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# AN ASYMPTOTIC PROPERTY FOR TAILS OF LIMIT PERIODIC CONTINUED FRACTIONS

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**1. Introduction.** In the present paper we study continued fractions of the forms

$$K_{n=1}^{\infty}\frac{a_n}{1} = K(a_n/1) = \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1} + \cdots, \tag{1.1}$$

$$a_n \in \mathbf{C}\setminus\{0\},$$

and

$$K_{n=1}^{\infty}\frac{1}{b_n} = K(1/b_n) = \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} + \cdots, \tag{1.2}$$

$$b_n \in \mathbf{C},$$

where the elements $\{a_n\}$, $\{b_n\}$ are *limit $k$-periodic* for a $k \in \mathbf{N}$, that is

$$a_{kn+p} = \tilde{a}_p + \delta_{kn+p} \text{ or } b_{kn+p} = \tilde{b}_p + \delta_{kn+p};\ \tilde{a}_p, \tilde{b}_p \in \mathbf{C}, \delta_n \to 0 \tag{1.3}$$

for $p = 1, \ldots, k$ and all $n \geq 0$. We also assume that these limit periodic continued fractions are of hyperbolic or loxodromic type. (For definition, see §2 and §3.) It is then well known that the continued fraction converges, at least generally. (For definition, see §3.) So do also all its *tails*

$$\begin{aligned} K_{n=1}^{\infty}\frac{a_{m+n}}{1} &= \frac{a_{m+1}}{1} + \frac{a_{m+2}}{1} + \cdots \\ K_{n=1}^{\infty}\frac{1}{b_{m+n}} &= \frac{1}{b_{m+1}} + \frac{1}{b_{m+2}} + \cdots \end{aligned} \tag{1.4}$$

for $m \in \mathbf{N}\cup\{0\}$. Let $f^{(m)}$ denote the value of the $m$-th tail (1.4). Then $\{f^{(m)}\}$ is also limit $k$-periodic [**3**, p. 96; **1**].

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We assume that $\{f^{(m)}\}$ has finite limit points, that is

$$f^{(kn+p)} = \Gamma_p + \varepsilon_{kn+p},\ \Gamma_p \in \mathbf{C} \quad \text{for } p = 1, \dots, k, n \geq 0, \varepsilon_n \to 0. \tag{1.5}$$

Then upper bounds for $|\varepsilon_n|$ in terms of the differences $|\delta_m|$ in (1.3) are well established. Here we go one step further, although, in a special case, we shall assume also that $\{\delta_{n+1}/\delta_n\}$ is limit $k$-periodic and see what effect that has on $\{\varepsilon_{n+1}/\varepsilon_n\}$. (Many continued fraction expansions of known functions satisfy this extra condition.)

The problem appeared in the following connection. Modified approximants

$$S_n(w_n) = \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1 + w_n}, \quad w_n \in \hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}, n \in \mathbf{N}, \tag{1.6}$$

converge faster to the value $f = f^{(0)}$ of (1.1) than the ordinary approximants $S_n(0)$ if we choose the modifying factors $\{w_n\}$ appropriately. In [**5**] some different choices for $\{w_n\}$ are compared. The asymptotic behavior of $\{\varepsilon_{n+1}/\varepsilon_n\}$ determines in some cases which one of the considered choices is the best one.

It is important to come up with good alternatives for the modifying factors $\{w_n\}$. Clearly $w_n = f^{(n)}$ is the "best" choice since $S_n(f^{(n)}) = f$, but $f^{(n)}$ is in general unknown. In view of (1.5), $w_{kn+p} = \Gamma_p$ seems to be a good choice, and indeed it is proved that if $f \neq \infty$ and $\Gamma_p \neq 0$, then

$$(f - S_{kn+p}(\Gamma_p))/(f - S_{kn+p}(0)) \to 0 \quad \text{as } n \to \infty \qquad [\mathbf{2}]. \tag{1.7}$$

It is shown later (in (2.4)) that, for $K(a_n/1)$,

$$\delta_{kn+p+1} = (1 + \Gamma_{p+1})\varepsilon_{kn+p} + \Gamma_p \varepsilon_{kn+p+1} + \varepsilon_{kn+p}\varepsilon_{kn+p+1}. \tag{1.8}$$

