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# SPECTRAL ALGEBRAS

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ABSTRACT. Spectral algebras are a class of abstract complex algebras which share many of the good properties of Banach algebras. In the commutative case they are precisely the class of abstract algebras having a full Gelfand theory. Any irreducible representation of a spectral algebra is strictly dense. Spectral algebras are defined and characterized in terms of spectral pseudo-norms and spectral subalgebras. Spectral algebras, spectral subalgebras and spectral pseudo-norms are shown to occur frequently in analysis.

It is also shown that when the spectral radius is finite valued, if it is either subadditive or submultiplicative, then it has both properties and that this occurs exactly for algebras which are spectral algebras and commutative modulo their Jacobson radicals.

The paper is written in an expository style.

**1. Introduction.** The spectrum is undoubtedly the most important concept in the theory of (linear, associative) algebras. This article is an exploration of those abstract (i.e., not necessarily normed) complex algebras in which the spectrum behaves as it does in Banach algebras. Examples which outline the boundaries of this theory are also given. The article is written in an expository fashion with minimal prerequisites. Proofs are complete and self contained except for some well-known arguments for which specific references are given.

We shall use $\mathbf{C}, \mathbf{R}, \mathbf{R}_+$ and $\mathbf{N}$ to denote the complex numbers, real numbers, nonnegative real numbers and natural numbers, respectively. In this article the word *algebra* will mean a ring, $\mathfrak{A}$, which is also a complex linear space under the same addition and satisfies $(\lambda a)b = \lambda(ab) = a(\lambda b)$ for all $a, b \in \mathfrak{A}$ and $\lambda \in \mathbf{C}$. In the introduction, for ease of exposition, we will also assume that $\mathfrak{A}$ contains a multiplicative identity, 1, (in this case $\mathfrak{A}$ is said to be *unital*), but this assumption will not be made later. For any element, $a \in \mathfrak{A}$, we define its *spectrum* and *spectral radius* by

$$\mathrm{Sp}(a) = \{\lambda \in \mathbf{C} : \lambda 1 - a \text{ has no inverse in } \mathfrak{A}\},$$

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and

$$\rho(a) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(a)\},$$

respectively. If the spectrum is empty or unbounded, then the spectral radius will have the value $-\infty$ or $\infty$, respectively. When no element has empty or unbounded spectrum, the spectral radius is a nonnegative real valued function on the algebra—a beginning for analysis and geometry.

Here are some elementary examples: (1) If $\mathfrak{A} = M_n$ is the algebra of all $n \times n$-matrices (for some positive integer $n$), then

$$\mathrm{Sp}(a) = \{\text{eigenvalues of } a\},$$

and

$$\rho(a) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } a\}.$$

(2) If $\mathfrak{A} = C(\Omega)$ is the algebra under pointwise operations of all continuous complex valued functions on some compact Hausdorff space $\Omega$ (such as $[0,1]$), then each function $f$ in $C(\Omega)$ satisfies

$$\mathrm{Sp}(f) = \{f(t) : t \in \Omega\} = (\text{range of } f) \equiv f(\Omega)$$

and

$$\rho(f) = \max\{|f(t)| : t \in \Omega\} \equiv ||f||_\infty,$$

where the last symbol is defined by the previous expression and is called the *supremum* or *uniform norm of* $f$.

An *algebra pseudo-norm*, $\sigma$, is a submultiplicative pseudo-norm, i.e., a function $\sigma : \mathfrak{A} \to \mathbf{R}_+$ satisfying

$$\begin{aligned} \sigma(a+b) &\le \sigma(a) + \sigma(b) && \text{(subadditive)} \\ \sigma(\lambda a) &= |\lambda|\sigma(a) && \text{(absolutely homogeneous)} \\ \sigma(ab) &\le \sigma(a)\sigma(b) && \text{(submultiplicative)} \end{aligned}$$

for all $a, b \in \mathfrak{A}$ and all $\lambda \in \mathbf{C}$. If, in addition, $\sigma(a) = 0$ implies $a = 0$, then we would call $\sigma$ an *algebra norm.* A *spectral pseudo-norm* is an algebra pseudo-norm, $\sigma$, satisfying

$$(1) \qquad \rho(a) \le \sigma(a) \qquad \forall a \in \mathfrak{A}.$$

In Theorem 3.1 and elsewhere we will show that this condition is equivalent to many others of which we mention two here:

$$\rho(a) = \lim_{n\to\infty} \sigma(a^n)^{1/n} \qquad \forall a \in \mathfrak{A}, \tag{2}$$

and

(3) the set of invertible elements in $\mathfrak{A}$ is open with respect to $\sigma$.

Algebra norms with property (3) were first studied by I. Kaplansky [**9**], who called algebras provided with such a norm, $Q$-algebras. (He actually considered rings which were not necessarily algebras.) E.A. Michael [**14**] and B. Yood [**28**] first noted the equivalence of these properties. Many examples of spectral pseudo-norms are known:

(i) Any complete algebra norm [**6**]. (An algebra together with a complete algebra norm is called a *Banach algebra*. The algebra, $C(\Omega)$, of example (2) above together with its uniform norm, $||f||_\infty$, is an example of a Banach algebra.)

(ii) Any algebra norm (not necessarily complete) on a completely regular semisimple commutative Banach algebra [**21**].

(iii) Any algebra norm (not necessarily complete) on a $B^*$-algebra [**21**], cf. [**3**].

(iv) Any algebra norm (not necessarily complete) on any two-sided ideal in $B(\mathfrak{X})$ or on any closed subalgebra of $B(\mathfrak{X})$ which includes all finite rank operators, where $B(\mathfrak{X})$ is the algebra of all bounded linear operators on a Banach space $\mathfrak{X}$ [**28**].

(v) Any algebra norm (not necessarily complete) on a modular annihilator algebra [**1**].

(vi) The operator norm on a full Hilbert algebra [**24**].

(vii) The Gelfand–Naimark pseudo-norm on a hermitian Banach $^*$-algebra ([**16**], cf. [**19** and **18**]).

(viii) Any algebra pseudo-norm on a Jacobson radical algebra.

Kaplansky [**10**] showed that an algebra pseudo-norm was spectral on $\mathfrak{A}$ if and only if for some (and hence for all) ideals $\mathfrak{I}$, both its restriction to $\mathfrak{I}$ and its quotient pseudo-norm on $\mathfrak{A}/\mathfrak{I}$ were spectral (cf. Theorem 3.4(v)).

A *spectral algebra* is an algebra on which some spectral pseudo-norm can be defined. We emphasize that it is not an algebra with a particular choice of spectral pseudo-norm, but merely an algebra on which it is possible to define at least one spectral pseudo-norm. Most spectral algebras have many spectral pseudo-norms, and no one spectral pseudo-norm is preferred above the others (cf. Example 3.6 and Proposition 3.7). The theorems in the body of the article will justify this definition. We will mention only a few of the simplest results in this introduction. Note that any Banach algebra is a spectral algebra.

A commutative algebra is spectral if and only if it satisfies all of the usual results of the Gelfand theory for commutative Banach algebras (Theorem 4.1). More explicitly, if all the elements of an algebra have nonempty bounded spectrum and satisfy

$$\begin{aligned} \mathrm{Sp}(a+b) &\subseteq \mathrm{Sp}(a) + \mathrm{Sp}(b) \\ \mathrm{Sp}(ab) &\subseteq \mathrm{Sp}(a)\mathrm{Sp}(b), \end{aligned}$$

then the algebra is commutative modulo its Jacobson radical (defined below), and the spectral radius itself is a spectral pseudo-norm, so that the algebra is a spectral algebra. Conversely, if an algebra which is commutative modulo its Jacobson radical is also spectral, then these results hold. Furthermore, these conditions on a unital algebra, $\mathfrak{A}$, are equivalent to the existence of a naturally defined, nonempty, compact space $\Gamma_{\mathfrak{A}}$ and a homomorphism $^\wedge$ from $\mathfrak{A}$ into the algebra $C(\Gamma_{\mathfrak{A}})$ of all continuous complex valued functions on $\Gamma_{\mathfrak{A}}$ satisfying

$$\mathrm{Sp}(a) = a^\wedge(\Gamma_{\mathfrak{A}}) \qquad \forall a \in \mathfrak{A}.$$

We will show by example (2.8) that the spectral radius need not be subadditive nor submultiplicative even when it is finite valued on a commutative algebra. However, if either of these conditions do hold on any algebra, then they both do, so the spectral radius is a spectral pseudo-norm and all the foregoing results also follow (Theorems 2.10 and 4.1).

For the purpose of this introduction, we may define the *Jacobson radical*, $\mathfrak{A}_J$ of an algebra, $\mathfrak{A}$, as the largest ideal of $\mathfrak{A}$ on which the spectral radius vanishes. (This agrees with any of the standard definitions, several of which have the virtue of applying to general rings, cf. 6.1, 6.5). Thus, it plays a distinguished role in the present theory. An algebra is said to be a *Jacobson* *radical algebra* if it is equal to its own Jacobson radical. Such an algebra is spectral since the identically zero function is a spectral pseudo-norm. (Caution: this is the only place in this introduction in which we speak of nonunital algebras; no radical algebra is unital.) Since the Jacobson radical is closed in any spectral pseudo-norm, any spectral pseudo-norm on a Jacobson semisimple algebra is actually a norm (Proposition 6.2 and Corollary 6.3).

The quotient of any spectral algebra by any ideal is spectral (Theorem 3.4). Stated another way, any homomorphic image of a spectral algebra is spectral. Thus, the quotient of a Banach algebra by a not necessarily closed ideal is a spectral algebra. In order to go further with categorical properties of spectral algebras, it is necessary to introduce the notion of spectral subalgebras.

A *spectral subalgebra* of an algebra is a subalgebra in which each element has the same nonzero numbers in its spectrum whether calculated relative to the subalgebra or the full algebra (cf. Section 5). A subalgebra is called a *unital subalgebra* if it contains the multiplicative identity element of the larger algebra. Bourbaki [**2**] calls unital subalgebras full (*pleine*) if they are spectral subalgebras, but many interesting cases of spectral subalgebras are not unital subalgebras as the following list of spectral subalgebras shows.

(i) Any one- or two-sided ideal.

(ii) The commutant of any subset. Hence, the center of an algebra and any maximal commutative subalgebra.

(iii) For any idempotent $e \in \mathfrak{A}$, the "corner" subalgebra, $e\mathfrak{A}e$.

(iv) Any modular annihilator or finite dimensional subalgebra of a normed algebra (Corollary 5.7).

(v) Any closed subalgebra of a Banach algebra in which each element of the subalgebra has nowhere dense spectrum relative to the subalgebra (Corollary 5.6).

(vi) Any closed $^*$-subalgebra of a Banach $^*$-algebra (Corollary 5.8).

(vii) The range of the left or right regular or extended regular representation.

(viii) Any intersection of spectral subalgebras or any spectral subalgebra of a spectral subalgebra.

As one would expect, a spectral subalgebra of a spectral algebra is again a spectral algebra in its own right. (However, a spectral subalgebra of a nonspectral algebra need not be a spectral algebra. Despite this difficulty, we believe the name "spectral subalgebra" is appropriate since spectral subalgebras are the appropriate kind of subobject in the category of spectral algebras. (The morphisms in this category are just algebra homomorphisms.)) This gives a long list of spectral algebras. For instance, any (not necessarily closed) ideal in a Banach algebra is spectral. In fact, one characterization of spectral algebras is that they are exactly those algebras, $\mathfrak{A}$, such that $\mathfrak{A}/\mathfrak{A}_J$ can be embedded as a spectral subalgebra of some Banach algebra (Theorem 6.10, cf. 6.11). This characterization remains valid even if the subalgebra is required to be dense and the Banach algebra is required to be semisimple.

In [**20**], C.E. Rickart extended Jacobson's result on strict density to all complex Banach algebras. This result also holds for spectral algebras (Theorem 6.7). Similarly, the theory of topological divisors of zero extends to spectral pseudo-norms, and hence its consequences are available for spectral algebras (Section 5).

