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REGULAR LATTICES AND WEAKLY REPLETE LATTICES

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ABSTRACT. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. The notion of $\mathcal{L}$ being regular or weakly replete are investigated. Also, spaces related to $X, \mathcal{L}$ are investigated in terms of the general Wallman space, and for $\mathcal{L}$ not necessarily disjunctive analogous of these spaces are constructed.

1. Introduction. It is well known that to each disjunctive lattice $\mathcal{L}$ of subsets of a set X there is associated the general Wallman space $I_R(\mathcal{L})$, $\tau W(\mathcal{L})$ (see below for definitions) which is a compact T_1 space and is T_2 if and only if $\mathcal{L}$ is normal. Moreover, if $\mathcal{L}$ is separating (T_1) then X is densely embedded in $I_R(\mathcal{L})$ and even homeomorphically if X carries the $\tau\mathcal{L}$ topology of closed sets. We will (see Section 4) carry out an analogous construction of an associated space in the case of a not necessarily disjunctive lattice $\mathcal{L}$ thereby extending the work of Illiadis [**5**] of an absolute closure.

Next, if $\mathcal{L}$ is disjunctive then associated with $X, \mathcal{L}$ is the pair $I_R^\sigma(\mathcal{L})$, $W_\sigma(\mathcal{L})$, where $W_\sigma(\mathcal{L})$ is a replete lattice, and which generalizes the notion of the usual real compactification of a Tychonoff space. We again generalize to the situation of a not necessarily disjunctive lattice and introduce the notion of a weakly replete lattice. This work extends some of the results of Liu.

We adhere to a measure theoretic point of view throughout since this is more natural in the case of restriction and extension problems, and since many extend to nonzero are valued measures. We first introduce some standard lattice terminology (see [**2, 3, 4, 6**]) and state the equivalent measure characterizations. In Section 3, we elaborate on some of these properties and show some relationships to τ-smooth

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measures. Sections 4 and 5 contain the constructions of the spaces indicated earlier and give the salient properties concerning weakly compact and weakly replete lattices.

2. Definitions and notations. a) Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. We shall assume, without loss of generality for our purposes, that $\phi, X \in \mathcal{L}$. The set whose general element L' is the complement of L of $\mathcal{L}$ is denoted by $\mathcal{L}'$. $\mathcal{L}$ is a complement generated if and only if, for every L of $\mathcal{L}$ there exists a sequence $\{L_n\}_{n=1}^{\infty}$ in $\mathcal{L}$ such that $L = \cap_{n=1}^{\infty} L_n'$. $\mathcal{L}$ is a delta-lattice (δ-lattice) if $\mathcal{L}$ is closed under countable intersections. $\mathcal{L}$ is a T_2-lattice if, for any $x, y \in X$, $x \neq y$, there exist $L_1, L_2 \in \mathcal{L}$ such that $x \in L_1'$, $y \in L_2'$ and $L_1' \cap L_2' = \phi$. $\mathcal{L}$ is regular if for every $x \in X$ and every $L \in \mathcal{L}$, if $x \notin L$ then there exist $L_1, L_2 \in \mathcal{L}$; $x \in L_1'$, $L \subset L_2'$ and $L_1' \cap L_2' = \phi$. $\mathcal{L}$ is a normal lattice, if for any $L_1, L_2 \in \mathcal{L}$, $L_1 \cap L_2 = \phi$, there exist $L_3, L_4 \in \mathcal{L}$ with $L_1 \subset L_3'$, $L_2 \subset L_4'$ and $L_3' \cap L_4' = \phi$. $\mathcal{L}$ is Lindelöf if and only if, for every $L_\alpha \in \mathcal{L}$, $\alpha \in A$, if $\cap_\alpha L_\alpha = \phi$, then for a countable subcollection $\{L_{\alpha_i}\}$ of $\{L_\alpha\}$, $\cap_{i=1}^{\infty} L_{\alpha_i} = \phi$. $\mathcal{L}$ is compact if and only if, for every $L_\alpha \in \mathcal{L}$, $\alpha \in A$, if $\cap_\alpha L_\alpha = \phi$ for finite subcollection $\{L_{\alpha_i}\}$ of $\{L_\alpha\}$, $\cap_{i=1}^{n} L_{\alpha_i} = \phi$. $\mathcal{L}$ is disjunctive if, for any $x \in X$ and every $L_1 \in \mathcal{L}$, if $x \notin L_1$, then there exists an $L_2 \in \mathcal{L}$ with $x \in L_2$ and $L_1 \cap L_2 = \phi$. Next, let $\mathcal{L}_1, \mathcal{L}_2$ be two lattices of subsets of X. $\mathcal{L}_1$ semi-separates $\mathcal{L}_2$ or for abbreviation ($\mathcal{L}_1$ s.s. $\mathcal{L}_2$) if and only if, for every $L_1 \in \mathcal{L}_1$ and every $L_2 \in \mathcal{L}_2$ if $L_1 \cap L_2 = \phi$ then there exists $\hat{L}_1 \in \mathcal{L}$, $L_2 \in \hat{L}_1$ and $L_1 \cap \hat{L}_1 = \phi$. $\mathcal{L}_1$ separates $\mathcal{L}_2$, if for any $L_2, \hat{L}_2 \in \mathcal{L}_2$, $L_2 \cap \hat{L}_2 = \phi$ then there exist $L_1, \hat{L}_1 \in \mathcal{L}_1$, $L_2 \subset L_1$, $\hat{L}_2 \subset \hat{L}_1$ and $L_1 \cap \hat{L}_1 = \phi$. $\mathcal{L}_2$ is countably bounded (countably paracompact) or simply ($\mathcal{L}_2$ is $\mathcal{L}_1$-cb (cp)), if given $B_n \downarrow \phi$, $B_n \in \mathcal{L}_2$, there exists $A_n \in \mathcal{L}_1$ with $B_n \subset A_n$ ($B_n \subset A_n'$) and $A_n \downarrow \phi$ ($A_n' \downarrow \phi$).