Dividing by $\varepsilon_{kn+p}$ we see that if the asymptotic behavior of $\{\varepsilon_{n+1}/\varepsilon_n\}$ is known, then so is the behavior of $\{\delta_{n+1}/\varepsilon_n\}$. Since $\{\delta_n\}$ is known, this gives us an estimate $\tilde{\varepsilon}_n$ for $\varepsilon_n$. One can then prove that

$$\begin{gathered}(f - S_{kn+p}(\Gamma_p + \tilde{\varepsilon}_{kn+p}))/(f - S_{kn+p}(\Gamma_p)) \to 0 \\ \text{as } n \to \infty \text{ if } f \neq \infty, \qquad [\mathbf{2}].\end{gathered} \tag{1.9}$$

In §2 our main results are presented and proved for the special case where $k = 1$. The more general results for $k \in \mathbf{N}$ are stated in §3. In

§4 we consider some other consequences of the techniques from §2 and §3.

**2. The case $\mathbf{k = 1}$.** $K(a_n/1)$ is limit 1-periodic of hyperbolic or loxodromic type if $a_n \to a$, where $|\arg(a + 1/4)| < \pi$. Then $K(a_n/1)$ converges to a value $f \in \hat{\mathbf{C}}$ and

$$f^{(n)} \to \Gamma = (\sqrt{1+4a} - 1)/2, \quad \text{where } \Re\sqrt{\phantom{x}} > 0 \qquad [\mathbf{3}, \text{p. } 96]. \tag{2.1}$$

With the notation $\delta_n = a_n - a,\ \varepsilon_n = f^{(n)} - \Gamma$ as in the introduction, we then have

THEOREM 2.1. *Let $K(a_n/1)$ satisfy $a_n \to a$ where $|\arg(a{+}1/4)| < \pi$. Then*

$$\lim_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n} = t \in \mathbf{C} \quad \textit{if and only if} \quad \lim_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} = t \in \mathbf{C}\,. \tag{2.2}$$

PROOF. Since $f^{(n)} \to \Gamma \neq \infty$, we can, without loss of generality assume that all $f^{(n)} \neq \infty$. (Otherwise we just consider a tail of $K(a_n/1)$.) From the relations

$$f^{(n-1)} = a_n/(1 + f^{(n)}), \qquad \Gamma = a/(1+\Gamma) \tag{2.3}$$

it then follows that

$$\delta_n = (1+\Gamma)\varepsilon_{n-1} + \Gamma\varepsilon_n + \varepsilon_{n-1}\varepsilon_n. \tag{2.4}$$

Assume first that $\lim \varepsilon_{n+1}/\varepsilon_n = t \in \mathbf{C}$. Since $\varepsilon_n \to 0$ we then know that $|t| \leq 1$. Then $\varepsilon_n \neq 0$ from some $n$ on. ($\varepsilon_n \neq \infty$ since we have assumed that all $f^{(n)}$ are finite.) Without loss of generality we assume that all $\varepsilon_n \neq 0$. From (2.4) we then get

$$\frac{\delta_{n+1}}{\delta_n} = \frac{\varepsilon_n}{\varepsilon_{n-1}} \cdot \frac{1 + \Gamma + \Gamma\varepsilon_{n+1}/\varepsilon_n + \varepsilon_{n+1}}{1 + \Gamma + \Gamma\varepsilon_n/\varepsilon_{n-1} + \varepsilon_n}, \tag{2.5}$$

which proves that $\delta_{n+1}/\delta_n \to t$. $((1 + \Gamma + \Gamma\varepsilon_{n+1}/\varepsilon_n + \varepsilon_{n+1}) \to 1 + \Gamma + t\Gamma \neq 0$ since $|t| \leq 1$ and $|1+\Gamma| > |\Gamma|$.)

Assume next that $\lim \delta_{n+1}/\delta_n = t \in \mathbf{C}$. Again $|t| \leq 1$ and $\delta_n \neq 0$ from some $n$ on. ($\delta_n \neq \infty$ by definition.) Without loss of generality we assume that all $\delta_n \neq 0$. By (2.4), we see that then no two consecutive $\varepsilon_n$ can both be zero and that all $\varepsilon_n \neq 0$ if $\Gamma = 0$. This means that (2.5) still holds (with the obvious interpretation if $\varepsilon_n$ or $\varepsilon_{n-1}$ is zero).