Although most of the results proved in this paper are new, most of the proofs are really not. The paper is an investigation of the appropriate choice of definitions to extend many known proofs to abstract algebras. We believe the results already stated, and those contained in the body of the article, show that these definitions provide the natural setting for these results. The theory of Banach algebras and complete norms will, of course, continue to dominate applications since it is so easy to complete an algebra with respect to an algebra norm, but the spectral properties investigated here lie behind the success of that theory. Because the paper is intended to be expository, an attempt is made to give elegant proofs for most results. Specific references are supplied for some extremely well-known statements.

Most of the results given here were found in 1972 but have remained unpublished (cf. remark (2) in Palmer [**16**]). The appearance of Wilansky's question [**27**] convinced the author to extract them from his unpublished book manuscript. The author thanks R.B. Burckel whose careful reading of a preliminary version of the manuscript allowed him to correct several errors. (See **Note added in proof** at the end.)

**2. Preliminary results and examples.** We begin with some simple results and counterexamples on the concept of spectrum in (complex) algebras.

If $\mathfrak{A}$ is not unital, let $\mathfrak{A}^1$ be its *unitization.* As a linear space, $\mathfrak{A}^1$ is $\mathbf{C} \oplus \mathfrak{A}$ and its multiplication is given by

$$(\lambda \oplus a)(\mu \oplus b) = \lambda\mu \oplus (\lambda b + \mu a + ab) \qquad \forall \lambda, \mu \in \mathbf{C}; \forall a, b \in \mathfrak{A}.$$

We regard $\mathbf{C}$ and $\mathfrak{A}$ as subalgebras of $\mathfrak{A}^1$ under the injections $\lambda \mapsto \lambda \oplus 0$ and $a \mapsto 0 \oplus a$, respectively. Henceforth, we replace the direct sum signs by ordinary plus signs. If $\mathfrak{A}$ is already unital, it is convenient to let $\mathfrak{A}^1$ denote $\mathfrak{A}$ itself and regard $\mathbf{C}$ as a subalgebra under the injection $\lambda \mapsto \lambda 1$. Then we may define the *spectrum* of an element $a$ in an arbitrary (not necessarily unital) algebra $\mathfrak{A}$ by

$$\mathrm{Sp}_{\mathfrak{A}}(a) = \{\lambda \in \mathbf{C} : \lambda - a \text{ has no inverse in } \mathfrak{A}^1\}$$

and define the *spectral radius* by $\rho_{\mathfrak{A}}(a) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(a)\}$ (with the usual convention if $\mathrm{Sp}_{\mathfrak{A}}(a)$ is empty or unbounded). When the algebra is clearly understood, we will drop the subscript on both of these notations.

Zero plays a special role in the spectrum and must be treated separately in the statement of many theorems. The problem is that this common notion of spectrum does not behave well with respect to homomorphisms or subalgebras unless they are unital. (A homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ is *unital* if $\mathfrak{A}$ and $\mathfrak{B}$ are unital and $\varphi(1)$ is the identity element in $\mathfrak{B}$; a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is *unital* if $\mathfrak{B}$ contains the identity element of $\mathfrak{A}$ so that the embedding homomorphism is unital.) For this reason, and in order to define the spectrum without mentioning $\mathfrak{A}^1$, it is better to consider quasi-invertibility rather than (ordinary) invertibility. We define the *quasi-product* of two elements by

$$a \circ b = a + b - ab$$

(so that $1 - (a \circ b) = (1 - a)(1 - b)$ holds in $\mathfrak{A}^1$), and define $a \in \mathfrak{A}$ to be *quasi-invertible* if there is an element $a^q$ (called its *quasi-inverse*) satisfying $a \circ a^q = a^q \circ a = 0$. Clearly, the quasi-inverse is unique if it exists and the set $\mathfrak{A}_{qG}$ of *quasi-invertible elements* is a group under the quasi-product with zero as group identity. Now we can give an equivalent definition of the spectrum:

> a nonzero $\lambda \in \mathbf{C}$ belongs to the spectrum of $a \in \mathfrak{A}$ if and only if $\lambda^{-1}$ is not quasi-invertible in $\mathfrak{A}$, and zero belongs to the spectrum if and only if $a$ is not invertible in $\mathfrak{A}$.

This definition is often easier to apply than the more intuitive one given previously despite its awkward appearance.

Since any homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ satisfies $\varphi(\mathfrak{A}_{qG}) \subseteq \mathfrak{B}_{qG}$ we see that $\mathrm{Sp}_{\mathfrak{B}}(\varphi(a)) \subseteq \mathrm{Sp}_{\mathfrak{A}}(a) \cup \{0\}$ holds for all $a \in \mathfrak{A}$. Similarly, if $\mathfrak{B}$ is any subalgebra of $\mathfrak{A}$, the inclusion $\mathfrak{B}_{qG} \subseteq \mathfrak{A}_{qG} \cap \mathfrak{B}$ implies $\mathrm{Sp}_{\mathfrak{A}}(b) \subseteq \mathrm{Sp}_{\mathfrak{B}}(b) \cup \{0\}$ for all $b \in \mathfrak{B}$. Simple examples show that the "$\cup\{0\}$" cannot be omitted from the right side of these inclusions.

We begin our serious discussion of the spectrum with the following general form of the spectral mapping theorem. This theorem includes the following kinds of results: $\mathrm{Sp}(a^2) = \mathrm{Sp}(a)^2$; $\mathrm{Sp}(a^{-1}) = \mathrm{Sp}(a)^{-1}$ for invertible $a \in \mathfrak{A}$; and $\mathrm{Sp}(a^q) = \{\lambda/(\lambda - 1) : \lambda \in \mathrm{Sp}(a)\}$ for quasi-invertible $a \in \mathfrak{A}$. Since the rational functions described in this theorem act in any algebra, this appears to be the natural form of the universal spectral mapping theorem, and it is surprising that it is not the form usually stated.

**Proposition 2.1.** *If $a$ is any element in any algebra $\mathfrak{A}$ and $f$ is any rational function with no poles in* $\mathrm{Sp}(a)$, *then $f(a)$ is defined in $\mathfrak{A}^1$. Unless* $\mathrm{Sp}(a)$ *is empty and $f$ is constant, $f(a)$ satisfies*

$$\mathrm{Sp}(f(a)) = \{f(\lambda) : \lambda \in \mathrm{Sp}(a)\} \equiv f(\mathrm{Sp}(a)).$$

*Furthermore, any $a, b \in \mathfrak{A}$ satisfy*

$$\mathrm{Sp}(ab) \cup \{0\} = \mathrm{Sp}(ba) \cup \{0\}.$$

*Proof.* If $\mathrm{Sp}(a)$ is nonempty and $f$ is a constant, $\lambda_0$, then they satisfy $f(a) = \lambda_0$ and $\mathrm{Sp}(f(a)) = \{\lambda_0\} = f(\mathrm{Sp}(a))$.

Henceforth, we assume that $f$ is not a constant and work in $\mathfrak{A}^1$. Let $p$ and $q$ be relatively prime polynomials satisfying $f = p/q$. In order to determine the invertibility of $\lambda_0 - f(a)$ in $\mathfrak{A}^1$ we factor $q$ and $\lambda_0 q - p$:

$$\begin{aligned} q(\lambda) &= \alpha(\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) \\ \lambda_0 q(\lambda) - p(\lambda) &= \beta(\mu_1 - \lambda)(\mu_2 - \lambda)\cdots(\mu_m - \lambda) \end{aligned}$$

where $\alpha$ and $\beta$ are not zero (the latter since $f = p/q$ is not a constant). No $\mu_j$ appears among the $\lambda_k$s since $p$ and $q$ were relatively prime. So

$$\lambda_0 - f(a) = \beta(\mu_1 - a)(\mu_2 - a)\cdots(\mu_m - a)/\alpha(\lambda_1 - a)(\lambda_2 - a)\cdots(\lambda_n - a)$$

is invertible unless some $\mu_j$ is in the spectrum of $a$, but the $\mu_j$s are just the roots of $\lambda_0 q(\mu) - p(\mu) = 0$. Another way to say the same thing is that $\lambda_0$ is in the spectrum of $f(a)$ if and only if an element of the spectrum $a$ is a root of $\lambda_0 q(\mu) - p(\mu)$. But this is what we wished to show.

The final remark follows from noting that $\lambda^{-1}b(c-1)a$ is a quasi-inverse for $\lambda^{-1}ba$ if $c$ is a quasi-inverse for $\lambda^{-1}ab$. $\square$

**Example 2.2.** Algebras may contain many elements with empty spectrum. Consider the function field $\mathbf{C}(z)$ in one variable consisting of all rational functions in $z$. Any nonconstant function has empty spectrum.

This situation cannot occur if any nontrivial algebra pseudo-norm can be defined on $\mathfrak{A}$. In this article an algebra pseudo-norm is called *nontrivial* unless it is identically zero *and* the algebra is unital. (Thus the identically zero function is a nontrivial algebra pseudo-norm on a nonunital algebra. The point is that we only need to insist that the extension of the algebra pseudo-norm $\sigma$ to $\sigma^1$ on $\mathfrak{A}^1$ is not identically zero, where $\sigma^1$ is defined by $\sigma^1(\lambda + a) = |\lambda| + \sigma(a)$ if $\mathfrak{A}$ is not unital.)

**Theorem 2.3.** *Let $\sigma$ be a nontrivial algebra pseudo-norm on an algebra, $\mathfrak{A}$. Then each element $a \in \mathfrak{A}$ satisfies*

$$\lim_{n\to\infty} \sigma(a^n)^{1/n} \le \rho(a).$$

*In particular, if such a pseudo-norm can be defined on an algebra, then no element of the algebra has empty spectrum.*

An instructive proof of this result can be derived from simple analytic function theory applied to the resolvent function $\lambda \mapsto (\lambda - a)^{-1}$. Instead, we refer the reader to C.E. Rickart's elementary proof [**21,22**] which can be readily adapted to the present case.

**Corollary 2.4** (Gelfand-Mazur Theorem). *Any division algebra on which a nontrivial algebra pseudo-norm can be defined is isomorphic to the complex field. The isomorphism is implemented by* $a \to \lambda$, *where* $\lambda$ *is the unique element in* $\mathrm{Sp}(a)$.

*Proof.* By the theorem there is some $\lambda \in \mathbf{C}$ in $\mathrm{Sp}(a)$. Since $\lambda 1 - a$ is not invertible in the division algebra $\mathfrak{A}^1 = \mathfrak{A}$, it is zero. $\square$

**Example 2.5.** Many normed algebras contain elements with unbounded spectrum. For instance, consider the unital, commutative, normed algebra $\mathbf{C}[z]$ of all polynomials with the norm $||p|| = \sup\{|p(\lambda)| : \lambda \in \mathbf{C}, |\lambda| \leq 1\}$. In this algebra, every element has closed spectrum since

$$\mathrm{Sp}(p) = \begin{cases} \mathbf{C} & \text{if } p \text{ is not constant,} \\ \lambda & \text{if } p \equiv \lambda. \end{cases}$$

**Example 2.6.** There exist unital, normed (even commutative) algebras in which some elements have unbounded and nonclosed spectrum. Consider $\mathfrak{R} = \{\text{rational functions with poles off } \{\lambda \in \mathbf{C} : |\lambda| < 1\}\}$. We may give this the norm $||f|| = \sup\{|f(\lambda)| : |\lambda| \leq 1/2\}$. The spectrum of each rational function $f \in \mathfrak{R}$ is just $\{f(\lambda) : |\lambda| < 1\}$.

If each element of an algebra has bounded spectrum, then each element has closed spectrum. Although this simple result must be widely known, we include a proof since we know of no appropriate reference.

**Proposition 2.7.** *If every element in an algebra has bounded spectrum, then every element has closed* (*hence, compact*) *spectrum.*

*Proof.* Suppose the algebra $\mathfrak{A}$ satisfies this hypothesis but $a \in \mathfrak{A}$ has nonclosed spectrum. Choose $\lambda$ in the closure of $\mathrm{Sp}(a)$ but not in $\mathrm{Sp}(a)$. Then $\mathrm{Sp}((\lambda - a)^{-1}) = \{(\lambda - \mu)^{-1} : \mu \in \mathrm{Sp}(a)\}$ is unbounded. $\square$

If every element in an algebra has nonempty, bounded spectrum, then the spectral radius is a nonnegative real valued function on the algebra. In a commutative Banach algebra the spectral radius is also subadditive and submultiplicative and hence is an algebra norm.