b) Let $\mathcal{A}$ be any algebra of subsets of X. A measure on $\mathcal{A}$ is defined to be a function μ from $\mathcal{A}$ to R such that μ is bounded and finitely additive. $\mathcal{A}(\mathcal{L})$ denotes the algebra of subsets of X generated by $\mathcal{L}$. If $x \in X$, then μ_x is the measure concentrated at x so that

$$\mu_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

where $A \in \mathcal{A}(\mathcal{L})$. $M(\mathcal{L})$ denotes the set whose general element is a measure on $\mathcal{A}(\mathcal{L})$. Since any $\mu \in M(\mathcal{L})$ can be split into its positive and negative pieces, then without loss of generality we may tacitly work with the nonnegative measures of $M(\mathcal{L})$. Let $\mu \in M(\mathcal{L})$, μ is $\mathcal{L}$-regular, if for any $A \in \mathcal{A}(\mathcal{L})$, $\mu(A) = \sup\{\mu(L); L \subset A, L \in \mathcal{L}\}$. $M_R(\mathcal{L})$ denotes the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$. $\mu \in M(\mathcal{L})$ is σ-smooth on $\mathcal{L}$, if $L_n \in \mathcal{L}$, $n = 1, 2, \dots$ and $L_n \downarrow \phi$, then $\mu(L_n) \to 0$. $M_\sigma(\mathcal{L})$ denotes the set of σ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$. $\mu \in M(\mathcal{L})$ is σ-smooth on $A(\mathcal{L})$, if $A_n \in \mathcal{A}(\mathcal{L})$, $n = 1, 2, \dots$ and $A_n \downarrow \phi$, then $\mu(A_n) \to 0$. $M^\sigma(\mathcal{L})$ denotes the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $M(\mathcal{L})$. $M^\sigma_R(\mathcal{L})$ denotes the set of $\mathcal{L}$-regular measures of $M^\sigma(\mathcal{L})$. It is easy to see that if $\mu \in M_R(\mathcal{L})$ then μ is σ-smooth on $\mathcal{A}(\mathcal{L})$ if and only if μ is σ-smooth on $\mathcal{L}$. $\mu \in M(\mathcal{L})$ is τ-smooth on $\mathcal{L}$ if for every $\{L_\alpha\}$, $L_\alpha \in \mathcal{L}$ such that $L_\alpha \downarrow \phi$, then $\mu(L_\alpha) \to 0$. $M_\tau(\mathcal{L})$ denotes the set of τ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$. $M^\tau_R(\mathcal{L})$ denotes the set of all $\mathcal{L}$-regular measures of $M(\mathcal{L})$ which are also τ-smooth on $\mathcal{L}$. Finally, $\mu \in M^\sigma(\mathcal{L})$ is strongly τ-smooth on $\mathcal{L}$, if for every $\{L_\alpha\}$, $L_\alpha \in \mathcal{L}$ such that $L_\alpha \downarrow$, then $\mu^*(\cap L_\alpha) = \inf \mu(L_\alpha)$ where μ^* is the induced outer measure. $I(\mathcal{L})$, $I_R(\mathcal{L})$, $I_\sigma(\mathcal{L})$, $I^\sigma(\mathcal{L})$, $I_\tau(\mathcal{L})$, and $I^\tau_R(\mathcal{L})$ are the subsets of the corresponding M's consisting of the nontrivial zero–one valued measures. For $\mu \in M(\mathcal{L})$, the support of μ, $S(\mu) = \cap\{L \in \mathcal{L} : \mu(L) = \mu(X)\}$. Consequently, if $\mu \in I(\mathcal{L})$, $S(\mu) = \cap\{L \in \mathcal{L}, \mu(L) = 1\}$. $\mathcal{L}$ is prime replete, if for each $\mu \in I_\sigma(\mathcal{L})$, $S(\mu) \neq \phi$. A premeasure on $\mathcal{L}$ is a function π from $\mathcal{L}$ to $\{0, 1\}$ such that $\pi(\phi) = 0$, $\pi(A) \leq \pi(B)$ for every $A \subset B$, $A, B \in \mathcal{L}$ and if $\pi(A) = \pi(B) = 1$, then $\pi(A \cap B) = 1$. $\Pi(\mathcal{L})$ denotes the set of all premeasures on $\mathcal{L}$. Note that, for every $\mu \in I(\mathcal{L})$, there exists a $\nu \in 1_R(\mathcal{L})$; $\mu \leq \nu$ on $\mathcal{L}$ or simply $(\mu \leq \nu(\mathcal{L}))$. Finally, if μ_x is the measure concentrated at x, then $\mu_x \in I_R(\mathcal{L})$ if and only if $\mathcal{L}$ is disjunctive.

3. Some further properties and measures. In this section we elaborate on certain lattice properties introduced in Section 2. In particular, we expand on the notion of a regular lattice and on τ-smooth measures, thereby showing some relationships to mildly normal lattices and to extensions of results on τ-smooth measures.

