ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 1, Winter 1990

THE RELATIVE CONSISTENCY OF A "LARGE CARDINAL" PROPERTY FOR ω_1

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ABSTRACT. There are two current methods for obtaining "Large Cardinal" properties for ω_1 and other small cardinals in the absence of AC (the Axiom of Choice): one is to work within ZF + AD (the Axiom of Determinateness); the other method is to prove the consistency of the desired property for ω_1 in ZF, assuming the consistency of ZF + AC + whatever other axiom is required. Both approaches yield ω_1 measurable. Using a straightforward modification of Jech's result, which showed ω_1 can be measurable, and assuming ZF + AC + there is a measurable cardinal, we prove ω_1 can be huge and, assuming ZF + AC +, there is a huge cardinal. In view of the well-known result that ω_1 is measurable in ZF + AD, our result should heighten interest in the problem of showing that ω_1 is huge in ZF + AD + DC, which is yet unresolved.

Introduction. A cardinal κ is called huge if $\kappa > \omega$ and, for some $\lambda > \kappa$, there is a κ-complete, fine, normal ultrafilter U on the field of all subsets of precisely the subsets of λ of order type κ. In ZF a huge cardinal is measurable. In fact, if κ is huge and $\lambda > \kappa$ is the necessary cardinal from the definition above, then κ is λ-supercompact (see [**5**]). With regard to the strength of huge cardinals for providing consistency results, it is known that, from the assumption of a huge cardinal in ZFC, the consistency of Vopěnka's principle follows, from which the existence of extendible cardinals and hence supercompact cardinals follows (see [**3**]and [**4**]). Indeed, in ZFC a huge cardinal is a very large cardinal.

It is interesting to investigate what the possibilities are in the absence of choice for small cardinals, in the partial ordering of cardinals in ZF, to possess "Large Cardinal" properties.

Currently, there are two major techniques for obtaining "Large Cardinal" properties in the absence of choice: one is to work within ZF + AD (the Axiom of Determinateness); the other is to prove the consistency of the desired property in ZF, assuming the consistency of ZF + AC +

Received by the editors on October 22, 1985.

whatever other axiom is required. Both approaches yield ω_1 measurable. In fact, the former technique yields ω_1 is ω_2-supercompact (see [**1**]).

The latter technique was used by Jech to show that ω_1 can be measurable, assuming ZF $+$ AC$+$ there is a measurable cardinal (see [**2**]). Employing a straightforward modification of Jech's proof, we prove ω_1 can be huge and, assuming ZF $+$AC $+$, there is a huge cardinal.

In view of the well known results mentioned above, that ω_1 is measurable and ω_2-supercompact in ZF $+$ AD $+ DC$, our result should heighten interest in the problem of showing ω_1 is huge in ZF $+$AD $+$DC , which is yet unresolved (see [**5**]).

The proof of the result employs the method of symmetric submodels of generic models (see [**3**]). Briefly, let M be the ground model, B be a complete Boolean algebra in M, H be a group of automorphisms of B (in M) and F a normal filter on H (in M): that is, for all subgroups S, T of H:

(i) $H \in F$;

(ii) if $S \in F$ and $S \subset T$, then $T \in F$;

(iii) if $S \in F$ and $K \in F$, then $H \cap K \in F$;

(iv) if $\pi \in H$ and $S \in F$, then $\pi S \pi^{-1} \in F$.

Given a name $\underline{x} \in V^B$ (the Boolean-valued model), then

$$\mathrm{sym}(\underline{x}) = \{\pi \in H : \pi(\underline{x}) = \underline{x}\}.$$

Now, $\underline{x} \in V^B$ is called symmetric if $\mathrm{sym}(\underline{x}) \in F$. And $A \subset B$ is called a symmetric subset of B if $\{\pi \in H : \pi a = a \text{ for all } a \in A\} \in F$. Define the class C of hereditarily symmetric names by induction on the rank of $\underline{x}$ as follows: if $\mathrm{dom}(\underline{x}) \subseteq C$ and $\underline{x}$ is symmetric, then $\underline{x} \in C$.

Let G be an M-generic ultrafilter on B. Define N to be a symmetric submodel of $M[G]$ just in case N is the class of all elements of $M[G]$ that have a hereditarily symmetric name. As a result of these definitions N is a transitive model of ZF , and $M \subseteq N \subseteq M[G]$.

The following notation is used:

- $|X|$ denotes the cardinality of X,

- $\overline{X}$ denotes the order-type of X, and
- $(\lambda)^\kappa$ denotes $\{X \subset \lambda : \overline{X} = \kappa\}$.

Let M be the ground model, B a complete Boolean algebra in M, V^B the Boolean-valued model, G an M-generic ultrafilter on B, and $M[G]$ the generic extension of M. Then

- $\underline{x}$ denotes a name in V^B for $x \in M[G]$, and
- $\check{x}$ denotes a canonical name in V^B for $x \in M$.

Finally, for any formula ζ,

$$||\zeta(\underline{x})|| \text{ denotes the Boolean value of}$$
$$\text{the statement } \zeta(\underline{x}), \text{ for } \underline{x} \in V^B.$$

Let M be a transitive model of ZFC, B a complete Boolean algebra in M, G an M-generic ultrafilter on B, H a group of automorphisms of B (in M), F a normal filter on H, $M[G]$ the generic extension of M, and N the symmetric submodel of $M[G]$ defined from B, H and F.