**Example 2.8.** There are unital commutative algebras in which the spectrum of every element is nonempty and bounded (hence compact) so that the spectral radius is a nonnegative real valued function, but on which the spectral radius is neither subadditive nor submultiplicative. In $\mathbf{C} \oplus \mathbf{C}(z)$ the spectrum of an arbitrary element is given by

$$\mathrm{Sp}(\lambda, f) = \begin{cases} \{\lambda\} & \text{if } f \text{ is not constant,} \\ \{\lambda, \mu\} & \text{if } f \equiv \mu, \end{cases}$$

so we have

$$\begin{aligned} \rho(0,1) &= 1 > 0 + 0 = \rho(0,z) + \rho(0, 1-z) \\ \rho(0,1) &= 1 > 0 \cdot 0 = \rho(0,z)\rho(0,z^{-1}). \end{aligned}$$

I am indebted to H.G. Dales for supplying the reference to Ptak and Zemanek [**18**] on which the next theorem is based. We will also need the following result of Claude Le Page [**12**]. Because of its elegance, we will give the proof of this result based on Marvin Rosenbloom [**25**].

**Lemma 2.9.** *A unital normed algebra, $\mathfrak{A}$, is commutative if and only if there is a constant, $C$, satisfying*

$$||ab|| \leq C||ba|| \qquad \forall a, b \in \mathfrak{A}.$$

*Proof.* If $\mathfrak{A}$ is commutative, we may choose $C = 1$. To prove the converse, first note that we may replace $\mathfrak{A}$ by its completion and thus assume that $\mathfrak{A}$ is complete. Define $e^{\lambda a}$ by $e^{\lambda a} = \sum_{n=0}^{\infty} (\lambda a)^n / n!$ so that

$e^{\lambda a}$ has inverse $e^{-\lambda a}$. Let $a, b \in \mathfrak{A}$ be arbitrary but fixed for the rest of this argument. For any fixed $\omega \in \mathfrak{A}^*$ (where $\mathfrak{A}^*$ is the Banach space dual of $\mathfrak{A}$) define $f : \mathbf{C} \to \mathbf{C}$ by

$$f(\lambda) = \omega(e^{\lambda a} b e^{-\lambda a}).$$

For suitable $c_n \in \mathfrak{A}$, $f(\lambda) = \sum_{n=0}^{\infty} \omega(c_n)\lambda^n$ is an entire function satisfying

$$\begin{aligned}|f(\lambda)| &\le ||\omega||\,||e^{\lambda a}(be^{-\lambda a})|| \\ &\le C||\omega||\,||be^{-\lambda a}e^{\lambda a}|| \\ &= C||\omega||\,||b||.\end{aligned}$$

Hence, by Liouville's theorem, $f$ is a constant function. Thus, the derivative, $f'$, of $f$ is identically zero. However, $f'(\lambda) = \omega(ae^{\lambda a}be^{-\lambda a} + e^{\lambda a}b(-a)e^{-\lambda a})$ implies $f'(0) = \omega(ab - ba)$. Since $\omega \in \mathfrak{A}^*$ was arbitrary, we conclude $ab = ba$ by the Hahn–Banach theorem. □

**Theorem 2.10.** *The following are equivalent for any algebra, $\mathfrak{A}$, in which the spectral radius is finite valued*:

(i) *There is a constant $C$ satisfying*

$$\rho(a + b) \le C(\rho(a) + \rho(b)) \qquad \forall a, b \in \mathfrak{A};$$

(ii) *There is a constant $C$ satisfying*

$$\rho(ab) \le C\rho(a)\rho(b) \qquad \forall a, b \in \mathfrak{A};$$

(iii) *$\mathfrak{A}$ is commutative modulo its Jacobson radical and the spectral radius is an algebra pseudo-norm.*

*Proof.* (i) $\Rightarrow$ (ii). Notice that the constants, $C$, necessarily satisfy $c \ge 1$. For $a, b \in \mathfrak{A}$, let $\lambda_0 \in \mathbf{C}$ satisfy $|\lambda_0| > 9C^2\rho(a)\rho(b)$. Then we may write $\lambda_0 = \mu\nu$ with $|\mu| > 3C\rho(a)$ and $|\nu| > 3C\rho(b)$. Hence, we get

$$\begin{aligned}\rho((\mu - a)^{-1}a) &= \sup\{|\lambda/(\mu - \lambda)| : \lambda \in \mathrm{Sp}(a)\} \\ &\le \sup\{|\lambda/(\mu - \lambda)| : |\lambda| \le \rho(a)\} < (2C)^{-1}\end{aligned}$$

and similarly $\rho((\nu - b)^{-1}b) < (2C)^{-1}$. Thus,

$$\lambda_0 - ab = (\mu - a)(1 + (\mu - a)^{-1}a + (\nu - b)^{-1}b)(\nu - b)$$

is invertible since

$$\rho((\mu - a)^{-1}a + (\nu - b)^{-1}b) \leq C(\rho((\mu - a)^{-1}a) + \rho((\nu - b)^{-1}b)) < 1$$

implies that the middle factor is invertible. We conclude that $\rho(ab) \leq 9C^2\rho(a)\rho(b)$.

(ii) $\Rightarrow$ (i). For $a, b \in \mathfrak{A}$, let $\lambda_0 \in \mathbf{C}$ satisfy $|\lambda_0| > \rho(a) + C\rho(b)$. We must consider the unital and nonunital cases separately. If $\mathfrak{A}$ is unital,

$$\lambda_0 - (a + b) = (\lambda_0 - a)(1 - (\lambda_0 - a)^{-1}b)$$

is invertible since

$$\rho((\lambda_0 - a)^{-1}b) \leq C\rho((\lambda_0 - a)^{-1})\rho(b) < C(C\rho(b))^{-1}\rho(b) = 1.$$

If $\mathfrak{A}$ is nonunital, $(\lambda_0 - a)^{-1}$ has the decomposition $\lambda_0^{-1} + c$ in $\mathfrak{A}^1$. Hence, in $\mathfrak{A}^1$,

$$\begin{aligned}\lambda_0 - (a + b) &= (\lambda_0 - a)(1 - (\lambda_0 - a)^{-1}b) \\ &= (\lambda_0 - a)(1 - \lambda_0^{-1}b - cb) \\ &= (\lambda_0 - a)(1 - cb(1 - \lambda_0^{-1}b)^{-1})(1 - \lambda_0^{-1}b)\end{aligned}$$

is invertible since

$$\rho(cb(1 - \lambda_0^{-1}b)^{-1}) \leq C\rho(c)\rho(b(1 - \lambda_0^{-1}b)^{-1}) < 1$$

follows from

$$\rho(c) = \rho((\lambda_0 - a)^{-1} - \lambda_0^{-1}) = \rho(a\lambda_0^{-1}(\lambda_0 - a)^{-1}) < |\lambda_0^{-1}|\rho(a)(C\rho(b))^{-1}$$

and

$$\rho(b(1 - \lambda_0^{-1}b)^{-1}) = |\lambda_0|\rho(b(\lambda_0 - b)^{-1}) < |\lambda_0|\rho(b)\rho(a)^{-1}.$$

We conclude

$$\rho(a + b) \leq \rho(a) + C\rho(b) \leq C(\rho(a) + \rho(b)).$$

(ii) $\Rightarrow$ (iii). By the last argument, we may assume $\rho(a + b) \leq \rho(a) + C\rho(b)$ for all $a, b \in \mathfrak{A}$. Define

$$p(a) = C^2 \sup\{\rho(a + b) - \rho(b) : b \in \mathfrak{A}\} \qquad \forall a \in \mathfrak{A}.$$

Then $p$ satisfies

$$\begin{aligned} p(a+c) &= C^2 \sup\{(\rho(a+c+b)-\rho(c+b))+(\rho(c+b)-\rho(b)) : b \in \mathfrak{A}\} \\ &\le p(a)+p(c) \qquad \forall a, c \in \mathfrak{A}, \\ C^2\rho(a) &\le p(a) \le C^3\rho(a) \qquad \forall a \in \mathfrak{A}, \\ p(ab) &\le C^3\rho(ab) \le C^4\rho(a)\rho(b) \le p(a)p(b) \qquad \forall a, b \in \mathfrak{A}, \end{aligned}$$

and thus is a spectral pseudo-norm (cf. Theorem 3.1 and Definition 3.2 below) which also satisfies

$$p(a)^2 \le C^6\rho(a)^2 = C^6\rho(a^2) \le C^4 p(a^2) \qquad \forall a \in \mathfrak{A}.$$

Hence, the ideal on which $p$ vanishes is the Jacobson Radical. Thus, $p$ induces a norm, $||\cdot||$, on $\mathfrak{A}/\mathfrak{A}_J$ defined by $||a+\mathfrak{A}_J|| = p(a)$ for all $a \in \mathfrak{A}$ which also satisfies the above inequality. However, this implies

$$\begin{aligned} ||(a+\mathfrak{A}_J)(b+\mathfrak{A}_J)|| &= ||ab+\mathfrak{A}_J|| = p(ab) \le C^3\rho(ab) = C^3\rho(ba) \\ &\le Cp(ba) = C||ba+\mathfrak{A}_J|| = C||(b+\mathfrak{A}_J)(a+\mathfrak{A}_J)|| \end{aligned}$$

so that $\mathfrak{A}/\mathfrak{A}_J$ is commutative by Lemma 2.9. However, since $\mathfrak{A}/\mathfrak{A}_J$ is a commutative spectral algebra, $\rho$ is subadditive so that $\rho = p$ by the definition of $p$. (The reader may check the proof of (i) $\Rightarrow$ (vi) in Theorem 4.1 to see that this argument is not circular.)

(iii) $\Rightarrow$ (i) and (ii). Since $\rho$ is an algebra pseudo-norm, we may choose $C = 1$. □

**3. Spectral pseudo-norms.** In this section we collect the most important results on spectral pseudo-norms. The reader should also consult Theorem 5.3, Propositions 5.10 and 6.2, and [**13**] for additional characterizations which depend on ideas not yet introduced. We begin by showing the equivalence of a large number of conditions for an algebra pseudo-norm.

**Theorem 3.1.** *The following are equivalent for an algebra pseudo-norm $\sigma$ on an algebra $\mathfrak{A}$:*

(i) *The set of quasi-regular elements of $\mathfrak{A}$ is open with respect to $\sigma$.*

(ii) *The set of quasi-regular elements of $\mathfrak{A}$ has nonempty interior with respect to $\sigma$.*

(iii) *For* $a \in \mathfrak{A}$, $\sigma(a) < 1$ *implies that* $a$ *is quasi-invertible in* $\mathfrak{A}$.

(iv) *Each* $a \in \mathfrak{A}$ *satisfies* $\rho(a) \leq \sigma(a)$.

(v) *Each* $a \in \mathfrak{A}$ *satisfies*

$$\rho(a) = \lim_{n\to\infty} \sigma(a^n)^{1/n}.$$

(vi) *There is a constant* $C \in \mathbf{R}_+$ *such that each* $a \in \mathfrak{A}$ *satisfies* $\rho(a) \leq C\sigma(a)$.

*Proof.* (i) $\Rightarrow$ (vi). Since zero is quasi-regular, there is some $\varepsilon > 0$ such that $\sigma(a) < \varepsilon$ implies $a \in \mathfrak{A}_{qG}$. Hence, $\sigma(a) < \varepsilon$ implies $\lambda^{-1}a \in \mathfrak{A}_{qG}$ for all $\lambda \in \mathbf{C}$ with $|\lambda| \geq 1$, which in turn implies $\rho(a) < 1$. Choose $C$ to be $\varepsilon^{-1}$.

(vi) $\Rightarrow$ (v). By Proposition 2.1 the spectrum of $a^n$ is just $\{\lambda^n : \lambda \in \mathrm{Sp}(a)\}$. Hence, $\rho(a^n) = \rho(a)^n$ holds for all $a \in \mathfrak{A}$ and $n \in \mathbf{N}$. Thus (vi) implies $\rho(a) = \rho(a^n)^{1/n} \leq C^{1/n}\sigma(a^n)^{1/n}$. Combining this with Theorem 2.3 gives the desired result.