Theorem 3.1. *$\mathcal{L}$ is regular if and only if for $\mu_1, \mu_2 \in I(\mathcal{L})$, $\mu_1 \leq \mu_2(\mathcal{L})$, then $S(\mu_1) = S(\mu_2)$.*

Proof. (i). If $S(\mu) = \cap\{L \in \mathcal{L}, \mu(L) = 1\}$ and $\mu_1 \leq \mu_2(\mathcal{L})$ then $S(\mu_2) \subset S(\mu_1)$ is trivial (it is not necessarily the condition "$\mathcal{L}$ is regular"). We now want to show that $S(\mu_1) \subset S(\mu_2)$; if not, there exists $x \in S(\mu_1)$, $x \notin S(\mu_2) = \cap_\alpha L_\alpha$, $L_a \in \mathcal{L}$, with $\mu_2(L_\alpha) = 1$. Therefore, there exists $L \in \mathcal{L}$, $\mu_2(L) = 1$ and $x \notin L$. Since $\mathcal{L}$ is regular, then by definition there exist $L_1, L_2 \in \mathcal{L}$, $x \in L_1'$, $L \subset L_2'$, $L_1 \cup L_2 = X$. Then either $\mu_1(L_1) = 0$ or $\mu_1(L_1) = 1$, but $\mu_2(L) = 1$ then $\mu_2(L_2') = 1$, $\mu_2(L_2) = 0$ and so $\mu_1(L_2) = 0$. Since $\mu_1 \leq \mu_2(\mathcal{L})$ then $\mu_1(L_1) = 1$ and $S(\mu_1) \subset L_1$, but $x \in S(\mu_1) \subset L_1$ and also $x \in L_1'$, which is a contradiction. Therefore, $S(\mu_1) \subset S(\mu_2)$ and, moreover, $S(\mu_1) = S(\mu_2)$.

(ii). Now we want to show that $\mathcal{L}$ is regular. If not, then there exist $x \in X$, $L \in \mathcal{L}$, $x \notin L$ and $H = \{\hat{L}' \in \mathcal{L}, x \in \hat{L}'$ or $L \subset \hat{L}'\}$ has the finite intersection property. Then there exists $\mu_1 \in I(\mathcal{L})$, $\mu_1(\hat{L}') = 1$ for all $\hat{L}' \in H$. If $\mu_1(\hat{L}) = 1$, $\hat{L} \in \mathcal{L}$, then $\mu_1(\hat{L}') = 0$, $L \subset \hat{L}'$ and so $\hat{L} \cap L \neq \phi$ for all $\hat{L} \in \mathcal{L}$. Therefore, there exists $\mu_2 \in I(\mathcal{L})$, $\mu_2(L) = 1$ and $\mu_1 \leq \mu_2(\mathcal{L})$. Also, since $\mu_1(\hat{L}) = 1$, then $\mu_1(\hat{L}') = 0$ and so $x \notin \hat{L}'$, $x \in \hat{L}$ for all $\hat{L} \in \mathcal{L}$ with $\mu(\hat{L}) = 1$. Therefore, $x \in S(\mu_1) = S(\mu_2)$, but $\mu_2(L) = 1$. Then $x \in L$ which is a contradiction. Thus, $\mathcal{L}$ must be regular. □

Definition 3.1. $\mathcal{L}$ is mildly normal, if for all $\mu \in I_\sigma(\mathcal{L})$, there exists a unique $\nu \in I_R(\mathcal{L})$, $\mu \leq \nu(\mathcal{L})$.

We have considered mildly normal lattices and their relationships to normality in [**1**]. Here, we want to consider mildly normal lattices and their relationships to regularity.

Theorem 3.2. *If $\mathcal{L}$ is regular and prime replete, then $\mathcal{L}$ is mildly normal.*

Proof. If $\mathcal{L}$ is not mildly normal, there exists $\mu \in I_\sigma(\mathcal{L})$, $\nu_1, \nu_2 \in I_R(\mathcal{L})$, $\nu_1 \neq \nu_2$ and $\mu \leq \nu_1(\mathcal{L})$, $\mu \leq \nu_2(\mathcal{L})$. Since $\nu_1 \neq \nu_2$, then there exists $L_1, L_2 \in \mathcal{L}$, $L_1 \cap L_2 = \phi$ and $\nu_1(L_1) = \nu_2(L_2) = 1$ and $\nu_1(L_2) = \nu_2(L_1) = 0$ but, since $\mathcal{L}$ is regular, $S(\mu) = S(\nu_1) \subset L_1$ and $S(\mu) = S(\nu_2) \subset L_2$. Then, $S(\mu) \subset L_1 \cap L_2 = \phi$ and so $S(\mu) = \phi$ which is a contradiction since $\mathcal{L}$ is prime replete. Thus, $\mathcal{L}$ must be mildly normal. □

Corollary 3.1. *If $\mathcal{L}$ is regular and Lindelöf, then $\mathcal{L}$ is mildly normal.*

Proof. Since $\mathcal{L}$ is Lindelöf, it follows immediately that $\mathcal{L}$ is prime replete and the result follows from Theorem 3.2. $\square$

Theorem 3.3. *If $\mathcal{L}$ is regular, $\mu \in I(\mathcal{L})$ and $\rho(L) = \sup_{\hat{L} \subset L} \mu(\hat{L})$, $\hat{L} \in \mathcal{L}$, then a) ρ is a premeasure on $\mathcal{L}$ and b) $S(\mu) = S(\rho)$.*

Proof. a) ρ is a premeasure on $\mathcal{L}$ since (i) clearly, $\rho(\phi) = 0$, (ii) if $L_1 \subset L_2$, then $\rho(L_1) = \sup_{\hat{L}_1 \subset L_1} \mu(\hat{L}_1) \leq \sup_{\hat{L}_2 \subset L_2} \mu(\hat{L}_2) = \rho(L_2)$, $\hat{L} \in \mathcal{L}$ and (iii) if $\rho(L_i) = 1$ for $i = 1, 2$, then there exist $\hat{L}_i \subset L_i$ and $\mu(\hat{L}_i) = 1$, $\hat{L}_i \in \mathcal{L}$ for $i = 1, 2$ and moreover, $\mu(\hat{L}_1 \cap \hat{L}_2) = 1$ and so $\rho(L_1 \cap L_2) = 1$.