*Case* 1. $\Gamma = 0$. Then (2.4) reduces to $\delta_n = \varepsilon_{n-1}(1+\varepsilon_n)$, and thus

$$\frac{\delta_{n+1}}{\delta_n} = \frac{\varepsilon_n}{\varepsilon_{n-1}} \frac{1+\varepsilon_{n+1}}{1+\varepsilon_n} \quad \text{where } \frac{1+\varepsilon_{n+1}}{1+\varepsilon_n} \to 1. \tag{2.5}$$

Hence, $\lim \varepsilon_n/\varepsilon_{n-1} = \lim \delta_{n+1}/\delta_n = t$.

*Case* 2. $\Gamma \neq 0$. Rearranging (2.5) we find that $\{g_n\}$, given by

$$g_n = \Gamma \varepsilon_{n+1}/\varepsilon_n, \quad \text{for } n = 0, 1, 2, \dots, \tag{2.6}$$

satisfies the recurrence relation

$$g_{n-1} = c_n/(d_n + g_n), \quad \text{for } n = 1, 2, 3, \dots, \tag{2.7}$$

where

$$c_n = \Gamma(1 + \Gamma + \varepsilon_n)\delta_{n+1}/\delta_n \to c = at \tag{2.8}$$

and

$$d_n = 1 + \Gamma + \varepsilon_{n+1} - \Gamma\delta_{n+1}/\delta_n \to d = 1 + \Gamma - \Gamma t. \tag{2.9}$$

$g_n$ is clearly well defined since $\Gamma \neq 0$ and $\varepsilon_n, \varepsilon_{n+1}$ are both finite and at least one of them non-zero. Since $\Gamma(1+\Gamma+\varepsilon_n) \to a = \Gamma(1+\Gamma) \neq 0$ and all $\delta_n \neq 0$, we can, without loss of generality, assume that all $c_n \neq 0$. Then $K(c_n/d_n)$ is a limit 1-periodic continued fraction. Every sequence $\{g_n^*\}$ satisfying (2.7) is called a sequence of right or wrong tails for $K(c_n/d_n)$. If we can prove that $K(c_n/d_n)$ is limit 1-periodic of hyperbolic or loxodromic type, i.e., that either

$$c = 0, \qquad d \neq 0 \tag{2.10}$$

or that the non-singular linear fractional transformation

$$s(w) = c/(d + w) \quad \text{where} \quad c \neq 0 \tag{2.11}$$

is of hyperbolic or loxodromic type, then we know that $\{g_n\}$ converges [**1**]. Since $\Gamma \neq 0$ we then have that $\{\varepsilon_{n+1}/\varepsilon_n\}$ converges. That $\lim \varepsilon_{n+1}/\varepsilon_n = t$ follows then by (2.5).

Clearly, if $t = 0$ then (2.10) holds. Assume that $t \neq 0$. Then $s(w)$ has the two fixed points

$$-(1+\Gamma) \quad \text{and} \quad \Gamma t. \tag{2.11}$$

Since $|d + (-1-\Gamma)| < |d + \Gamma t|$, it follows that $s(w)$ is of hyperbolic or loxodromic type. □

It is interesting to note that a slightly weaker version of the non-trivial part (the if-part) of (2.2) can be proved by using a formula for a linear approximation of the value $f$ of $K((a+\delta_n)/1)$ if all $|\delta_n| \leq \rho$ for $\rho > 0$ sufficiently small:

$$f = \Gamma + \frac{1}{1+\Gamma} \sum_{m=0}^{\infty} \left(\frac{-\Gamma}{1+\Gamma}\right)^m \delta_{m+1} + O(\rho^2) \qquad [\mathbf{4}]. \tag{2.12}$$

The $O$-term is dominated by $K\rho^2$ for some $K > 0$ depending only upon $a$ [**4**].

If we assume that $\{|\delta_n|\}$ is a decreasing sequence from some $n$ on, the if-part of (2.2) follows easily, since, by (2.12),

$$\varepsilon_n = f^{(n)} - \Gamma = \frac{1}{1+\Gamma} \sum_{m=0}^{\infty} \left(\frac{-\Gamma}{1+\Gamma}\right)^m \delta_{n+m+1} + O(|\delta_{n+1}|^2); \tag{2.13}$$

that is (since $\delta_{n+1} \neq 0$ and $|d_n|$ decreases),

$$\begin{aligned} \frac{\varepsilon_n}{\delta_{n+1}} &= \frac{1}{1+\Gamma} \sum_{m=0}^{\infty} \left(\frac{-\Gamma}{1+\Gamma}\right)^m \frac{\delta_{n+m+1}}{\delta_{n+1}} + O(|\delta_{n+1}|) \\ &\underset{n\to\infty}{\longrightarrow} \quad \frac{1}{1+\Gamma} \sum_{m=0}^{\infty} \left(\frac{-\Gamma}{1+\Gamma}\right)^m t^m = \frac{1}{1+\Gamma+t\Gamma} \end{aligned} \tag{2.14}$$

or $\lim \delta_{n+1}/\varepsilon_n = 1 + \Gamma + t\Gamma$. Inserting the expression (2.4) for $\delta_{n+1}$ then gives the result.