(v) $\Rightarrow$ (iv). Immediate since $\lim \sigma(a^n)^{1/n} \leq \sigma(a)$.

(iv) $\Rightarrow$ (iii). Immediate since $\rho(a) < 1$ implies $1 \notin \mathrm{Sp}(a)$.

(iii) $\Rightarrow$ (ii). Zero is in the interior of $\mathfrak{A}_{qG}$.

(ii) $\Rightarrow$ (i). Let $c \in \mathfrak{A}_{qG}$ be arbitrary and let $d \in \mathfrak{A}_{qG}$ be an interior point of $\mathfrak{A}_{qG}$. It is enough to show that $c$ is an interior point. Define a map $L : \mathfrak{A} \to \mathfrak{A}$ by $L(a) = d \circ c^q \circ a$. Clearly, $L$ is continuous and maps both $c$ onto $d$ and $\mathfrak{A}_{qG}$ onto $\mathfrak{A}_{qG}$. Hence, $\mathfrak{A}_{qG} = L^{-1}(\mathfrak{A}_{qG})$ is a neighborhood of $c$. $\square$

**Definition 3.2.** An algebra pseudo-norm satisfying one (and hence all) of the conditions in the theorem above is called a *spectral pseudo-norm*.

A spectral pseudo-norm which is actually a norm will, of course, be called a *spectral norm*. In the introduction we provided a list of spectral pseudo-norms and norms. Spectral pseudo-norms are always nontrivial since 1 is never quasi-invertible.

When dealing with algebra norms rather than pseudo-norms, Fuster and Marquina [**5**] noted the following equivalence. The result could be stated informally as follows: An algebra norm is spectral if and only if all the geometric series which should converge have limits.

**Proposition 3.3.** *The following conditions are equivalent for an algebra norm, $\sigma$, on an algebra $\mathfrak{A}$:*

(i) *$\sigma$ is spectral.*

(ii) *The geometric series $\sum_{n=1}^{\infty} a^n$ has a sum in $\mathfrak{A}$ if it satisfies $\sum_{n=1}^{\infty} \sigma(a^n) < \infty$.*

(iii) *The geometric series $\sum_{n=1}^{\infty} a^n$ has a sum in $\mathfrak{A}$ if $\sigma(a) < 1$.*

*Proof.* (i) $\Rightarrow$ (ii). For a spectral norm $\sigma, \lim \sigma(a^n) = 0$ implies that $a \circ (-\sum_{n=1}^{N} a^n) = (-\sum_{n=1}^{N} a^n) \circ a = a^{N+1}$ is quasi-invertible in $\mathfrak{A}$ for large enough $N$. Hence $a$ is quasi-invertible in $\mathfrak{A}$. We will show that $-a^q$ is the sum of the series $\sum a^n$. This follows from the estimate:

$$\begin{aligned}\sigma\left(a^q + \sum_{n=1}^{N} a^n\right) &= \sigma\left((1-a^q)(1-a)\left(a^q + \sum_{n=1}^{N} a^n\right)\right) \\ &\leq \sigma^1(1-a^q)\sigma(a^q - aa^q + a - a^{N+1}) \\ &= (1+\sigma(a^q))\sigma(a^{N+1}).\end{aligned}$$

(ii) $\Rightarrow$ (iii). Immediate.

(iii) $\Rightarrow$ (i). If the series converges, then

$$a \circ \left(-\sum_{n=1}^{\infty} a^n\right) = a - \sum_{n=1}^{\infty} a^n + \sum_{n=1}^{\infty} a^{n+1} = 0,$$

so $a$ is quasi-invertible, verifying (iii) of the previous theorem. □

Notice that the proof of the implication (i) $\Rightarrow$ (ii) remains correct when $\sigma$ is merely a pseudo-norm. However, the proof of the opposite implication establishes only $\sigma(a \circ (-\sum_{n=1}^{\infty} a^n)) = 0$. Since the set of elements on which any spectral pseudo-norm vanishes is included in the Jacobson radical (Corollary 5.3 below), this implies that $a \circ (-\sum_{n=1}^{\infty} a^n)$ is quasi-invertible when $\sigma$ is a spectral pseudo-norm. Hence, the second condition of Proposition 2.13 could be included in

Theorem 2.11 by restating it as follows: "For $a \in \mathfrak{A}$, $\sigma(a) = 0$ implies $a \in \mathfrak{A}_{qG}$ (or alternatively, $a \in \mathfrak{A}_J$) and $\sigma(a) < 1$ implies that there is a sum in $\mathfrak{A}$ for $\sum_{n=1}^{\infty} a^n$," but this was omitted because of its inelegance.

Next we mention the stability properties of spectral pseudo-norms and some related results. Properties (iv) and (v) are due to Kaplansky [**10**]. Spectral subalgebras are defined in the Introduction and Definition 5.1. The quotient pseudo-norm induced by $\sigma$ on $\mathfrak{A}/\mathfrak{I}$ is defined by

$$\sigma(a + \mathfrak{I}) = \inf\{\sigma(a + b) : b \in \mathfrak{I}\} \qquad \forall a \in \mathfrak{A}$$

**Theorem 3.4.** *The following are equivalent for an algebra pseudo-norm $\sigma$ on an algebra $\mathfrak{A}$.*

(i) *$\sigma$ is spectral.*

(ii) *The restriction of $\sigma$ to each spectral subalgebra is spectral.*

(iii) *The restriction of $\sigma$ to each maximal commutative subalgebra is spectral.*

(iv) *The quotient pseudo-norm induced by $\sigma$ on $\mathfrak{A}/\mathfrak{I}$ is spectral for each ideal $\mathfrak{I}$ of $\mathfrak{A}$.*

(v) *There is some ideal $\mathfrak{I}$ for which both $\sigma$ and its quotient pseudo-norm are spectral on $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$, respectively.* (*When this condition holds for some ideal, of course, it holds for all ideals.*)

(vi) *The quotient norm induced by $\sigma$ on $\mathfrak{A}/\mathfrak{A}_J$ is spectral where $\mathfrak{A}_J$ is the Jacobson radical* (*defined below*).

*Proof.* (i) $\Rightarrow$ (ii). Since the nonzero spectrum of an element in a spectral subalgebra is the same whether calculated in the subalgebra or the full algebra, this is immediate from Theorem 3.1 (iv).

(ii) $\Rightarrow$ (iii). Maximal commutative subalgebras are spectral subalgebras.

(iii) $\Rightarrow$ (i). Each element in the algebra belongs to some maximal commutative subalgebra, so this also follows from Theorem 3.1(iv).

(i) $\Rightarrow$ (iv). The inequality $\rho_{\mathfrak{A}/\mathfrak{I}}(a+\mathfrak{I}) \leq \rho_{\mathfrak{A}}(a)$ for all $a \in \mathfrak{A}$ and Theorem 3.1 (iv) give this.

(iv) $\Rightarrow$ (vi) $\Rightarrow$ (v). Immediate since $\mathfrak{A}_J$ is always spectral.

(v) $\Rightarrow$ (i). It is enough to show $\sigma(a) < 1/3$ implies that $a$ is quasi-invertible in $\mathfrak{A}$. This implies $\sigma(a+\mathfrak{I}) < 1/3$ and, hence, that the quasi-inverse, $b+\mathfrak{I}$, of $a+\mathfrak{I}$ satisfies $\sigma(b+\mathfrak{I}) < (1/3)/(1-1/3) = 1/2$. Hence, $b$ can be chosen in $\mathfrak{A}$ to satisfy $\sigma(b) < 1/2$ so that $\sigma(a \circ b) \leq \sigma(a) + \sigma(b) + \sigma(ab) < 1$; then $a \circ b$ is quasi-invertible in $\mathfrak{I}$ (where $\sigma$ is spectral) and hence $a \circ b$ and $a$ are quasi-invertible in $\mathfrak{A}$. $\square$

**Example 3.5.** Spectral pseudo-norms are not unique in any sense even in unital, semisimple commutative Banach algebras. Consider $\ell^1(\mathbf{Z})$ with convolution multiplication. Both $\rho$ and $||\cdot||_1$ are spectral norms. Nevertheless, Proposition 1.3 of Palacios [**15**] can be restated as follows.

**Proposition 3.6.** *If a sequence converges in two different spectral norms on an algebra, the two limits differ by an element of the Jacobson radical. Hence, in a semisimple algebra the two limits are equal.*

Of course, the above result has Johnson's uniqueness of norm theorem for semisimple Banach algebras as an immediate corollary.

Finally, we formally define spectral algebras.

**Definition 3.7.** An algebra is called *spectral* if some spectral pseudo-norm can be defined on it.

We wish to emphasize again that a spectral algebra is not an algebra together with some particular spectral pseudo-norm but is rather an abstract algebra on which some spectral pseudo-norm can be defined.

As Rickart points out in his book [**23**, 1.4.1], the limit, $\lim \sigma(a^n)^{1/n}$, has many desirable properties. Theorem 3.1(v) immediately shows that in any spectral algebra the spectral radius has all these properties. We have already shown by example that subadditivity and submultiplicativity on commuting elements is not a property of the spectral radius (even when it is finite valued) in arbitrary commutative algebras.

We can express this definition in a more geometric way.

**Proposition 3.8.** *An algebra is spectral if and only if there is some balanced, convex, absorbing semigroup included in the set of quasi-regular elements.*

*Proof.* The open unit ball of a spectral pseudo-norm would be such a set, and conversely given such a set, its Minkowski functional would be a spectral pseudo-norm. □

We will need a simple technical result.

**Proposition 3.9.** *An algebra is spectral if and only if its unitization is spectral.*

*Proof.* If $\mathfrak{A}$ is nonunital and $\sigma$ is a spectral pseudo-norm on $\mathfrak{A}$, then $\sigma^1 : \mathfrak{A}^1 \to \mathbf{R}_+$ defined by $\sigma^1(\lambda + a) = |\lambda| + \sigma(a)$ is also spectral since $\rho(\lambda + a) \leq |\lambda| + \rho(a)$ is obvious. If $\sigma$ is a spectral pseudo-norm on $\mathfrak{A}^1$, then its restriction to $\mathfrak{A}$ is spectral. □

**Example 3.10.** Here is a typical example of a nonspectral algebra norm. Let $\mathfrak{A}$ be the disc algebra: the algebra under pointwise operations of functions continuous on the closed unit disc in $\mathbf{C}$ and holomorphic on its interior, $\mathbf{D}$. Let $S$ be any subset of $\mathbf{D}$ with a limit point in $\mathbf{D}$. Then $||\cdot||_S$ defined by $||f||_S = \sup\{|f(\lambda)| : \lambda \in S\}$ is a norm on $\mathfrak{A}$. It is not spectral unless the closure of $S$ includes the boundary of the disc.

**Question 3.11.** Recall that an algebra pseudo-norm is spectral if its restriction to each maximal commutative subalgebra is spectral. Is an algebra spectral if each of its maximal commutative subalgebras is?

**Question 3.12.** There is some evidence (and several plausible but fallacious "proofs" based on the isomorphism between $(\mathfrak{I}_1 + \mathfrak{I}_2)/\mathfrak{I}_1$ and $\mathfrak{I}_2/(\mathfrak{I}_1 \cap \mathfrak{I}_2)$ and Theorem 3.4(v)) that in any algebra, or at least in any commutative algebra, the sum of two ideals is a spectral algebra if each of the ideals are spectral algebras in their own right. A similar variant relates to a pseudo-normed algebra $(\mathfrak{A}, \sigma)$ and asks whether $\sigma$ is spectral on the sum of two ideals when it is spectral on each ideal separately. This last result would imply that each pseudo-normed algebra has a largest ideal on which the algebra pseudo norm is spectral. The commutative result implies this largest spectral ideal exists in any commutative algebra. Are any of these results true?

**4. Commutative spectral algebras.** In this short section we show that a commutative algebra is spectral if and only if it has a complete Gelfand theory. We discuss the well-known Gelfand construction only enough to fix our notation. In fact, we require only commutativity modulo the Jacobson radical. Finally, we note that every commutative algebra has a largest ideal which satisfies the Gelfand theory.