LEMMA. *Let κ be huge in M. If every symmetric subset of B has cardinality less than κ, then κ is huge in N.*

PROOF. Let U be, in M, a κ-complete, normal ultrafilter over $(\lambda)^\kappa$ for some $\lambda > \kappa$. The ultrafilter U will generate a κ-complete, normal ultrafilter over $(\lambda)^\kappa$ in N. This will be accomplished if it can first be demonstrated that

if $\gamma < \kappa$ and $\{X_\alpha : \alpha < \gamma\}$ is a partition of $(\lambda)^\kappa$ in N,
then, for some $\alpha < \gamma, Y \subset X_\alpha$ for some $Y \in U$.

Next, in order that the ultrafilter U^i in N, generated from U in M, be normal in N, both (1) $N \models \{x \in (\lambda)^\kappa : \alpha \in x\} \in U^i$, for all $\alpha \in \lambda$, and (2) $N \models f : (\lambda)^\kappa \to \lambda$ and $\{x \in (\lambda)^\kappa : f(x) \in x\} \in U^i$ implies, for some $\gamma < \lambda$, $\{x \in (\lambda)^\kappa : f(x) = \gamma\} \in U^i$ must be proved.

The first statement is proved for $\gamma = 2$. The proof for $\gamma = 2$ generalizes in a straightforward manner to $\gamma < \kappa$. Let $X \in N$ be a subset of $(\lambda)^\kappa$ and $\underline{X}$ a symmetric name for X. Let

$$A = \{||\check{x} \in \underline{X}|| : x \in (\lambda)^\kappa\}.$$

If $\pi \in H$ is such that $\pi(\underline{x}) = \underline{x}$, since $\check{x}$ is a canonical name for something in the kernel, then $\pi(\check{x}) = \check{x}$ for every $x \in (\lambda)^\kappa$. This gives

$$\pi(||\check{x} \in \underline{X}||) = ||\pi(\check{x}) \in \pi(\underline{X})|| = ||\check{x} \in \underline{X}||.$$

So $\text{sym}(\underline{X}) \subset \text{sym}(||\check{x} \in \underline{X}||)$, for every $X \in (\lambda)^\kappa$. This gives

$$\text{sym}(\underline{X}) \subset \{\pi \in H : \pi a = a \text{ for every } a \in A\}.$$

Hence A is a symmetric subset of B. By hypothesis $|A| < \kappa$. For each $a \in A$, let

$$Y_a = \{x \in (\lambda)^\kappa ||\check{x} \in \underline{X}|| = a\}.$$

Given any $x \in (\lambda)^\kappa, ||\check{x} \in \underline{X}|| = a$, and if $a \neq a'$, then $Y_a \cap Y_{a'} = \varnothing$. So $\{Y_a : a \in A\}$ is a partition of $(\lambda)^\kappa$ into $|A|$-many subsets. Since $|A| < \kappa$ and U is a κ-complete ultrafilter over $(\lambda)^\kappa$, for some $a \in A, Y_a \in U$. Now

$$\begin{aligned}
N \models Y_a \subset X \quad & \text{iff} \quad ||\check{Y}_a \subset \underline{X}||_{HS} \in G \\
& \text{iff} \quad a = \prod_{\check{x} \in \check{Y}_a} ||\check{x} \in \underline{X}||_{HS} \in G \\
& \text{iff} \quad a = \prod_{\check{x} \in \check{Y}_a} ||\check{x} \in \underline{X}||_{HS} \in G \\
& \text{iff} \quad a \in G.
\end{aligned}$$

And

$$\begin{aligned}
N \models Y_a \subset (\lambda)^\kappa - X \text{ iff } \quad & ||\check{Y}_a \subset (\check{\lambda})^\kappa - \underline{X}||_{HS} \in G \\
\text{iff} \quad & \prod_{x \in Y_a} ||\check{x} \in (\check{\lambda})^\kappa - \underline{X}||_{HS} \in G \\
\text{iff} \quad & \prod_{\check{x} \in \check{Y}_a} ||\check{x} \notin \underline{X}||_{HS} \in G \\
\text{iff} \quad & 1 - a \notin G \\
\text{iff} \quad & a \notin G.
\end{aligned}$$

Therefore, either X or $(\lambda)^\kappa - X$ has a subset in U.

Finally, the fact that U^i is generated from a normal filter determines that U^i is normal: $\{x \in (\lambda)^\kappa : \alpha \in x\} \in U^i$, since it is in the filter which generates U^i, and the generating filter is closed under diagonal intersections; so U^i must be closed under diagonal intersection, which yeilds the normality of U^i

THEOREM. *If* ZFC + *"a huge cardinal exists" is consistent, then* ZF + *"$\aleph_1$ is huge" is consistent.*

The theorem follows from the lemma and a result of Jech (see [**2**]) which proves: if P is a set of conditions used in the Levy collapse to turn κ into a countable ordinal; B the Boolean algebra generated from P; H the group of automorphisms of B induced by all permutations of κ; F the normal filter on H generated by subgroups H_γ of H whose members leave ordinals less than γ fixed, for each $\gamma < \kappa$; N the symmetric model corresponding to B, G, F, H; then $\kappa = (\aleph_1)^N$ and, for every symmetric $A \subseteq B, |A| < \kappa$.

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