b) Suppose $\mathcal{L}$ is regular, since $\rho \leq \mu(\mathcal{L})$, then $S(\mu) \subset S(\rho)$. If $S(\mu) \neq S(\rho)$, then there exists an $x \in S(\rho)$ but $x \notin S(\mu)$, then there exists $L \in \mathcal{L}$, $x \notin L$ with $\mu(L) = 1$, therefore $x \in L'$ and so $x \in L_1' \subset L_2 \subset L'$ where $L_1, L_2 \in \mathcal{L}$ therefore $L \subset L_2' \subset L_1$, $\mu(L_2') = 1$ and $\rho(L_1) = 1$ (by the definition of ρ) and since $x \in S(\rho)$ then $x \in L_1$, but $x \in L_1'$ which is a contradiction. Thus, $S(\mu) = S(\rho)$. $\square$

To prove the next theorem, we note that if $\mu \in I_\sigma(\mathcal{L}')$ and $\mathcal{L}$ is a complement generated, then $\mu \in I_R(\mathcal{L})$. This result is not difficult and is also true, if $\mu \in M_\sigma(\mathcal{L}')$ in which case $\mu \in M_R(\mathcal{L})$.

Theorem 3.4. *Suppose $\mu \in I^\sigma(\mathcal{L})$ or just $I_\sigma(\mathcal{L}')$ and $\mathcal{L}$ is a complement generated, then if $S(\mu) \neq \phi$, $\mu = \mu_x$ for some $x \in X$.*

Proof. Suppose $\mu \in I_\sigma(\mathcal{L}')$ and $\mathcal{L}$ is complement generated, then $\mu \in I_R^\sigma(\mathcal{L})$, and if $S(\mu) \neq \phi$, then there exists $x \in S(\mu)$ and so $\mu \leq \mu_x(\mathcal{L})$, but $\mu \in I_R^\sigma(\mathcal{L})$. Thus, $\mu = \mu_x$ for some $x \in X$. $\square$

As an immediate consequence of Theorem 3.5, we have the following corollary, where $\mathcal{Z}$ is the lattice of zero sets of a topological space.

Corollary 3.2. *Suppose $\mu \in I^\sigma(\mathcal{Z})$ and $I_\tau(\mathcal{Z})$, then $S(\mu) \neq \phi$ and $\mu = \mu_x$ for $x \in X$.*

Proof. Since $\mathcal{Z}$ is complement generated, $\mu \in I_R^\sigma(\mathcal{Z})$ and since $\mu \in I_\tau(\mathcal{Z})$, $S(\mu) \neq \phi$; therefore, there exists an $x \in S(\mu)$ and $\mu \le \mu_x(\mathcal{Z})$, but $\mu \in I_R^\sigma(\mathcal{Z})$. Thus, $\mu = \mu_x$. □

Now we list the following immediate observations (in this connection see [**7**]): (1) If $\mu \in M_\tau(\mathcal{L})$ then $S(\mu) \neq \phi$. (2) If $\mathcal{L}$ is a δ-lattice, then $M_R^\sigma(\mathcal{L}) = M_R^\tau(\mathcal{L})$ if and only if $S(\mu) \neq \phi$ for all $\mu \in M_R^\sigma(\mathcal{L})$ where μ is nontrivial. Also (3) if $\mathcal{L}$ is a δ-lattice, then $\mu \in M_R^\tau(\mathcal{L})$ if for any $\{L_\alpha\}$ of $\mathcal{L}$ with $L_\alpha \downarrow$, $\mu^*(\cap L_\alpha) = \operatorname{Inf} \mu(L_\alpha)$.

Theorem 3.5. *Let $\mathcal{L}$ be T_2. Suppose $\mu \in I^\sigma(\mathcal{L})$ and strongly τ-smooth, then $S(\mu) = \{x\}$ and $\mu_x = \mu$ for some $x \in X$.*

Proof. Since $\mu \in I^\sigma(\mathcal{L})$ and strongly τ-smooth, then $S(\mu) \neq \phi$ and by T_2, $S(\mu) = \{x\}$, $\mu \le \mu_x$ and so $S(\mu) = S(\mu_x) = \{x\}$; therefore, $\{x\} = \cap L_\alpha$, $\mu(L_\alpha) = 1$, $L_\alpha \in \mathcal{L}$ and by the strongly τ-smoothness of μ, $\mu^*(\{x\}) = \operatorname{Inf} \mu(L_\alpha) = 1$. If $\mu_x(L) = 1$, $L \in \mathcal{L}$, then $x \in L$; hence, $1 = \mu^*(\{x\}) \le \mu(L)$, $\mu(L) = 1$ and so $\mu_x \le \mu(\mathcal{L})$. Thus, $\mu_x = \mu$. □

As an immediate consequence of Theorem 3.5, we obtain the following corollary where $\mathcal{F}$ is the lattice of all closed sets of a topological space.

Corollary 3.4. *Let X be a T_2-topological space and $\mathcal{F}$ the lattice of closed sets. If $\mu \in I^\sigma(\mathcal{F})$ and strongly τ-smooth then $S(\mu) = \{x\}$ and $\mu = \mu_x$ for some x.*

4. Spaces related to $X, \mathcal{L}$. We first consider the case where $\mathcal{L}$ is disjunctive and give a brief review of the most important properties of the associated Wallman space. In fact, the Wallman topology is obtained by taking the totality of all $W(L) = \{\mu \in I_R(\mathcal{L}), \mu(L) = 1, L \in \mathcal{L}\}$ as a base for the closed set on $I_R(\mathcal{L})$. And, for a disjunctive $\mathcal{L}$, $I_R(\mathcal{L})$ with the topology $\tau W(\mathcal{L})$ of closed sets is a compact T_1 space and will be T_2 if and only if $\mathcal{L}$ is normal and is called the general

Wallman space associated with X and $\mathcal{L}$.