Let $\mathfrak{A}$ be an arbitrary algebra and let $\Gamma_{\mathfrak{A}}$ be the set of all nonzero (hence surjective) algebra homomorphisms from $\mathfrak{A}$ into $\mathbf{C}$. (Until we provide criteria for this set to be nonempty we tacitly assume that it is nonempty for each algebra $\mathfrak{A}$ under discussion.) For each $a \in \mathfrak{A}$ we define $\hat{a} : \Gamma_{\mathfrak{A}} \to \mathbf{C}$ by $\hat{a}(\gamma) = \gamma(a)$ for all $\gamma \in \Gamma_{\mathfrak{A}}$. It is easy to see that the map $a \mapsto \hat{a}$ is an algebra homomorphism of $\mathfrak{A}$ into the algebra of all complex-valued functions on $\Gamma_{\mathfrak{A}}$. We will call the kernel of this homomorphism the *Gelfand radical of* $\mathfrak{A}$. In a commutative algebra it equals the Jacobson radical of $\mathfrak{A}$.

Consider the set $\Gamma_{\mathfrak{A}}$ of maps from $\mathfrak{A}$ into $\mathbf{C}$ as a subset of the Cartesian product $\prod_{a\in\mathfrak{A}} \mathbf{C}$. When $\Gamma_{\mathfrak{A}}$ is provided with the relativized product topology, we call it the Gelfand space of $\mathfrak{A}$. This topology is precisely the weakest topology such that $\hat{a} : \Gamma_{\mathfrak{A}} \to \mathbf{C}$ is continuous for each $a \in \mathfrak{A}$. The continuous function $\hat{a} : \Gamma_{\mathfrak{A}} \to \mathbf{C}$ is called the *Gelfand transform of* $a$, and the homomorphism $a \mapsto \hat{a}$ is called the *Gelfand homomorphism.*

It is easy to see that the range of the Gelfand transform, $\hat{a}$, of $a \in \mathfrak{A}$ is included in the spectrum of $a$. (If $b$ is an inverse for $\gamma(a) - a$ in $\mathfrak{A}^1$, then $1 = \gamma(1) = \gamma(b(\gamma(a) - a)) = \gamma(b)(\gamma(a) - \gamma(a)) = 0$, where $\gamma$ is extended (if necessary) to $\mathfrak{A}^1$ by $\gamma(\lambda+a) = \lambda+\gamma(a)$.) If each element of $\mathfrak{A}$ has bounded spectrum, then $\Gamma_{\mathfrak{A}}$ is included in the compact product $\prod_{a\in\mathfrak{A}} \mathbf{C}_{\rho(a)}$, where $\mathbf{C}_{\rho(a)} = \{\lambda \in \mathbf{C} : |\lambda| \le \rho(a)\}$. If $\mathfrak{A}$ is unital, $\Gamma_{\mathfrak{A}}$ is closed in this product and, hence, compact. If $\mathfrak{A}$ is nonunital,

$\Gamma_{\mathfrak{A}} \cup \{0\}$ (where 0 represents the identically zero homomorphism) is closed. Hence, $\Gamma_{\mathfrak{A}} \cup \{0\}$ is the one-point compactification of $\Gamma_{\mathfrak{A}}$, which must, therefore, be locally compact even when $\mathfrak{A}$ is nonunital. Furthermore, $\hat{a}$ vanishes at infinity on $\Gamma_{\mathfrak{A}}$.

So far, we have provided no conditions to force the Gelfand space to be nonempty. Clearly, the Gelfand transform of any commutator is zero, so that we cannot expect the Gelfand space to be large enough to be useful unless $\mathfrak{A}$ is almost commutative. In this case that turns out to mean that $\mathfrak{A}$ is commutative modulo the Jacobson radical. Also, $\Gamma_{\mathfrak{A}}$ would be empty if any element of $\mathfrak{A}$ had empty spectrum. In fact, we only get a useful theory when there are enough elements in the Gelfand space so that the obvious inclusions

$$\hat{a}(\Gamma_{\mathfrak{A}}) \subseteq \mathrm{Sp}_{C(\Gamma_{\mathfrak{A}})}(\hat{a}) \subseteq \mathrm{Sp}_{\mathfrak{A}}(a) \cup \{0\}$$

can be reversed. (As usual, zero plays a special role in the spectrum here, so that it must be omitted from the reversed inclusions.) It is only at this stage of the construction that we need to introduce the spectral condition as a substitute for completeness.

**Theorem 4.1.** *The following are equivalent for an algebra, $\mathfrak{A}$.*

(i) *$\mathfrak{A}$ is a spectral algebra which is commutative modulo its Jacobson radical.*

(ii) *The spectral radius on $\mathfrak{A}$ is an algebra pseudo-norm.*

(iii) *The spectral radius is finite valued and there is a constant $C$ satisfying $\rho(a+b) \leq C(\rho(a)+\rho(b))$, $a, b \in \mathfrak{A}$.*

(iv) *The spectral radius is finite valued and there is a constant $C$ satisfying $\rho(ab) \leq C\rho(a)\rho(b)$, $a, b \in \mathfrak{A}$.*

(v) *Every element of $\mathfrak{A}$ has nonempty bounded spectrum and any two elements satisfy:*

$$Sp(a+b) \subseteq Sp(a) + Sp(b)$$
$$Sp(ab) \subseteq Sp(a)Sp(b)$$

(vi) *Either $\mathfrak{A}$ is radical or the Gelfand space $\Gamma_{\mathfrak{A}}$, of $\mathfrak{A}$, is nonempty and locally compact in the Gelfand topology, and each element $a \in \mathfrak{A}$*

*satisfies*

$$\hat{a} \in C_0(\Gamma_{\mathfrak{A}})$$
$$\hat{a}(\Gamma_{\mathfrak{A}}) \subseteq \mathrm{Sp}(a) \subseteq \hat{a}(\Gamma_{\mathfrak{A}}) \cup \{0\}.$$

*If $\mathfrak{A}$ is unital, $\Gamma_{\mathfrak{A}}$ is compact and each $a \in \mathfrak{A}$ satisfies*

$$\hat{a}(\Gamma_{\mathfrak{A}}) = \mathrm{Sp}(a).$$

*Proof.* Theorem 2.10 shows that conditions (iii) and (iv) are equivalent to each other and to (i) and (ii) combined. Hence, (ii) is also equivalent to (iii) and (iv) and they imply (i). Since the implications (vi) $\Rightarrow$ (v) $\Rightarrow$ (ii) are clear, it only remains to prove (i) $\Rightarrow$ (vi). (In the nonunital case it appears at first that (vi) would imply the two inclusions of (v) only with "$\cup\{0\}$" added to the right side, but this addition is not needed since the spectrum of any element in a nonunital algebra contains zero.)

(i) $\Rightarrow$ (vi). If $\mathfrak{A}$ is not a radical algebra, (i.e., $\mathfrak{A}$ is not equal to its Jacobson radical), there is some element $b \in \mathfrak{A}$ which has a nonzero number $\lambda$ in its spectrum. We will show that, for any such nonzero $\lambda$ in the spectrum of any element $b \in \mathfrak{A}$, there is some $\gamma$ in $\Gamma_{\mathfrak{A}}$ satisfying $\lambda = \gamma(b)$. Since the discussion given before the statement of this theorem established the topological properties of $\Gamma_{\mathfrak{A}}$ and of $\hat{a}$ and the inclusion $\hat{a}(\Gamma_{\mathfrak{A}}) \subseteq \mathrm{Sp}(a)$, this will conclude the proof. (If $\mathfrak{A}$ is unital and $a$ is not invertible, so that its spectrum contains zero, to establish the last sentence we must also show that some $\gamma \in \Gamma_{\mathfrak{A}}$ satisfies $\gamma(a) = 0$, but the argument is entirely analogous to the one we are about to give.)

The set $\mathfrak{A}(\lambda - b)$ is a proper modular left ideal with right relative identity $\lambda^{-1}b$. By Zorn's lemma it is included in a maximal modular left ideal, $\mathfrak{M}$, which does not contain $\lambda^{-1}b$. However, since any maximal modular left ideal such as $\mathfrak{M}$ includes the Jacobson radical $\mathfrak{A}_J$ of $\mathfrak{A}$, and since $\mathfrak{A}/\mathfrak{A}_J$ is commutative, $\mathfrak{M}$ is in fact a two-sided ideal. Choose a spectral pseudo-norm $\sigma$ on $\mathfrak{A}$. If any $c \in \mathfrak{M}$ satisfies $\sigma(\lambda^{-1}b - c) < 1$, then $\lambda^{-1}b - c$ has a quasi-inverse, $a$. However, this implies $\lambda^{-1}b = -(a - a\lambda^{-1}b) + c - ac \in \mathfrak{M}$, contrary to the construction of $\mathfrak{M}$. Thus, the $\sigma$-closure of $\mathfrak{M}$ is a proper ideal and, hence, equals $\mathfrak{M}$ by maximality. Therefore, $\mathfrak{M}$ is closed with respect to $\sigma$ which thus induces a norm on the division algebra $\mathfrak{A}/\mathfrak{M}$. By the Gelfand-Mazur theorem, there is an isomorphism of $\mathfrak{A}/\mathfrak{M}$ onto $\mathbf{C}$. Define $\gamma$ to be the composition of this isomorphism with the natural map of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{M}$. The equality $\gamma(b) = \lambda$ is now obvious. $\square$

**5. Spectral subalgebras and topological quasi-divisors of zero.** In the introduction we listed a number of sufficient conditions for a subalgebra to be a spectral subalgebra. Of these, conditions (i), (ii), (iii), (vii) and (viii) have easy algebraic proofs which we omit. The other three all depend on the theory of topological quasi-divisors of zero, introduced by Kaplansky [**11**] following Silov [**26**]. Topological quasi-divisors of zero cannot be defined in a spectral algebra, since the algebra has no particular spectral pseudo-norm. However, the most useful consequences of this concept are independent of the particular choice of norm, so that they are available in the theory of spectral algebras. We will sketch these results very briefly, since they do not differ appreciably from the Banach algebra case. We begin with a formal definition of the concept informally defined in the introduction.

**Definition 5.1.** A subalgebra $\mathfrak{B}$ of an algebra $\mathfrak{A}$ is said to be a spectral subalgebra if it satisfies

$$\mathfrak{B}_{qG} = \mathfrak{B} \cap \mathfrak{A}_{qG}.$$

It is immediate that this is equivalent to the condition

$$\mathrm{Sp}_{\mathfrak{B}}(b) \cup \{0\} = \mathrm{Sp}_{\mathfrak{A}}(b) \cup \{0\} \qquad \forall b \in \mathfrak{B}.$$

Since $\mathrm{Sp}_{\mathfrak{A}}(b) \subseteq \mathrm{Sp}_{\mathfrak{B}}(b) \cup \{0\}$ holds for any element in an arbitrary subalgebra, we may write

$$\mathrm{Sp}_{\mathfrak{B}}(b) \subseteq \mathrm{Sp}_{\mathfrak{A}}(b) \cup \{0\} \qquad \forall b \in \mathfrak{B},$$

for the equivalent condition. (At the end of this section we will discuss when the stronger inclusion

$$\mathrm{Sp}_{\mathfrak{B}}(b) \subseteq \mathrm{Sp}_{\mathfrak{A}}(b) \qquad \forall b \in \mathfrak{B},$$

holds.) A unital subalgebra is spectral if and only if it is full (pleine) in the sense of Bourbaki [**2**].

**Definition 5.2.** An element, $a$, in an algebra, $\mathfrak{A}$, with an algebra pseudo-norm, $\sigma$, is called a left topological quasi-divisor of zero if there exists a sequence $\{b_n\}_{n\in\mathbf{N}} \subseteq \mathfrak{A}$ satisfying both

$$\sigma(b_n) = 1, \qquad \forall n \in \mathbf{N}, \quad \text{and} \quad \sigma(b_n - ab_n) \to 0.$$

Right topological quasi-divisors of zero are defined similarly, and an element which is both a right and a left topological quasi-divisor of zero is called a two-sided topological quasi-divisor of zero.

