_k(s)x \\
&= \sum_{k=0}^{T-1} \int_0^{2\pi((T-k)/T)} e^{-ins-i((2\pi kn)/T)}\Phi(ds)a_k\left(s + \frac{2\pi k}{T}\right)x \\
&= \sum_{k=0}^{T-1} \int_0^{2\pi} e^{-ins-i((2\pi kn)/T)}\Phi(ds)a_k\left(s + \frac{2\pi k}{T}\right)x \\
&= \int_0^{2\pi} e^{-ins} \sum_{k=0}^{T-1} e^{-i((2\pi kn)/T)}\Phi(ds)a_k\left(s + \frac{2\pi k}{T}\right)x,
\end{aligned}$$

giving the result (5.16). Equation (5.17) follows from (5.16) and (5.9).

REFERENCES

1. G.T. Adams, P.J. McGuire and V.I. Paulsen, *Analytic reproducing kernels and multiplication operators*, Illinois J. Math. **36** (1992), 404–419.

2. N. Dunford and J.T Schwartz, *Linear operator* I, Interscience, New York, 1958.

3. W.A. Gardner, *Cyclostationarity in communications and signal processing*, IEEE Press, New York, 1994.

4. E.G. Gladyshev, *Periodically correlated random sequences*, Soviet Math. Dokl. **2** (1961), 385–388.

5. H.L. Hurd, *An investigation of periodically correlated stochastic processes*, Ph.D. Dissertation, Duke University, Durham, North Carolina, 1969.

6. ———, *Representation of strongly harmonizable periodically corrolated processes and their covariance*, J. Multivariate Anal. **29** (1989), 53–67.

7. V. Mandredkar and H. Salehi, *On singularity and Lebesgue type decomposition for operator-valued measures*, J. Multivariate Anal. **2** (1971), 167–185.

8. A.G. Miamee and H. Salehi, *On the prediction of periodically correlated stochastic processes*, in *Multivariate analysis*–V (P.R. Krishnaiah, ed.), Amsterdam, North Holland Publ. Co., 1980, pp. 167–179.

9. A.G. Miamee and A.R. Soltani, *On spectral analysis of periodically correlated stochastic processes*, Tech. Report, Hampton University, Virginia, 1995.

10. A.R. Nematollahi and A.R. Soltani, *Discrete time periodically correlated Markov processes*, Probab. Math. Statist. **20** (2000), 1–14.

11. M. Pourahmadi and H. Salehi, *On subordination and linear transformation of harmonizable and periodically correlated processes*, in *Probability theory on vector spaces*, III (Lublin), Springer, Berlin, 1983, pp. 195–213.

12. Yu A. Rozanov, *Some approximation problems in the theory of stationary processes*, J. Multivariate Anal. **2** (1972), 135–144.

13. ———, *Stationary random processes*, Holden-Day, San Francisco, 1967.

14. W. Rudin, *Functional analysis*, McGraw Hill, New York, 1976.

15. H. Salehi and A.R. Soltani, *On regularity of homogeneous random fields*, in *Prediction theory and harmonic analysis* (V. Mandrekar and H. Salehi, eds.), North-Holland Publ. Co., Amsterdam, 1983.

16. A.R. Soltani, *Extrapolation and moving average representation for stationary random fields and Beurling's theorem*, Ann. Probab. **12** (1984), 120–132.

17. A.R. Soltani and A. Parvardeh, *Simple random measures and simple processes*, Teor. Veroyatnos. i Primenen. **50** (2005), 533–548.

18. A.R. Soltani and Z. Shishebor, *A spectral representation for weakly periodic sequences of bundled linear transformations*, Acta Math. Hungar. **80** (1998), 265–270.

19. ———, *Weakly periodic sequences of bounded linear transformations*: *A spectral characterization*, Georgian Math. J. **6** (1999), 91–95.

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