Another interesting observation is that Gauss' continued fractions $1 + K(a_n z/1)$ for hypergeometric functions ${}_2F_1$ satisfy the conditions of Theorem 2.1 with $t = -1$ [**3**, p. 123].

We can obtain a similar result for continued fractions $K(1/b_n)$. $K(1/b_n)$ is limit 1-periodic of hyperbolic or loxodromic type if $b_n \to b$, where $b \in \mathbf{C} \setminus i[-2, 2]$. In this case $K(1/b_n)$ converges to a value $f \in \hat{\mathbf{C}}$ and

$$f^{(n)} \to \Gamma = (\sqrt{1 + 4/b^2} - 1)b/2 \quad \text{where } \Re\sqrt{\ } > 0. \tag{2.15}$$

Using the notation $\delta_n = b_n - b$ and $\varepsilon_n = f^{(n)} - \Gamma$, we then have

THEOREM 2.2. *Let $K(1/b_n)$ satisfy $b_n \to b \in \mathbf{C} \setminus i[-2, 2]$. Then*

$$\lim_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n} = t \in \mathbf{C} \quad \textit{if and only if} \quad \lim_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} = t \in \mathbf{C}\,. \tag{2.16}$$

PROOF. The proof follows the one of Theorem 2.1 with some modifications. First of all the recurrence relations for the tails now become

$$f^{(n-1)} = 1/(b_n + f^{(n)}), \qquad \Gamma = 1/(b + \Gamma) \tag{2.3$'$}$$

such that we get

$$(\Gamma + \varepsilon_n)\delta_{n+1} = -(b + \Gamma)\varepsilon_n - \Gamma\varepsilon_{n+1} - \varepsilon_n\varepsilon_{n+1} \tag{2.4$'$}$$

and thus

$$\frac{\delta_{n+1}}{\delta_n} = \frac{\Gamma + \varepsilon_{n-1}}{\Gamma + \varepsilon_n} \cdot \frac{\varepsilon_n}{\varepsilon_{n-1}} \cdot \frac{b + \Gamma + \Gamma\varepsilon_{n+1}/\varepsilon_n + \varepsilon_{n+1}}{b + \Gamma + \Gamma\varepsilon_n/\varepsilon_{n-1} + \varepsilon_n}. \tag{2.5$'$}$$

This proves the only if part of (2.16).

To prove the if part, we observe that $\Gamma \neq 0$, and that rearranging (2.5$'$) gives that $\{g_n\}$ (defined by (2.6)) satisfies the recurrence relation (2.7) with

$$c_n = \Gamma(b + \Gamma + \varepsilon_n)\frac{\delta_{n+1}}{\delta_n} \cdot \frac{\Gamma + \varepsilon_n}{\Gamma + \varepsilon_{n-1}} \to c = at. \tag{2.8$'$}$$

and

$$d_n = b + \Gamma + \varepsilon_{n+1} - \Gamma\frac{\delta_{n+1}}{\delta_n} \cdot \frac{\Gamma + \varepsilon_n}{\Gamma + \varepsilon_{n-1}} \to d = b + \Gamma - \Gamma t. \tag{2.9$'$}$$

For sufficiently large $N, K_{n=N}^{\infty}(c_n/d_n)$ is then a limit 1-periodic continued fraction of hyperbolic or loxodromic type, and the result follows. □

**3. The case $k \in \mathbf{N}$.** The result in §2 can be extended to the more general case where $K(a_n/1)$ or $K(1/b_n)$ is limit $k$-periodic of hyperbolic or loxodromic type. This is important since also this class contains continued fraction expansions of many interesting functions. For instance, the $C$-fraction expansion of $(1-z)_2F_1(a,1;c;z^2)$ is limit 4-periodic with limit 4-periodic $\delta_{n+1}/\delta_n$. Also, cases where $K(a_n/1)$ (or $K(1/b_n)$) is limit 1-periodic and $\{\delta_{n+1}/\delta_n\}$ is limit $k$-periodic for some $k > 1$ are interesting. Then $K(a_n/1)$ can be regarded as a limit $k$-periodic continued fraction in order to apply the results from this paper.