The utility of this concept arises from the obvious fact that even a one-sided topological quasi-divisor of zero cannot have a quasi-inverse in any larger pseudo-normed algebra (i.e., $1 = \sigma(b_n) = \sigma(a^q \circ a \circ b_n) \leq \sigma(b_n - ab_n) + \sigma(a^q)\sigma(b_n - ab_n) \to 0$) and the following theorem. The sufficiency of this condition was first noted by V. Mascioni in [**13**].

**Theorem 5.3.** *Let $\mathfrak{A}$ be an algebra, and let $\sigma$ be an algebra pseudo-norm on $\mathfrak{A}$. Then $\sigma$ is a spectral pseudo-norm if and only if every element in the boundary of $\mathfrak{A}_{qG}$ is a two-sided topological quasi-divisor of zero.*

*Proof.* To establish the necessity of the condition adapt [**23**, 1.5.9]. For the sufficiency, simply note that if $\sigma$ is not spectral, then some elements of the boundary of $\mathfrak{A}_{qG}$ belong to $\mathfrak{A}_{qG}$, contradicting the remark preceding this theorem. □

**Corollary 5.4.** *Let $\sigma$ be an algebra pseudo-norm on an algebra $\mathfrak{A}$. If $\mathfrak{B}$ is a subalgebra on which $\sigma$ is a spectral pseudo-norm, then every element of $\mathfrak{B}$ satisfies*

$$\partial \mathrm{Sp}_{\mathfrak{B}}(b) \subseteq \mathrm{Sp}_{\mathfrak{A}}(b) \cup \{0\} \subseteq \mathrm{Sp}_{\mathfrak{B}}(b) \cup \{0\}.$$

*Proof.* The second inclusion holds for any subalgebra and the first follows from the theorem and preceding remarks. □

**Corollary 5.5.** *Let $\mathfrak{A}$ be a spectral algebra. Then the following are equivalent for any subalgebra $\mathfrak{B}$:*

(i) $\rho_{\mathfrak{A}}(b) = \rho_{\mathfrak{B}}(b)$, $\forall b \in \mathfrak{B}$.

(ii) $\mathfrak{B}$ *is a spectral algebra satisfying*

$$\partial \mathrm{Sp}_{\mathfrak{B}}(b) \subseteq \partial \mathrm{Sp}_{\mathfrak{A}}(b) \cup \{0\} \subseteq \mathrm{Sp}_{\mathfrak{A}}(b) \cup \{0\} \subseteq \mathrm{Sp}_{\mathfrak{B}}(b) \cup \{0\} \qquad \forall b \in \mathfrak{B}.$$

(iii) *The restriction to* $\mathfrak{B}$ *of some spectral pseudo-norm on* $\mathfrak{A}$ *is a spectral pseudo-norm on* $\mathfrak{B}$.

(iv) *The restriction to* $\mathfrak{B}$ *of every spectral pseudo-norm on* $\mathfrak{A}$ *is a spectral pseudo-norm on* $\mathfrak{B}$.

*Proof.* (i) $\Rightarrow$ (iv). Use Theorem 3.1(v).

(iv) $\Rightarrow$ (iii). Obvious, since $\mathfrak{A}$ is spectral.

(iii) $\Rightarrow$ (ii). This follows from the previous corollary and the fact that the spectrum is closed in a spectral algebra.

(ii) $\Rightarrow$ (i). Obvious. $\square$

**Corollary 5.6.** *A closed subalgebra of a Banach algebra is a spectral subalgebra if every element in the subalgebra has a nowhere dense spectrum in the subalgebra.*

*Proof.* A closed subalgebra of a Banach algebra obviously satisfies Corollary 5.5(iii). If $\mathrm{Sp}_{\mathfrak{B}}(b)$ is nowhere dense, then it is equal to its own boundary, so Corollary 5.5(ii) shows that $\mathfrak{B}$ is a spectral subalgebra when every element of $\mathfrak{B}$ has this property. $\square$

**Corollary 5.7.** *Any finite dimensional or modular annihilator subalgebra of a normed algebra is a spectral subalgebra.*

*Proof.* The restriction of the norm on $\mathfrak{A}$ to the subalgebra $\mathfrak{B}$ is a spectral norm by completeness and [**1**], respectively, in these two cases. In both cases, every element of the subalgebra has nowhere dense spectrum. Apply Corollary 5.4. $\square$

**Corollary 5.8.** *Any closed* $^*$*-subalgebra of a Hermitian Banach* $^*$*-algebra is a spectral subalgebra.*

*Proof.* If $b$ has no quasi-inverse in $\mathfrak{B}$, then either $b \circ b^*$ or $b^* \circ b$ has none either. Since these elements are hermitian, their spectra are purely real. Hence, Corollary 5.5 gives

$$\mathrm{Sp}_{\mathfrak{B}}(b \circ b^*) = \partial\mathrm{Sp}_{\mathfrak{B}}(b \circ b^*) \subseteq \partial\mathrm{Sp}_{\mathfrak{A}}(b \circ b^*) \cup \{0\} = \mathrm{Sp}_{\mathfrak{A}}(b \circ b^*) \cup \{0\}$$

which shows that $b \circ b^*$ (or $b^* \circ b$, as the case may be) has no quasi-inverse in $\mathfrak{A}$. The same follows for $b$. $\square$

The next result is a corollary of Theorem 3.4 which provides innumerable examples of spectral algebras which are not Banach algebras.

**Corollary 5.9.** *Any spectral subalgebra and any quotient of a spectral algebra is a spectral algebra.*

The following criterion will be used in the next section.

**Proposition 5.10.** *Let $||\cdot||$ be an algebra norm on an algebra $\mathfrak{A}$, and let $\mathfrak{A}^-$ be the completion of $\mathfrak{A}$ with respect to $||\cdot||$. Then the following are equivalent*:

(i) *$||\cdot||$ is a spectral norm on $\mathfrak{A}$,*

(ii) $\rho_{\bar{\mathfrak{A}}}(a) = \rho_{\mathfrak{A}}(a)$, $\forall a \in \mathfrak{A}$;

(iii) *$\mathfrak{A}$ is a spectral subalgebra of $\bar{\mathfrak{A}}$.*

*Proof.* (i) $\Rightarrow$ (iii). Suppose $a \in \mathfrak{A}$ has a quasi-inverse $b \in \bar{\mathfrak{A}}$. Choose a sequence $\{b_n\}_{n\in\mathbf{N}}$ in $\mathfrak{A}$ converging to $b$. This sequence satisfies $a \circ b_n \to a \circ b = 0$ and $b_n \circ a \to b \circ a = 0$. Hence, $a \circ b_n$ and $b_n \circ a$ are eventually quasi-invertible in $\mathfrak{A}$. This implies that $a$ is quasi-invertible in $\mathfrak{A}$.

(iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i). Immediate. $\square$

In the list of spectral pseudo-norms given in the introduction, (ii), (iii), (iv), (v) and (viii) assert that all algebra norms on certain algebras are spectral. In [**28**] Yood called algebras with this property permanent $Q$-algebras and noted some of the following equivalent conditions.

**Proposition 5.11.** *The following are equivalent for an algebra $\mathfrak{A}$:*

(i) *Every algebra norm on $\mathfrak{A}$ is spectral.*

(ii) *If $\mathfrak{B}$ is a Banach algebra and $\varphi : \mathfrak{A} \to \mathfrak{B}$ is an injective* *homomorphism, then*

$$\rho_{\mathfrak{B}}(\varphi(a)) = \rho_{\mathfrak{A}}(a) \qquad \forall a \in \mathfrak{A}.$$

(iii) *If $\mathfrak{B}$ is a Banach algebra and $\varphi : \mathfrak{A} \to \mathfrak{B}$ is an injective homomorphism with dense range, then $\varphi(\mathfrak{A})$ is a spectral subalgebra of $\mathfrak{B}$.*

*Proof.* (i) $\Rightarrow$ (iii). Proposition 5.10.

(iii) $\Rightarrow$ (ii). Any $a \in \mathfrak{A}$ satisfies $\rho_{\mathfrak{A}}(a) = \rho_{\overline{\varphi(\mathfrak{A})}}(\varphi(a)) = \lim ||\varphi(a)^n||^{1/n} = \rho_{\mathfrak{B}}(\varphi(a))$, where $||\cdot||$ is the complete norm of $\mathfrak{B}$ and $\overline{\varphi(\mathfrak{A})}$ is the closure of $\varphi(\mathfrak{A})$ in $\mathfrak{B}$.

(ii) $\Rightarrow$ (i). If $||\cdot||$ is an algebra norm on $\mathfrak{A}$, let $\mathfrak{B}$ be the completion of $\mathfrak{A}$, and let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be the natural embedding. Each $a \in \mathfrak{A}$ satisfies $\rho_{\mathfrak{A}}(a) = \rho_{\mathfrak{B}}(a) \le ||a||$, so $||\cdot||$ is spectral. □

**Question 5.12.** Do simple unital Banach algebras satisfy these conditions? More generally, do (not necessarily commutative) completely regular Banach algebras satisfy them?

We have noted that a subalgebra $\mathfrak{B}$ of an algebra $\mathfrak{A}$ is a spectral subalgebra if and only if it satisfies

$$\mathrm{Sp}_{\mathfrak{B}}(b) \subseteq \mathrm{Sp}_{\mathfrak{A}}(b) \cup \{0\} \qquad \forall b \in \mathfrak{B}.$$

In most examples the stronger inclusion

$$\mathrm{Sp}_{\mathfrak{B}}(b) \subseteq \mathrm{Sp}_{\mathfrak{A}}(b), \qquad \forall b \in \mathfrak{B},$$

actually holds. This simply requires that, whenever an element of $\mathfrak{B}$ is invertible in $\mathfrak{A}$, then it is invertible in $\mathfrak{B}$. In fact, we have not succeeded in finding any example where this does not hold. Perhaps this is because the stronger inclusion holds in at least the following cases: $\mathfrak{B}$ is a unital subalgebra of $\mathfrak{A}$, $\mathfrak{A}$ is nonunital, $\mathfrak{B}$ is a one- or two-sided ideal; $\mathfrak{B}$ is the commutant of any subset; $\mathfrak{B}$ is a corner subalgebra $e\mathfrak{A}e$ for some idempotent $e \in \mathfrak{A}$; $\mathfrak{B}$ is a finite dimensional subalgebra of $\mathfrak{A}$. (In this case suppose $b \in \mathfrak{B}$ is invertible in $\mathfrak{A}$. By finite dimensionality, $\sum_{k=1}^{n} \lambda_k b^k = 0$ holds for some positive integer $n$ and $\lambda_k \in \mathbf{C}$. Multiplying by suitable powers of $b^{-1}$ we conclude $1 = bp(b)$ and $b^{-1} = p(b)$ both belong to $\mathfrak{B}$, where $p$ is a polynomial.) Here is a general, but not very useful result.

**Proposition 5.13.** *The following are equivalent for any subalgebra, $\mathfrak{B}$, of any algebra, $\mathfrak{A}$:*

(i) $\mathrm{Sp}_{\mathfrak{B}}(b) \subseteq \mathrm{Sp}_{\mathfrak{A}}(b)$, $\forall b \in \mathfrak{B}$;

(ii) $\mathfrak{B}$ *is a spectral subalgebra of* $\mathfrak{A}$ *satisfying either*:

(a) $\mathfrak{B}$ *is a unital subalgebra of* $\mathfrak{A}$, *or*

(b) *no element of* $\mathfrak{B}$ *has an inverse in* $\mathfrak{A}$.

*Proof.* (i) $\Rightarrow$ (ii). Clearly, $\mathfrak{B}$ is a spectral subalgebra of $\mathfrak{A}$ since it satisfies $\mathrm{Sp}_{\mathfrak{B}}(b) \sim \{0\} \subseteq \mathrm{Sp}_{\mathfrak{A}}(b) \sim \{0\}$ for all $b \in \mathfrak{B}$. Suppose $b \in \mathfrak{B}$ has an inverse, $b^{-1}$, in $\mathfrak{A}$. Then $0 \notin \mathrm{Sp}_{\mathfrak{A}}(b)$ implies $0 \notin \mathrm{Sp}_{\mathfrak{B}}(b)$ by (i) so that $\mathfrak{B}$ is a unital algebra. Then we have

$$1_{\mathfrak{B}} = 1_{\mathfrak{B}} 1_{\mathfrak{A}} = 1_{\mathfrak{B}} b b^{-1} = b b^{-1} = 1_{\mathfrak{A}},$$

so $\mathfrak{B}$ is a unital subalgebra of $\mathfrak{A}$.