Also, for a disjunctive $\mathcal{L}$ and $A, B \in \mathcal{A}(\mathcal{L})$, $W(A)$ is a lattice with respect to union and intersection. Moreover, $W(A') = (W(A))'$, $W(\mathcal{A}(\mathcal{L})) = \mathcal{A}(W(\mathcal{L}))$, $W(A) = W(B)$ if and only if $A = B$ and $W(A) \subset W(B)$ if and only if $A \subset B$. Now we note that, if $\mathcal{L}$ is disjunctive so is $W(\mathcal{L})$, and in addition to each $\mu \in M(\mathcal{L})$, there exists a $\hat{\mu} \in M(W(\mathcal{L}))$ defined by $\mu(A) = \hat{\mu}(W(A))$ for all $A \in \mathcal{A}(\mathcal{L})$ such that the map $\mu \to \hat{\mu}$ is one-to-one and onto; moreover, $\mu \in M_R(\mathcal{L})$ if and only if $\hat{\mu} \in M_R(W(\mathcal{L}))$.

We next introduce the notion of an $\mathcal{L}$-convergent measure $\mu \in I(\mathcal{L})$ and list several properties.

Definition 4.1. $\mu \in I(\mathcal{L})$ is $\mathcal{L}$-convergent, if there exists an $x \in X$ such that $\mu_x \leq \mu(\mathcal{L})$.

Some properties are (1) μ is $\mathcal{L}$-convergent if and only if $S(\mu) \neq \phi$ on $\mathcal{L}'$, for all $\mu \in I(\mathcal{L})$; (2) if $\mu_1 \leq \mu_2(\mathcal{L})$, where $\mu_1, \mu_2 \in I(\mathcal{L})$, then a) if μ_1 is $\mathcal{L}$-convergent so is μ_2, and b) if $\mathcal{L}'$ is regular and μ_2 is $\mathcal{L}$-convergent, then μ_1 is $\mathcal{L}$-convergent; (3) if $\mathcal{L}'$ is T_2 and μ is $\mathcal{L}$-convergent where $\mu \in I(\mathcal{L})$, then there exists a unique $x \in X$ such that $\mu_x \leq \mu(\mathcal{L})$.

The proofs of these properties are not difficult and will not be given.

Definition 4.2. $\mathcal{L}$ is weakly compact if, for all $\mu \in I_R(\mathcal{L})$, μ is $\mathcal{L}$-convergent.

Remark 1. Note that if $\mathcal{L}$ is compact, then for $\mu \in I_R(\mathcal{L})$, $S(\mu) \neq \phi$. Let $x \in S(\mu)$. Then, $\mu \leq \mu_x$ on $\mathcal{L}$. Thus, $\mu = \mu_x$ and so $\mathcal{L}$ is weakly compact.

Definition 4.3. $\mathcal{L}$ is almost compact if, for any $\mu \in I_R(\mathcal{L}')$, $S(\mu) \neq \phi$ on $\mathcal{L}$.

The next theorem is not difficult to prove.

Theorem 4.1. *$\mathcal{L}$ is weakly compact if and only if $\mathcal{L}'$ is almost compact.*

Remark 2. Now we note that a topological space X is absolutely closed (generalized absolutely closed) if and only if the lattice $\mathcal{O}$ of open sets is $T_2(T_0)$ and weakly compact.

Let $\mathcal{L}$ be a lattice of subsets of X, and define $U(\mathcal{L})$ as the collection of all: $\{\cup L_\alpha; L_\alpha \in \mathcal{L}\}$.

Theorem 4.2. a) *If* $\mathcal{L}_1 \subset \mathcal{L}_2$ *and* $\mathcal{L}_2$ *is weakly compact, then* $\mathcal{L}_1$ *is weakly compact.* b) *Suppose* $\mathcal{L}_1 \subset \mathcal{L}_2 \subset U(\mathcal{L}_1)$ *and* $\mathcal{L}_1$ *s.s.* $\mathcal{L}_2$*, then, if* $\mathcal{L}_1$ *is weakly compact,* $\mathcal{L}_2$ *is weakly compact.*

Proof. a) Let $\nu \in I_R(\mathcal{L}_2)$ be an extension of $\mu \in I_R(\mathcal{L}_1)$ and by the weakly compactness of $\mathcal{L}_2$, there exists an x, $\mu_x \leq \nu(\mathcal{L}_2)$, $\mu_x \leq \mu(\mathcal{L}_1)$. Thus, $\mathcal{L}_1$ is weakly compact.

b) Since $\mathcal{L}_1$ s.s. $\mathcal{L}_2$, $\mu \in I_R(\mathcal{L}_1)$ and $\mu_x \leq \mu(\mathcal{L}_1)$ where μ is the restriction of $\nu \in I_R(\mathcal{L}_2)$ to $\mathcal{A}(\mathcal{L}_1)$. Let $L_2 \in \mathcal{L}_2$ and $\mu_x(L_2) = 1$, then $x \in L_2 = \cup L_{1\alpha}$, $L_{1\alpha} \in \mathcal{L}_1$ and so $x \in L_{1\alpha}$ and $\mu_x(L_{1\alpha}) = 1$ for some $L_{1\alpha}$. Furthermore, since $\mu_x \leq \mu(\mathcal{L}_1)$, $\mu(L_{1\alpha}) = 1$, but $L_{1\alpha} \subset L_2$ then $\nu(L_2) = 1$ and so $\mu_x \leq \nu(\mathcal{L}_2)$. Thus, $\mathcal{L}_2$ is weakly compact. □

Remark 3. Let X be a topological space and $\mathcal{O}$ the collection of open sets, then by Theorem 4.1, $\mathcal{O}$ is weakly compact if and only if $\mathcal{F} = \mathcal{O}'$ is almost compact.