A limit $k$-periodic continued fraction $K(a_n/b_n)$ is said to be of *hyperbolic* or *loxodromic type* if the limits

$$\lim_{n\to\infty} a_{kn+p} = \tilde{a}_p, \lim_{n\to\infty} b_{kn+p} = \tilde{b}_p \text{ for } p = 1, \dots, k \tag{3.1}$$

are finite and the linear fractional transformation

$$\tilde{S}_k(w) = \frac{\tilde{a}_1}{\tilde{b}_1} + \frac{\tilde{a}_2}{\tilde{b}_2} + \cdots + \frac{\tilde{a}_k}{\tilde{b}_k + w} = \frac{\tilde{A}_k + \tilde{A}_{k-1}w}{\tilde{B}_k + \tilde{B}_{k-1}w}, \tag{3.2}$$

where $\tilde{A}_m, \tilde{B}_m$ satisfy the recurrence relation

$$\begin{gathered} \tilde{A}_m = \tilde{b}_m\tilde{A}_{m-1} + \tilde{a}_m\tilde{A}_{m-2}, \quad \tilde{B}_m = \tilde{b}_m\tilde{B}_{m-1} + \tilde{a}_m\tilde{B}_{m-2} \\ \text{for } \ m = 1, \dots, k, \\ \tilde{A}_0 = \tilde{B}_{-1} = 0, \qquad \tilde{A}_{-1} = \tilde{B}_0 = 1, \end{gathered} \tag{3.3}$$

satisfies

$$\begin{aligned} &|\tilde{A}_{k-1} + \tilde{B}_k + \sqrt{R}| \neq |\tilde{A}_{k-1} + \tilde{B}_k - \sqrt{R}|, \\ &\text{where } R = (\tilde{A}_{k-1} - \tilde{B}_k)^2 + 4\tilde{A}_k\tilde{B}_{k-1}. \end{aligned} \tag{3.4}$$

$\tilde{S}_k$ is non-singular if and only if all $\tilde{a}_n \neq 0$. It can be proved that if $\tilde{S}_k$ is non-singular, then $\tilde{S}_k$ is hyperbolic or loxodromic if and only if (3.4) holds [**1**].

It does not change anything if we instead regard a tail of $K(a_n/b_n)$. For $n \in \mathbf{N}$ let

$$(3.2') \qquad \tilde{S}_k^{(n)}(w) = \frac{\tilde{a}_{n+1}}{\tilde{b}_{n+1}} + \frac{\tilde{a}_{n+2}}{\tilde{b}_{n+2}} + \cdots + \frac{\tilde{a}_{n+k}}{\tilde{b}_{n+k} + w} = \frac{\tilde{A}_k^{(n)} + \tilde{A}_{k-1}^{(n)} w}{\tilde{B}_k^{(n)} + \tilde{B}_{k-1}^{(n)} w},$$

where $\tilde{a}_{kn+p} = \tilde{a}_p$, $\tilde{b}_{kn+p} = \tilde{b}_p$ for $p = 1, \dots, k$ and all $n \geq 0$. Then one can prove that $\tilde{S}_k^{(n)}$ is non-singular if and only if $\tilde{S}_k = \tilde{S}_k^{(0)}$ is non-singular, and $\tilde{S}_k^{(n)}$ is of hyperbolic or loxodromic type if and only if $\tilde{S}_k$ is of hyperbolic or loxodromic type, [**1**].

Let $\Gamma_n$ and $y_n$ denote the attractive and repulsive fixed point of $\tilde{S}_k^{(n)}$. (If $\tilde{S}_k$ is singular, then $\Gamma_n = \tilde{S}_k^{(n)}(w)$ and

(3.5)
$$y_n = \begin{cases} -\tilde{B}_k^{(n)}/\tilde{B}_{k-1}^{(n)} & \text{if } \tilde{a}_{n+1} = 0, \\ \tilde{a}_{n+1}/(\tilde{b}_{n+1} + y_{n+1}) & \text{if } \tilde{a}_{n+1} \neq 0, \end{cases} \quad \text{for } n = p, p-1, \dots, 0,$$

starting with a $p \in \{k, k+1, \dots, 2k-1\}$ such that $\tilde{a}_{p+1} = 0$. Further, the relation $y_{n+k} = y_n$ allows us to define $\{y_n\}$ for all $n \in \mathbf{N}$.) $\Gamma_n$ is then the same $\Gamma_n$ as in the introduction.