(ii) $\Rightarrow$ (i). If $\mathfrak{B}$ is a unital subalgebra as well as a spectral subalgebra of $\mathfrak{A}$, then (i) is obvious. Suppose $\mathfrak{B}$ is a spectral subalgebra satisfying (b). Since it is a spectral subalgebra we need only check that $0 \in \mathrm{Sp}_{\mathfrak{B}}(b)$ implies $0 \in \mathrm{Sp}_{\mathfrak{A}}(b)$ for all $b \in \mathfrak{B}$. But (b) implies $0 \in \mathrm{Sp}_{\mathfrak{A}}(b)$ for all $b \in \mathfrak{B}$. $\square$

**Question 5.14.** Find an example of a spectral subalgebra which does not satisfy the last proposition.

**6. The Jacobson radical, strict density and some characterizations of spectral algebras.** The Jacobson radical plays the same role in the theory of spectral algebras that it does in Banach algebra theory. In the introduction we defined the Jacobson radical as the largest ideal on which the spectral radius vanished. That definition was convenient for the early sections of this paper, but here we will derive it from more common definitions and derive the properties of the Jacobson radical in a spectral algebra. Jacobson's result [**8**] on strict density, which was extended to Banach algebras by Rickart [**20**], also extends to spectral algebras. At the end of this section, the proof of Theorem 6.10 uses virtually all of the results already proved to give several characterizations of spectral algebras.

**Definition 6.1.** An algebra is said to be *primitive* if it has a faithful (algebraically) irreducible representation. An ideal is said to be *primitive* if it is the kernel of some irreducible representation. The *Jacobson radical*, $\mathfrak{A}_J$, of an algebra $\mathfrak{A}$ is the intersection of all the primitive ideals of $\mathfrak{A}$. (Hence, if there are no primitive ideals, $\mathfrak{A}_J$ equals $\mathfrak{A}$.) The algebra $\mathfrak{A}$ is called *Jacobson radical* or *semisimple* if it satisfies $\mathfrak{A} = \mathfrak{A}_J$ or $\mathfrak{A}_J = \{0\}$, respectively.

We need to review some standard material to establish notation. If $T : \mathfrak{A} \to \mathfrak{L}(\mathfrak{X})$ is an irreducible representation (where $\mathfrak{X}$ is a linear space and $\mathfrak{L}(\mathfrak{X})$ is the algebra of all linear maps of $\mathfrak{X}$ into $\mathfrak{X}$), and $z \in \mathfrak{X}$ is an arbitrary nonzero element, then there is some $e \in \mathfrak{A}$ satisfying $T_e z = z$. Hence, the ideal

$$\mathfrak{L} = \{a \in \mathfrak{A} : T_a z = 0\}$$

is a maximal (by the irreducibility of $T$), modular left ideal with right relative unit $e$. Define the quotient of $\mathfrak{L}$ by

$$\mathfrak{L} : \mathfrak{A} = \{a \in \mathfrak{A} : a\mathfrak{A} \subseteq \mathfrak{L}\}.$$

It is easy to see that $\mathrm{Ker}(T) = \mathfrak{L} : \mathfrak{A}$. Conversely, if $\mathfrak{L}$ is any maximal modular left ideal, then the left regular representation of $\mathfrak{A}$ on $\mathfrak{A}/\mathfrak{L}$ is irreducible with kernel $\mathfrak{L} : \mathfrak{A}$. Hence, the primitive ideals are characterized as the quotients of maximal modular left ideals.

**Proposition 6.2.** *An algebra pseudo-norm is spectral if and only if every maximal modular one-sided ideal is closed. Hence, primitive ideals and the Jacobson radical are closed in any spectral pseudo-norm.*

*Proof.* Let $\sigma$ be a spectral pseudo-norm on $\mathfrak{A}$ and let $\mathfrak{L}$ be a maximal modular left ideal with right relative unit $e$. The closure $\bar{\mathfrak{L}}$ of $\mathfrak{L}$ relative to $\sigma$ is certainly a modular left ideal with $e$ as a right relative unit. Since

$\sigma(e-b) < 1$ implies that $e-b$ is quasi-invertible, which in turn implies the contradiction $e = b + (e-b)^q(e-1) - (e-b)^q b \in \mathfrak{L}$, $e$ does not belong to $\bar{\mathfrak{L}}$ which is therefore proper. By maximality, $\mathfrak{L}$ equals $\bar{\mathfrak{L}}$ and is therefore closed. The argument for right ideals is entirely similar.

Conversely, suppose $\sigma$ is an algebra pseudo-norm on $\mathfrak{A}$ which is not spectral. Theorem 3.1(iii) implies that there is some nonquasi-invertible $a \in \mathfrak{A}$ with $\sigma(a) < 1$. By symmetry, we may assume $a$ is not left invertible so that $\mathfrak{A}(1-a)$ is a proper ideal. By Zorn's lemma, there is a maximal modular left ideal $\mathfrak{L}$ which includes $\mathfrak{A}(1-a)$. However, $a - a^n = (\sum_{k=1}^n a^k)(1-a)$ shows that $a$ belongs to the $\sigma$-closure of $\mathfrak{L}$. Since this implies $\bar{\mathfrak{L}} = \mathfrak{A}$, $\mathfrak{L}$ is not closed.

The definitions of $\mathfrak{L} : \mathfrak{A}$ and $\mathfrak{A}_J$ show that they are closed when $\mathfrak{L}$ is. (Alternatively, $\mathfrak{L} : \mathfrak{A} = \cap \mathfrak{L}'$, where the intersection extends over all maximal modular left ideals $\mathfrak{L}'$ satisfying $\mathfrak{L} : \mathfrak{A} = \mathfrak{L}' : \mathfrak{A}$, and hence $\mathfrak{A}_J = \cap\{\mathfrak{L} : \mathfrak{L}$ is a maximal modular left ideal of $\mathfrak{A}\}$, so $\mathfrak{L} : \mathfrak{A}$ and $\mathfrak{A}_J$ are closed.) □

**Corollary 6.3.** *If $\sigma$ is a spectral pseudo-norm on an algebra $\mathfrak{A}$, then the set of elements on which $\sigma$ vanishes is included in the Jacobson radical of $\mathfrak{A}$. Hence, any spectral pseudo-norm on a semisimple algebra is a norm.*

**Theorem 6.4.** *An element in an algebra is quasi-invertible if and only if it is quasi-invertible modulo each primitive ideal of $\mathfrak{A}$.*

*Proof.* See Rickart [**23**, 2.2.9(v)]. □

**Corollary 6.5.** *The Jacobson radical is the largest ideal $\mathfrak{I}$ in an algebra $\mathfrak{A}$ satisfying any (hence all) of the following equivalent conditions*:

(i) $\mathrm{Sp}_{\mathfrak{A}/\mathfrak{I}}(a+\mathfrak{I}) \subseteq \mathrm{Sp}_{\mathfrak{A}}(a) \subseteq \mathrm{Sp}_{\mathfrak{A}/\mathfrak{I}}(a+\mathfrak{I}) \cup \{0\}$, $\forall a \in \mathfrak{A}$;

(ii) $\mathrm{Sp}_{\mathfrak{A}}(b) = \{0\}$, $\forall b \in \mathfrak{I}$;

(iii) $\rho_{\mathfrak{A}}(b) = 0$, $\forall b \in \mathfrak{I}$.

We generalize a result of Kaplansky [**11**] for Banach algebras.

**Proposition 6.6.** *A spectral pseudo-normed algebra $\mathfrak{A}$ which contains no topological quasi-divisors of zero* (*except possibly* 1) *is either a Jacobson radical algebra or isomorphic to* $\mathbf{C}$. *The former case occurs if and only if $\mathfrak{A}$ is nonunital and the latter if and only if some element of $\mathfrak{A}$ has nonzero spectrum.*

*Proof.* Suppose some element $a \in \mathfrak{A}$ has spectrum not equal to $\{0\}$. Then $\partial\mathrm{Sp}(a)$ must contain a nonzero complex number since the spectrum is bounded and nonempty (by Theorem 2.3). Theorem 5.3 shows that $\lambda^{-1}a$ is a topological quasi-divisor of zero and hence equals 1. Now let $b \in \mathfrak{A}$ be arbitrary and choose $t > \sigma(b)$, so that $\sigma(t - b)$ contains a nonzero element, $\mu$, in its boundary. The argument just given for $\lambda^{-1}a$ implies $\mu^{-1}(t - b) = 1$. Hence, the map $b = (t - \mu)1 \to t - \mu$ is an isomorphism of $\mathfrak{A}$ onto $\mathbf{C}$.

If every element of $\mathfrak{A}$ has spectrum $\{0\}$, then $\mathfrak{A}$ is a Jacobson radical algebra and thus has no identity element. $\square$

**Theorem 6.7.** *Let $\mathfrak{A}$ be a spectral algebra. Then any irreducible representation of $\mathfrak{A}$ is strictly dense. The representation space may be given a norm relative to which each representing operator is bounded. If a spectral pseudo-norm is chosen on $\mathfrak{A}$, the norm on $\mathfrak{X}$ may be chosen so that the representation is a contraction.*

*Proof.* Choose a spectral pseudo-norm $\sigma$ on $\mathfrak{A}$. Let $T : \mathfrak{A} \to \mathfrak{L}(\mathfrak{X})$ be an irreducible representation of $\mathfrak{A}$. Let $z \in \mathfrak{X}$ be any nonzero element. Choose $e \in \mathfrak{A}$ satisfying $T_e z = z$ and define $\mathfrak{L}$ by $\mathfrak{L} = \{a \in \mathfrak{A} : T_a z = 0\}$. Then $\mathfrak{L}$ is a maximal modular left ideal of $\mathfrak{A}$ with $e$ as right relative unit and hence is closed in $\sigma$. Thus, we may define a norm on $\mathfrak{X}$ by

$$||x|| = \inf\{\sigma(a) : T_a z = x\}.$$

This is just the quotient norm on $\mathfrak{A}/\mathfrak{L}$ transferred to $\mathfrak{X}$. It is easy to see that the operator norm on $T_a$ for any $a \in \mathfrak{A}$ satisfies $||T_a|| \le \sigma(a)$. By Schur's lemma [**23**, 2.4.4], the commutant $T'_{\mathfrak{A}}$ is a division algebra. Suppose $S$ belongs to this commutant and $x = T_a z$ is an arbitrary element of $\mathfrak{X}$. Then we find $||Sx|| = ||T_a S z|| \le ||T_a||\,||Sz|| \le \sigma(a)||Sz||$. By taking the infimum over all $a \in \mathfrak{A}$ satisfying $x = T_a z$ we get $||Sx|| \le ||Sz||\,||x||$. Hence, $T'_{\mathfrak{A}}$ is included in the set of bounded linear operators from $\mathfrak{X}$ into $\mathfrak{X}$, so $T'_{\mathfrak{A}}$ is normed. Thus $T'_{\mathfrak{A}}$ is isomorphic to $\mathbf{C}$ by the Gelfand–Mazur theorem. Now Jacobson's original argument [**8**] shows that $T_{\mathfrak{A}}$ is strictly dense. $\square$

**Corollary 6.8.** *A primitive algebra is spectral if and only if it can be faithfully represented as a strictly dense, spectral normed algebra of bounded linear operators on a normed linear space.*

**Question 6.9.** Is the operator norm, which was defined in the theorem, a spectral norm? Is the algebra of representing operators a spectral subalgebra of $\mathfrak{L}(\mathfrak{X})$? The answer is often affirmative. When does $T_{\mathfrak{A}}$ act topologically irreducibly on the completion $\bar{\mathfrak{X}}$ of $\mathfrak{X}$? (This happens unless $\bar{\mathfrak{X}}$ has a $T$ invariant subspace which has intersection $\{0\}$ with $\mathfrak{X}$.)