Now, consider X and suppose $\mathcal{L}$ is nondisjunctive and define $\hat{I} = \{\mu_x; x \in X\} \cup \{\mu \in I_R(\mathcal{L}); \mu \text{ is not } \mathcal{L} - \text{convergent}\}$ and $\hat{W}(A) = \{\mu \in \hat{I}; \mu(A) = 1, A \in \mathcal{A}(\mathcal{L})\}$. We also assume that $\mathcal{L}$ is T_0, so $x, y \in X$ and $x \neq y$ implies $\mu_x \neq \mu_y$. Then, for $A, B \in \mathcal{A}(\mathcal{L})$, we have a) $A = B$ if and only if $\hat{W}(A) = \hat{W}(B)$, b) $\hat{W}(A \cup B) = \hat{W}(A) \cup \hat{W}(B)$ and $\hat{W}(A \cap B) = \hat{W}(A) \cap \hat{W}(B)$, and c) $\hat{W}(A') = (\hat{W}(A))'$ and $\hat{W}(\mathcal{A}(\mathcal{L})) = \mathcal{A}(\hat{W}(\mathcal{L}))$. Finally, let $\mu \in I(\mathcal{L})$ and define $\hat{\mu} \in I(\hat{W}(\mathcal{L}))$ to be $\hat{\mu}(\hat{W}(A)) = \mu(A)$, $A \in \mathcal{A}(\mathcal{L})$. Also, it can be easily shown that the map $\mu \to \hat{\mu}$ is one-to-one and onto from $I(\mathcal{L})$ to $I(\hat{W}(\mathcal{L}))$ and, moreover $\mu \in I_R(\mathcal{L})$ if and only if $\hat{\mu} \in I_R(\hat{W}(\mathcal{L}))$.

Theorem 4.3. $\hat{W}(\mathcal{L})$ *is weakly compact and* T_0.

Proof. a) Suppose $\hat{\mu} \in I_R(\hat{W}(\mathcal{L}))$, then $\mu \in I_R(\mathcal{L})$. (i) If μ is $\mathcal{L}$-

convergent, then $\mu_x \leq \mu(\mathcal{L})$ and so $\hat{\mu}_x \leq \hat{\mu}(\hat{W}(\mathcal{L}))$. Also, for $A \in \mathcal{A}(\mathcal{L})$,

$$\mu_x(A) = \hat{\mu}_x(\hat{W}(A)) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}, \text{ and } \mu_x \in \hat{W}(A)$$

if and only if $\mu_x(A) = 1$. Thus, $\hat{\mu}_x$ is the measure concentrated at μ_x and $\hat{\mu}$ is $\hat{W}(\mathcal{L})$-convergent and so $\hat{W}(\mathcal{L})$ is weakly compact. (ii) If μ is not $\mathcal{L}$-convergent, then $S(\hat{\mu}) = \cap \hat{W}(L)$, $\hat{\mu}(\hat{W}(L)) = 1 = \mu(L)$, hence $\mu \in \hat{W}(L)$, $\mu \in \hat{I}$, $\mu \in S(\hat{\mu})$ and $\hat{\mu}$ is the measure concentrated at μ, and so $\hat{W}(\mathcal{L})$ is weakly compact. b) Let $\mu_1, \mu_2 \in \hat{I}$, $\mu_1 \neq \mu_2$, then there exists on $L \in \mathcal{L}$, say, $\mu_1(L) = 1$, $\mu_2(L) = 0$. Therefore, $\mu_1 \in \hat{W}(L)$, $\mu_2 \in \hat{W}(L')$ and so $\hat{W}(L)$ is T_0. □

Theorem 4.4. *If $\mathcal{L} = U(\mathcal{L})$, then $\hat{W}(\mathcal{L})$ separates $U(\hat{W}(\mathcal{L}))$.*

Proof. Suppose

$$(*) \qquad \left(\bigcup_\alpha \hat{W}(L_\alpha)\right) \cap \left(\bigcup_\beta \hat{W}(L_\beta)\right) = \phi,$$

$A = (\cup_\alpha L_\alpha) \in \mathcal{L}$, and $B = (\cup_\beta L_\beta) \in \mathcal{L}$. Since $\mathcal{L} = U(\mathcal{L})$, then $(\cup_\alpha \hat{W}(L_\alpha)) \subset \hat{W}(A)$ and $(\cup_\beta \hat{W}(L_\beta)) \subset \hat{W}(B)$. Also, if $A \cap B \neq \phi$, then $L_\alpha \cap L_\beta \neq \phi$ for some α, β. Let $x \in L_\alpha \cap L_\beta$, then $\mu_x \in \hat{I}$, $\mu_x \in \hat{W}(L_\alpha)$ and $\mu_x \in \hat{W}(L_\beta)$ that contradicts $(*)$. Thus, $A \cap B = \phi$ and the desired result is now clear. □

Now we easily note that if $\mathcal{L} = U(\mathcal{L})$, then $\hat{I}$ with $\mathcal{O} = \hat{W}(\mathcal{L})$ is generalized absolutely closed and is absolutely closed if $\mathcal{L}'$ is T_2. Thus, if we consider X and let $\mathcal{L} = U(\mathcal{L}) = \mathcal{O}_X$ the lattice of open sets on X and T_2, then $\hat{I}, \mathcal{O}$ is an absolute closure of X since one can easily observe that $\overline{X} = \hat{W}(X)$.

5. Further spaces associated with $X, \mathcal{L}$. Again, we start with the case where $\mathcal{L}$ is a disjunctive lattice, only now we consider $I_R^\sigma(\mathcal{L})$ and take all $W_\sigma(\mathcal{L}) = \{\mu \in I_R^\sigma(\mathcal{L}); \mu(L) = 1, \mu \in \mathcal{L}\}$ as a base for the closed sets.