With this notation we know that if $K(a_n/b_n)$ is limit $k$-periodic of hyperbolic or loxodromic type and *all* $y_p \neq \infty$, then $K(a_n/b_n)$ converges to a value $f \in \hat{\mathbf{C}}$ [**1**]. If $y_p = \infty$ for one or more $p \in \{0, \dots, k-1\}$, then $K(a_n/b_n)$ may diverge, but it will always converge generally to a value $f \in \hat{\mathbf{C}}$ [**1**]. By general convergence we mean

DEFINITION 3.1. A continued fraction $K(a_n/b_n)$ is said to converge generally to a value $f \in \hat{\mathbf{C}}$, if there exist two sequences $\{u_n\}$, $\{v_n\}$ of elements from $\hat{\mathbf{C}}$ such that

(3.6)
$$\lim S_n(u_n) = \lim S_n(v_n) = f, \qquad \liminf \frac{|u_n - v_n|}{\sqrt{1+|u_n|^2}\sqrt{1+|v_n|^2}} > 0.$$

The (general) value $f$ of a generally convergent continued fraction is unique. If $K(a_n/b_n)$ converges to $f$, then it also converges generally to $f$.

We shall assume that all $\Gamma_n \neq \infty$, but we allow $y_n = \infty$. $f^{(n)} = \Gamma_n + \varepsilon_n$ therefore denotes the general values of the tails of $K(a_n/1)$ or $K(1/b_n)$ in cases where $K(a_n/1)$ or $K(1/b_n)$ diverges in the ordinary sense. Under our conditions we still have that $\varepsilon_n \to 0$.

THEOREM 3.2. *Let $K(a_n/1)$ be a limit $k$-periodic continued fraction of hyperbolic or loxodromic type, and let $\Gamma_p \neq \infty$ for $p = 0, \dots, k-1$. Then the following hold.*

A. *If, for an $m \in \{1, \dots, k\}$,*

$$\lim_{n\to\infty} \varepsilon_{kn+p+1}/\varepsilon_{kn+p} = s_p \in \mathbf{C}\,, \quad \text{for } p = m, m-1, \tag{3.7}$$

*and $s_p \neq -(1+\Gamma_{p+1})/\Gamma_p$ for at least one of the indices $p = m, m-1$, then*

$$\lim_{n\to\infty} \frac{\delta_{kn+m+1}}{\delta_{kn+m}} = t_m = s_{m-1} \frac{1+\Gamma_{m+1}+\Gamma_m s_m}{1+\Gamma_m+\Gamma_{m-1}s_{m-1}}. \tag{3.8}$$

B. *If*

$$\lim_{n\to\infty} \delta_{kn+p+1}/\delta_{kn+p} = t_p \in \mathbf{C}\,, \quad \text{for } p = 1, \dots, k, \tag{3.9}$$

*then*

$$\lim_{n\to\infty} \varepsilon_{kn+p+1}/\varepsilon_{kn+p} = s_p \neq -\frac{1+\Gamma_{p+1}}{\Gamma_p}, \quad \text{for } p = 0, \dots, k-1. \tag{3.10}$$

REMARKS 3.3. 1. If (3.10) holds, then $\prod_{p=0}^{k-1} |s_p| \leq 1$ since $\varepsilon_n \to 0$. Likewise, if (3.9) holds, then $\prod_{p=1}^{k} |t_p| \leq 1$.

2. The implication in part A also involves the existence of $\delta_{kn+m+1}/\delta_{kn+m}$ from some $n$ on. Likewise, if (3.9) holds, then $\varepsilon_{n+1}/\varepsilon_n$ is well-defined from some $n$ on.

3. Clearly, the connection (3.8) between $t_p$ and $s_p, s_{p-1}$ also holds in part $B$. Moreover,

$$\prod_{p=0}^{k-1} s_p = \prod_{p=1}^{k} t_p. \tag{3.11}$$

A proof of Theorem 3.2 will not be included here. It can be proved following the same idea as in the proof of Theorem 2.1.