We will now give a theorem which characterizes spectral algebras in a number of ways. First, we define ample, normed, subdirect products of normed algebras. Let $\mathfrak{A}$ and $\mathfrak{A}^\alpha$ be algebras for each $\alpha$ in some index set $A$. Then the Cartesian product $\prod_{\alpha\in A}\mathfrak{A}^\alpha$ is an algebra under pointwise operations. Any homomorphism $\varphi : \mathfrak{A} \to \prod_{\alpha\in A}\mathfrak{A}^\alpha$ induces homomorphisms $\varphi^\alpha : \mathfrak{A} \to \mathfrak{A}^\alpha$ for all $a \in A$ by $\varphi^\alpha(a) = \varphi(a)_\alpha$ (the component of $\varphi(a)$ in $\mathfrak{A}^\alpha$). A homomorphism $\varphi : \mathfrak{A} \to \prod_{\alpha\in A}\mathfrak{A}^\alpha$ is called a *subdirect product decomposition* if (1) the kernel of $\varphi$ is zero, and (2) $\varphi^\alpha : A \to A^\alpha$ is surjective for each $\alpha \in A$. This subdirect product decomposition is said to be *ample* if $a \in \mathfrak{A}$ is quasi-invertible if and only if $\varphi^\alpha(a)$ is quasi-invertible in $\mathfrak{A}^\alpha$ for each $\alpha \in A$. If each $(\mathfrak{A}^\alpha, ||\cdot||_\alpha)$ is a normed algebra, then the subdirect product decomposition is said to be *normed* if

$$||a||_\infty = \sup\{||\varphi_\alpha(a)||_\alpha : \alpha \in A\}$$

is finite for each $a \in \mathfrak{A}$. In this case $||\cdot||_\infty$ is an algebra norm on $\mathfrak{A}$.

**Theorem 6.10.** *The following are equivalent for an algebra,* $\mathfrak{A}$:

(i) *It is spectral*;

(ii) *There is some Banach algebra,* $\mathfrak{B}$, *and some homomorphism* $\varphi : \mathfrak{A} \to \mathfrak{B}$ *with the Jacobson radical as kernel satisfying*

$$\rho_{\mathfrak{B}}(\varphi(a)) = \rho_{\mathfrak{A}}(a) \qquad \forall a \in \mathfrak{A};$$

(iii) *The algebra modulo its Jacobson radical can be embedded as a dense spectral subalgebra of a semisimple Banach algebra;*

(iv) *The algebra modulo its Jacobson radical is isomorphic to an ample, normed, subdirect product of spectral normed algebras. The factors in this subdirect product may be chosen as strictly dense, spectral normed algebras of bounded operators on normed linear spaces.*

*Proof.* (i) $\Rightarrow$ (iii). Let $\sigma'$ be an arbitrary spectral pseudo-norm on $\mathfrak{A}$. Corollary 6.3 shows that $\sigma'$ induces a spectral norm $|||\cdot|||$ on $\mathfrak{A}/\mathfrak{A}_J$. Let $(\bar{\mathfrak{A}}, |||\cdot|||)$ be the completion of $\mathfrak{A}/\mathfrak{A}_J$ in this norm. The Jacobson radical, $\bar{\mathfrak{A}}_J$, of $\bar{\mathfrak{A}}$ has intersection $\{0\}$ with $\mathfrak{A}/\mathfrak{A}_J \subseteq \bar{\mathfrak{A}}$ since the intersection would be an ideal in which each element has spectral radius zero. (Note that the quotient spectral norm $||\cdot||$ on $\bar{\mathfrak{A}}/\bar{\mathfrak{A}}_J$, when restricted to $\mathfrak{A}/\mathfrak{A}_J$, induces a pseudo-norm $\sigma$ on $\mathfrak{A}$ which is spectral, since we have

$$\begin{aligned}\rho_{\mathfrak{A}}(a) &= \rho_{\mathfrak{A}/\mathfrak{A}_J}(a+\mathfrak{A}_J) = \rho_{\bar{\mathfrak{A}}}(a+\mathfrak{A}_J)\\ &= \rho_{\bar{\mathfrak{A}}/\bar{\mathfrak{A}}_J}((a+\mathfrak{A}_J)+\bar{\mathfrak{A}}_J) \le ||(a+\mathfrak{A}_J)+\bar{\mathfrak{A}}_J|| = \sigma(a),\end{aligned}$$

where we have used Corollary 6.5(i) and Proposition 5.10.) Therefore, $\bar{\mathfrak{A}}/\bar{\mathfrak{A}}_J$ is the completion of $(\mathfrak{A}/\mathfrak{A}_J, ||\cdot||)$ and $\mathfrak{A}/\mathfrak{A}_J$ is a dense subalgebra. Proposition 5.11 shows that $\mathfrak{A}/\mathfrak{A}_J$ is a spectral subalgebra of $\bar{\mathfrak{A}}/\bar{\mathfrak{A}}_J$.

(iii) $\Rightarrow$ (iv). Let $\prod$ be the set of primitive ideals of $\mathfrak{A}$. Then there is a natural injective homomorphism of $\mathfrak{A}/\mathfrak{A}_J$ into $\prod_{\mathfrak{P}\in\Pi} \mathfrak{A}/\mathfrak{P}$ built from the natural maps $\varphi_{\mathfrak{P}} : \mathfrak{A}/\mathfrak{A}_J \to \mathfrak{A}/\mathfrak{P}$ for each $\mathfrak{P} \in \prod$. Clearly, this is a subdirect product representation. Condition (iii) and Corollary 5.9 show that we may choose a spectral norm, $||\cdot||$, on $\mathfrak{A}/\mathfrak{A}_J$. Provide each quotient $\mathfrak{A}/\mathfrak{P}$ with the quotient spectral norm, $||\cdot||_{\mathfrak{P}}$. Then this is a normed subdirect product. (Clearly the norm $||\cdot||_\infty$ on the subdirect product induces a spectral pseudo-norm on $\mathfrak{A}$.) Finally, the subdirect product is ample by Theorem 6.4. (Note that $\{\mathfrak{P}/\mathfrak{A}_J : \mathfrak{P} \in \prod\}$ is precisely the set of primitive ideals of $\mathfrak{A}/\mathfrak{A}_J$. This is obvious from the definition of primitive ideals given here.) The factors $\mathfrak{A}/\mathfrak{P}$ can be given the asserted form by Corollary 6.8.

(iv) $\Rightarrow$ (ii). Complete $\mathfrak{A}/\mathfrak{A}_J$ with respect to the norm $||\cdot||_\infty$ induced by the semidirect product. This is a spectral norm since the semidirect product is ample and its factors are spectral normed. Hence, the completion $\bar{\mathfrak{A}}$ satisfies $\rho_{\mathfrak{A}}(a) = \rho_{\mathfrak{A}/\mathfrak{A}_J}(a+\mathfrak{A}_J) = \lim ||(a+\mathfrak{A}_J)^n||^{1/n} = \rho_{\bar{\mathfrak{A}}}(a+\mathfrak{A}_J)$.

(ii) $\Rightarrow$ (i). Corollary 5.5 shows this. $\square$

**Remark 6.11.** Since any radical algebra is spectral, the example of H.G. Dales [**4**] shows that not every spectral algebra can be embedded in a Banach algebra.

**Note added in proof:** A lot has happened since this article was submitted in 1986. The book manuscript from which the article was taken has been prepared for publication and will soon appear as a two-volume work, *Banach algebras and the general theory of* $^*$*-algebras*, in the Cambridge University Encyclopedia of Mathematics series. That work extends some of the results in the present article. The author has used T.J. Ransford's beautiful arguments in [*A short proof of Johnson's uniqueness of the norm theorem*, Bull. London Math. Soc. **21** (1989), 487–488, MR 90g 46069] to give a simple direct proof of Proposition 3.6 above.

The ideas of the present article help to clarify the concepts of local Banach algebras (i.e., not necessarily complete normed algebras in which each element has a full analytic functional calculus) and of local Banach subalgebras, which have become more popular recently, particularly in $K$-theory applied to $C^*$-algebras and Banach algebras.

## REFERENCES


**1.** B.A. Barnes, *Modular annihilator algebras*, Canad. J. Math. **18** (1966), 566–578, MR 33 2681.

**2.** N. Bourbaki, *Theories Spectrales*, 1 *and* 2, Elem. Math. Fasc. **23**, Hermann, Paris, 1967, MR 35 4725.

**3.** S.B. Cleveland, *Homomorphisms of non-commutative* $^*$*-algebras*, Pacific J. Math. **13** (1963), 1097–1109, MR 28 1500.

**4.** H.G. Dales, *Norming nil algebras*, Proc. Amer. Math. Soc. **83** (1981), 71–74, MR 82j 46069.

**5.** R. Fuster and A. Marquina, *Geometric series in incomplete normed algebras*, Amer. Math. Monthly **91** (1984), 49–51, MR 85g 46059.

**6.** I.M. Gelfand, *Normierte Ringe*, Mat. Sbornik (N.S.) **9** (51) (1941), 3–24, MR 3 51.

**7.** N. Jacobson, *The radical and semisimplicity for arbitrary rings*, Amer. J. Math. **67** (1945), 300–320, MR 7 2.

**8.** ———, *Structure theory of simple rings without finiteness assumptions*, Trans. Amer. Math. Soc. **57** (1945), 228–245, MR 6 200.

**9.** I. Kaplansky, *Topological rings*, Amer. J. Math. **69** (1947), 153–183, MR 8 434.

**10.** ———, *Locally compact rings*, Amer. J. Math. **70** (1948), 447–459, MR 9 562.

**11.** ———, *Normed algebras*, Duke Math. J. **16** (1949), 399–418, MR 11 115.

**12.** C. Le Page, *Sur quelques conditions entrainant la commutativité dans les algèbres de Banach*, C.R. Acad. Sci. Paris Ser. A-B **265** (1967), 235–237, MR 37 1999.

**13.** V. Mascioni, *Some characterizations of complex $Q$-algebras*, El. Math. **42** (1987), 10–14, MR 88k 46053.

**14.** E.A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. No. 11 (1952), 79 pp, MR 14 482.

**15.** Angel Rodriquez Palacios, *The uniqueness of the complete norm topology in complete normed nonassociative algebras*, J. Funct. Anal. **60** (1985), 1–15, MR 86f 46053.

**16.** T.W. Palmer, *Hermitian Banach $^*$-algebras*, Bull. Amer. Math. Soc. **78** (1972), 522–524, MR 45 7481.

**17.** Vlastimil Ptak, *On the spectral radius in Banach algebras with involution*, Bull. London Math. Soc. (2) (1970), 327–334, MR 43 932.

**18.** Vlastimil Ptak and Jaroslav Zemanek, *On uniform continuity of the spectral radius in Banach algebras*, Manuscripta Math. **20** (1977), 177–189, MR 56 1065.

**19.** D.A. Raikov, *To the theory of normed rings with involution*, C.R. (Doklady) Acad. Sci. URSS (N.S.) **54** (1946), 387–390, MR 8 469.

**20.** C.E. Rickart, *The uniqueness of norm problem in Banach algebras*, Ann. of Math. (2) **51** (1950), 615–628, MR 11 670.

**21.** ———, *On spectral permanence for certain Banach algebras*, Proc. Amer. Math. Soc. **4** (1953), 191–196, MR 14 660.

**22.** ———, *An elementary proof of the fundamental theorem in the theory of Banach algebras*, Michigan Math. J. **5** (1958), 75–78, MR 20 4786.

**23.** ———, *General theory of Banach algebras*, van Nostand, Princeton, 1960. (Reprint: Krieger, Huntington, NY, 1974), MR 22 A 5903.

**24.** M.A. Rieffel, *Square-integrable representations of Hilbert algebras*, J. Funct. Anal. **3** (1969), 265–300, MR 39 6094.

**25.** M. Rosenbloom, *On a theorem of Fuglede and Putnam*, J. London Math. Soc. **33** (1958), 376–377, MR 20 6037.

**26.** G.E. Šilov, *On the extension of maximal ideals*, C.R. (Doklady) Acad. Sci. URSS (N.S.) **29** (1940), 83-84, MR 2 314.

**27.** A. Wilansky, *Letter to the editor*, Amer. Math. Monthly **91** (1984), 531.

**28.** B. Yood, *Homomorphisms on normed algebras*, Pacific J. Math. **8** (1958), 373–381, MR 21 2924.

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