The corresponding properties listed at the beginning of Section 4 for $W(L)$ sets hold for $W_\sigma(L)$ sets. Also, to each $\mu \in M(\mathcal{L})$ there corresponds a $\mu' \in M(W_\sigma(\mathcal{L}))$ defined by $\mu'(W_\sigma(A)) = \mu(A)$ for all $A \in \mathcal{A}(\mathcal{L})$. The map $\mu \to \mu'$ is one-to-one, onto between $M_R(\mathcal{L})$ and $M_R(W_\sigma(\mathcal{L}))$ and also between $M_R^\sigma(\mathcal{L})$ and $M_R^\sigma(W_\sigma(\mathcal{L}))$, since $\mu' \in M_R^\sigma(W_\sigma(\mathcal{L}))$ if and only if $\mu \in M_R^\sigma(\mathcal{L})$.

It is known that $W_\sigma(\mathcal{L})$ is replete [**8**] and that $I_R^\sigma(\mathcal{L})$ with $W_\sigma(\mathcal{L})$ as base for the closed sets is a T_1 space. To show that $W_\sigma(\mathcal{L})$ is replete, let $\mu' \in I_R^\sigma(W_\sigma(\mathcal{L}))$, then $S(\mu') = \cap\{W_\sigma(L_\alpha); \mu'(W_\sigma(L_\alpha) = 1, L_\alpha \in \mathcal{L}\}$, but $\mu'(W_\sigma(L_\alpha)) = \mu(L_\alpha)$ where $\mu \in I_R^\sigma(\mathcal{L})$. Hence, $\mu \in S(\mu')$ and $W_\sigma(\mathcal{L})$ is replete.

As in Section 4, we now proceed to the case of a not necessarily disjunctive lattice $\mathcal{L}$.

Definition 5.1. $\mathcal{L}$ is weakly replete if, for any $\mu \in I_R^\sigma(\mathcal{L})$, μ is $\mathcal{L}$-convergent.

Remark 4. Note that if $\mathcal{L}$ is replete, then for $\mu \in I_R^\sigma(\mathcal{L})$, $S(\mu) \neq \phi$. Let $x \in S(\mu)$, then $\mu \leq \mu_x$ on $\mathcal{L}$. Thus, $\mu = \mu_x$ and so $\mathcal{L}$ is weakly replete.

Note that a topological space X is an α-space if and only if it is T_2 and the lattice $\mathcal{O}$ of open sets is weakly replete.

Theorem 5.1. *Suppose $\mathcal{L}_2$ is weakly replete and $\mathcal{L}_2$ is $\mathcal{L}_1$-cb or $\mathcal{L}_1$-cp, then $\mathcal{L}_1$ is weakly replete.*

Proof. Extend $\mu_1 \in I_R^\sigma(\mathcal{L}_1)$ to $\mu_2 \in I_R(\mathcal{L}_2)$. Since $\mathcal{L}_2$ is $\mathcal{L}_1$-cb or $\mathcal{L}_1$-cp, then $\mu_2 \in I_R^\sigma(\mathcal{L}_2)$ and since $\mathcal{L}_2$ is weakly replete there exists an $x \in X$, $\mu_x \leq \mu_2(\mathcal{L}_2)$; therefore, $\mu_x \leq \mu_1(\mathcal{L}_1)$ and so $\mathcal{L}_1$ is weakly replete. □

Theorem 5.2. *Suppose $\mathcal{L}_1$ s.s. $\mathcal{L}_2$ and $\mathcal{L}_2 \subset U(\mathcal{L}_1)$. If $\mathcal{L}_1$ is weakly replete, then $\mathcal{L}_2$ is weakly replete.*

Proof. Let $\mu_2 \in I_R^\sigma(\mathcal{L}_2)$ since $\mathcal{L}_1$ s.s. $\mathcal{L}_2$, then $\mu_1 = \mu_2|_{\mathcal{A}(\mathcal{L})}$, $\mu_1 \in I_R(\mathcal{L}_1)$ and since $\mathcal{L}_1$ is weakly replete, there exists an $x \in X$, $\mu_x \leq \mu_1(\mathcal{L}_1)$. Now, if $L_2 \in \mathcal{L}_2$ and $\mu_x(L_2) = 1$, then $x \in L_2$ and $L_2 = \cup_\alpha L_{1\alpha}$, $L_{1\alpha} \in \mathcal{L}_1$, then $x \in L_{1\alpha}$, $\mu_x(L_{1\alpha}) = 1$ and so $\mu_1(L_{1\alpha}) = 1$, but $L_{1\alpha} \subset L_2$, then $\mu_2(L_2) = 1$ and so $\mu_x \leq \mu_2(\mathcal{L}_2)$. Thus, $\mathcal{L}_2$ is weakly replete. □

Now, we proceed to generalize some of the work of Liu. Consider X and suppose $\mathcal{L}$ is T_0 so if $x \neq y$, then $\mu_x = \mu_y$ and define $\hat{I}^\sigma = \{\mu_x; x \in X\} \cup \{\mu \in I_R^\sigma(\mathcal{L}); \mu \text{ is not } \mathcal{L}\text{-convergent}\}$ and $\hat{W}^\sigma(A) = \{\mu \in \hat{I}^\sigma, \mu(A) = 1, A \in \mathcal{A}(\mathcal{L})\} = \hat{W}(A) \cap \hat{I}^\sigma$.