For the choice $k = 1$, we have that $y_n \neq \infty$ and Theorem 3.2 reduces to Theorem 2.1. For the choice $k = 2$ we also have $y_n \neq \infty$ such that $K(a_n/1)$ converges. The connection between $(s_0, s_1)$ and $(t_1, t_2)$ is then given by

$$(3.12) \qquad t_p = s_{p-1}\frac{s_p\Gamma_p + 1 + \Gamma_{p-1}}{s_{p-1}\Gamma_{p-1} + 1 + \Gamma_p}, \quad \text{for } p = 1, 2 \quad (s_2 = s_0),$$

and thus

$$(3.13) \qquad s_p = t_{p+1}\frac{1 + \Gamma_{p+1} - \Gamma_{p+1}t_p}{1 + \Gamma_p - \Gamma_p t_{p+1}}, \quad \text{for } p = 0, 1 \quad (t_0 = t_2).$$

For continued fractions $K(1/b_n)$ we have, similarly,

THEOREM 3.4. *Let $K(1/b_n)$ be a limit $k$-periodic continued fraction of hyperbolic or loxodromic type, and let $\Gamma_p \neq \infty$ for $p = 0, \dots, k-1$. Then the following hold.*

A. *If, for an $m \in \{1, \dots, k\}$,*

$$(3.14) \qquad \lim_{n\to\infty} \varepsilon_{kn+p+1}/\varepsilon_{kn+p} = s_p \in \mathbf{C}, \quad \text{for } p = m,\ m-1,$$

*and $s_p = -(\tilde{b}_{p+1} + \Gamma_{p+1})/\Gamma_p$ does not occur for both indices $p = m, m-1$, then*

$$(3.15) \qquad \lim_{n\to\infty}\frac{\delta_{kn+m+1}}{\delta_{kn+m}} = t_m = s_{m-1}\frac{\Gamma_{m-1}}{\Gamma_m}\cdot\frac{\tilde{b}_{m+1} + \Gamma_{m+1} + \Gamma_m s_m}{\tilde{b}_m + \Gamma_m + \Gamma_{m-1}s_{m-1}}.$$

B. *If*

$$(3.16) \qquad \lim_{n\to\infty} \delta_{kn+p+1}/\delta_{kn+p} = t_p \in \mathbf{C}, \quad \text{for } p = 1, \dots, k,$$

*then*

$$(3.17) \qquad \lim_{n\to\infty} \varepsilon_{kn+p+1}/\varepsilon_{kn+p} = s_p \neq -(\tilde{b}_{p+1} + \Gamma_{p+1})/\Gamma_p \quad \text{for } p = 0, \dots, k-1.$$

Remarks 3.3 still hold, and Theorem 3.4 reduces to Theorem 2.2 for the choice $k = 1$.

**4. Some other results.** Reading the proofs of Theorem 2.1 and Theorem 2.2 we see that they depend on

(i) the recurrence relations (2.3) and (2.3′),

(ii) the fact that $f^{(n)} = \Gamma + \varepsilon_n$ where $\varepsilon_n \to 0, \Gamma \neq \infty$, and

(iii) the continued fraction $K(c_n/d_n)$, given by (2.8)–(2.9) or

(2.8′)–(2.9′), being limit 1-periodic of hyperbolic or loxodromic type.

It is well known that if $K(a_n/1)$ or $K(1/b_n)$ is limit $k$-periodic of hyperbolic or loxodromic type, then every sequence $\{g_n\}$ of $g$-wrong tails (i.e., $\{g_n\}$ satisfies (2.3) or (2.3′) with $g_0 \neq f$) is limit $k$-periodic such that

$$\lim_{n\to\infty} g_{kn+p} = y_p, \quad \text{for } p = 0, \dots, k-1 \qquad [\mathbf{1}]. \tag{4.1}$$

For $k = 1$ we have $y_p = y \neq \infty$ such that $\{g_n\}$ satisfies (i) and (ii) above with $\Gamma$ replaced by $y$. The similarity goes further. We have

THEOREM 4.1. *Let $K(a_n/1)$ satisfy $a_n \to a$ where $|\arg(a+1/4)| < \pi$, and let $\{g_n\}$ be an arbitrary sequence of $g$-wrong tails for $K(a_n/1)$. Further let $\varepsilon_n = g_n - y$, and let $t \in \mathbf{C}$ satisfy $|t| \neq |1+y|/|y|$. Then*

$$\lim_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n} = t \quad \textit{if and only if} \quad \lim_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} = t. \tag{4.2}$$

REMARKS 4.2. 1. The conclusion (4.2) is empty for $|t| > 1$. The extra condition $|t| \neq |1+y|/|y|$ is vital since $|1+y| < |y|$. In Theorem 2.1 no such condition was needed since $|1+\Gamma| > |\Gamma|$.