Theorem 5.3. *$\hat{I}^\sigma, \hat{W}^\sigma(\mathcal{L})$ is weakly replete.*

Proof. Let $\hat{\mu} \in I_R^\sigma(\hat{W}^\sigma(\mathcal{L}))$, then $\mu \in I_R^\sigma(\mathcal{L})$ since $L_n \downarrow \phi$ is equivalent to $\hat{W}^\sigma(L_n) \downarrow \phi$. Now the proof will be completed by considering two cases: a) If μ is $\mathcal{L}$-convergent, then there exists an $x \in X : \mu_x \leq \mu(\mathcal{L})$, hence $\hat{\mu}_x \leq \hat{\mu}(\hat{W}^\sigma(\mathcal{L}))$ where $\hat{\mu}_x$ is the measure concentrated at μ_x, so $\hat{\mu}$ is $\hat{W}^\sigma(\mathcal{L})$-convergent. Thus, $\hat{I}^\sigma, \hat{W}^\sigma(\mathcal{L})$ is weakly replete. b) If μ is not $\mathcal{L}$-convergent, then $\mu \in \hat{I}^\sigma$ and so $S(\hat{\mu}) = \cap \hat{W}^\sigma(L)$ but $\hat{\mu}(\hat{W}^\sigma(L)) = 1$ implies that $\mu \in \hat{W}^\sigma(\mathcal{L})$ and $\hat{\mu} \in \hat{I}^\sigma$, then $\mu \in S(\hat{\mu})$ and $\hat{\mu}$ is the measure concentrated at μ. Thus, $\hat{I}^\sigma, \hat{W}^\sigma(\mathcal{L})$ is weakly replete. □

Now we may easily note that if $\hat{W}^\sigma(\mathcal{L})$ is taken as a base for open sets on $\hat{I}^\sigma$ so $\mathcal{O}_\sigma = \cup \hat{W}^\sigma(\mathcal{L})$, then $\mathcal{O}_\sigma$ is T_0.

Theorem 5.4. *If $\mathcal{L} = U(\mathcal{L})$, then $\hat{I}^\sigma, \mathcal{O}_\sigma$ is a generalized α-space and is an α-space if $\mathcal{L}'$ is T_2.*

Proof. a) $\hat{W}^\sigma(\mathcal{L})$ separates $\cup \hat{W}^\sigma(\mathcal{L}) = \mathcal{O}_\sigma$ for if

$$(*) \qquad \left(\bigcup_\alpha \hat{W}^\sigma(L_\alpha)\right) \cap \left(\bigcup_\beta \hat{W}^\sigma(L_\beta)\right) = \phi.$$

Let $A = \cup_\alpha L_\alpha$, $B = \cup_\beta L_\beta$, $A, B \in \mathcal{L}$ since $\mathcal{L} = U(\mathcal{L})$, also $\cup_\alpha \hat{W}^\sigma(L_\alpha) \subset \hat{W}^\sigma(A)$ and $\cup_\beta \hat{W}^\sigma(L_\beta) \subset \hat{W}^\sigma(B)$, and if $A \cap B \neq \phi$, then $L_\alpha \cap L_\beta \neq \phi$ for some α, β. Let $x \in L_\alpha \cap L_\beta$. Then, $\mu_x \in \hat{I}^\sigma$, $\mu_x \in \hat{W}^\sigma(L_\alpha)$ and $\mu_x \in \hat{W}^\sigma(L_\beta)$ which contradicts $(*)$. Thus, $\hat{W}^\sigma(\mathcal{L})$ separates $\cup \hat{W}^\sigma(\mathcal{L})$ and since $\hat{W}^\sigma(\mathcal{L})$ is weakly replete, so is $\mathcal{O}_\sigma = \cup \hat{W}^\sigma(\mathcal{L})$ by Theorem 5.2. Also, $\mathcal{O}_\sigma$ is T_0 by the above note.

Thus, $\hat{I}^\sigma$, $\mathcal{O}_\sigma$ is a generalized α-space. b) (i) Suppose $\mu_x \neq \mu_y$ so $x \neq y$. Then, since $\mathcal{L}'$ is T_2, there exists $L_1, L_2 \in \mathcal{L}$ such that $x \in L_1$, $y \in L_2$, $L_1 \cap L_2 = \phi$. Then $\mu_x \in \hat{W}^\sigma(L_1)$ and $\mu_y \in \hat{W}^\sigma(L_2)$, $\hat{W}^\sigma(L_1) \cap \hat{W}^\sigma(L_2) = \phi$. Thus, $\hat{W}^\sigma(\mathcal{L})$ and therefore $\mathcal{O}_\sigma$ is T_2. (ii) Also, if $\mu_1, \mu_2 \in \hat{I}^\sigma$ and μ_1, μ_2 are not $\mathcal{L}$-convergent and $\mu_1 \neq \mu_2$, then there exist $L_1, L_2 \in \mathcal{L}$ such that $\mu_1(L_1) = \mu_2(L_2) = 1$, $\mu_1(L_2) = \mu_2(L_1) = 0$ and $L_1 \cap L_2 = \phi$, then $\mu_1 \in \hat{W}^\sigma(L_1)$, $\mu_2 \in \hat{W}^\sigma(L_2)$ and $\hat{W}^\sigma(L_1) \cap \hat{W}^\sigma(L_2) = \phi$. Thus, $\hat{W}^\sigma(\mathcal{L})$, and therefore $\mathcal{O}_\sigma$ is T_2. (iii) Finally, if $\mu_x \in \hat{I}^\sigma$ and $\mu \in \hat{I}^\sigma$ and μ is not $\mathcal{L}$-convergent, then since $\mu_x \not\leq \mu(\mathcal{L})$ we can find $L_1, L_2 \in \mathcal{L}$ as before. Thus, $\hat{W}^\sigma(\mathcal{L})$ and therefore $\mathcal{O}_\sigma$ is T_2. □

Remark 5. Now we may easily note that $\bar{\hat{I}}^\sigma = \hat{I}$.

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