2. Also Theorem 3.2 has a parallel for $g$-wrong tails $\{g_n\}$, with $\Gamma_n$ replaced by $y_n$. The extra condition on $\{t_n\}$ then takes the form

$$\prod_{n=1}^{k} |t_n| \neq \prod_{n=1}^{k} \left| \frac{1+y_n}{y_n} \right|. \tag{4.3}$$

It is well known that the conclusions are empty if $\prod_{n=1}^{k}|t_n| > 1$ and that, for limit $k$-periodic continued fractions of hyperbolic or loxodromic type,

$$(4.4) \qquad \prod_{n=1}^{k}\left|\frac{1+y_n}{y_n}\right| < 1 < \prod_{n=1}^{k}\left|\frac{1+\Gamma_n}{\Gamma_n}\right| \text{ if all } y_n,\, \Gamma_n \neq \infty.$$

3. (4.2) also holds for continued fractions $K(1/b_n)$, where $b_n \to b \in \mathbf{C} \setminus i[-2,2]$, when $|t| \neq |1+y|/|y|$.

PROOF. The proof goes through just as before, since $1+y+y\varepsilon_{n+1}/\varepsilon_n+\varepsilon_{n+1} \to 1+y+yt \neq 0$ if $\varepsilon_{n+1}/\varepsilon_n \to t$. If $\delta_{n+1}/\delta_n \to t$, we need to show that $K_{n=N}^{\infty}(c_n/d_n)$, where

$$(2.8'') \qquad c_n = y(1+y+\varepsilon_n)\delta_{n+1}/\delta_n \to c = at$$

and

$$(2.9'') \qquad d_n = 1+y+\varepsilon_{n+1} - y\delta_{n+1}/\delta_n \to d = 1+y-yt$$

is a limit 1-periodic continued fraction of hyperbolic or loxodromic type for sufficiently large $N$. This happens if and only if $|t| \neq |1+y|/|y|$ . □

Another observation is that the proofs of Theorem 3.2 and 3.4 do not really depend on $K(a_n/1)$ or $K(1/b_n)$ to be of hyperbolic or loxodromic type. This means that Theorem 3.2 and 3.4 also holds for $K(a_n/1)$ or $K(1/b_n)$ being of the elliptic or parabolic type as long as $\{f^{(n)}\}$ (or $\{g_n\}$) is limit $k$-periodic with finite limits and $\prod_{p=1}^{k}|t_p| < 1$.

If $a_n \to -1/4$ and

$$(4.5) \qquad |a_n| - \Re(a_n e^{-i2\alpha}) \leq 2q_{n-1}(1-q_n)\cos^2\alpha \quad \text{from some } n \text{ on},$$

where $-\pi/2 < \alpha < \pi/2$ is a fixed constant and $0 < q_n \to 1/2$, then one can prove that every sequence of right or wrong tails of $K(a_n/1)$ converges to $-1/2$. We therefore have, in particular,

THEOREM 4.3. *Let $K(a_n/1)$, where $a_n = -1/4+\delta_n, \delta_n \to 0$ satisfies (4.5), be given, and let $\{g_n\}$ be a sequence of right or wrong tails of $K(a_n/1)$. Then the following hold.*

A. *Let* $t \in \mathbf{C}, |t| < 1$. *Then*

$$\lim \delta_{n+1}/\delta_n = t \Longleftrightarrow \lim(g_{n+1} + 1/2)/(g_n + 1/2) = t. \tag{4.6}$$

B. *Let* $t_1, t_2 \in \mathbf{C}, |t_1 t_2| < 1$. *Then*

$$\lim_{n\to\infty} \delta_{2n+p+1}/\delta_{2n+p} = t_p, \quad \text{for } p = 1, 2, \tag{4.7}$$

*if and only if*

$$\lim_{n\to\infty} \frac{g_{2n+p+1} + 1/2}{g_{2n+p} + 1/2} = s_p = \frac{1 + t_p}{1 + t_{p+1}} t_{p+1}, \quad \text{for } p = 0, 1 \quad (t_0 = t_2). \tag{4.8}$$

**Added in Proof**. See also P. Levrie, *Improving a method for computing non-dominant solutions of certain second-order recurrence relations of Poincaré-type*, Numer. Math., to appear.

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