ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 4, Fall 1992

GENERALIZED FREE PRODUCTS OF π_c GROUPS

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ABSTRACT. A group G is said to be π_c if and only if for every pair of elements g_1 and g_2 of G either $g_1 = g_2^k$ for some integer k or there exists a normal subgroup N of finite index in G such that $g_1 \not\equiv g_2^z \bmod N$ for all integers z. In this note we prove that a certain generalized free product amalgamating a cyclic subgroup is π_c and then apply the result to show some one-relator groups are π_c.

1. Introduction. A group G is termed π_c if and only if for every pair of elements g_1 and g_2 of G either $g_1 = g_2^k$ for some integer k or there exists a normal subgroup N of finite index in G such that $g_1 \not\equiv g_2^z \bmod N$ for all integers z. Clearly, a π_c group is residually finite. However, the one-relator group $G = \langle a, b; a^{-1}ba = b^2\rangle$ is residually finite but not π_c [**1**]. Examples of π_c groups are the finite groups, free groups and finitely generated torsion-free nilpotent groups [**4, 5**].

In [**7**], Stebe proved that the generalized free products of isomorphic π_c groups amalgamating a cyclic subgroup are again π_c. In this note, we shall prove the following theorem.

Theorem. *Let A and B be π_c groups and let $a \in A$ and $b \in B$ where a and b have the same order. If A is $\langle a\rangle$-Pot and B is $\langle b\rangle$-Pot, then the generalized free product G of A and B amalgamating the subgroups $\langle a\rangle$ and $\langle b\rangle$ with $a = b$, is π_c.*

Modifying [**2**], a group G is termed $\langle x\rangle$-potent (or $\langle x\rangle$-Pot for short) for a nontrivial element x of G if and only if for every positive integer n (dividing the order of x if the order is finite) there exists a normal subgroup N of finite index in G such that xN has order exactly n in G/N. G is termed potent if it is $\langle x\rangle$-Pot for every nontrivial element x

Received by the editors on July 31, 1989, and in revised form on June 15, 1990.
1980 AMS *Mathematics Subject Classification.* Primary 20E25, 20E30; Secondary 20F05.
Key words and phrases. Generalized free product, one-relator group residually finite, π_c, potent.

of G. Free groups and finitely generated torsion-free nilpotent groups are examples of potent groups [**8**].

The following notations will be used for any group G: $N \triangleleft_f G$ means N is a normal subgroup of finite index in G. If G is a generalized free product and $x \in G$, then $||x||$ will denote the usual reduced length of x.

2. Proof of the Theorem. We shall use the following two lemmas in the proof of our theorem.

Lemma 1. *Let G be the generalized free product of groups A and B with amalgamated subgroup H. Let $g_1, g_2 \in G$, where g_1 is reduced and g_2 is cyclically reduced. Then $g_1 \in \langle g_2 \rangle$ only under the following conditions*:

(a) *If $||g_2|| = 0$ or 1, say $g_2 = a$, where $a \in A$ or B, then $g_1 = a^k$ for some integer k.*

(b) *If $||g_2|| = m$, $m > 1$, say $g_2 = y_1 y_2 \cdots y_m$, then $g_1 = x_1 x_2 \cdots x_n$ in reduced form with $n = km$ for some positive integer k. In this case, $g_1 = g_2^z$ if and only if $z = \pm k$. If $g_1 = g_2^k$, then there is a finite sequence of elements $c_0, c_1, \dots, c_{km}$ in H such that $y_j^{-1} c_{j-1} x_j = c_j$, $j = 1, 2, \dots, km$, where $y_i = y_{i+m} = \cdots = y_{i+(k-1)m}$, $i = 1, 2, \dots, m$, and $c_0 = c_{km} = 1$. If $g_1 = g_2^{-k}$, similar equalities hold with y_{km}^{-1} replacing y_1, y_{km-1}^{-1} replacing y_2, etc.*

Proof. Straightforward. □

Lemma 2. *Let G be the generalized free product of groups A and B with amalgamated subgroup H. If A and B are finite, then G is π_c.*

Proof. Since the generalized free product G with finite factors A and B is free-by-finite [**6**] and free-by-finite groups are π_c [**7**], the result follows. □

Now we can prove the Theorem.

Proof of Theorem. Let g_1 and g_2 be elements of G such that $g_1 \notin \langle g_2 \rangle$. To prove the theorem, it is sufficient for us to find a π_c group $\overline{G}$ and a homomorphism θ from G to $\overline{G}$ such that $g_1\theta \notin \langle g_2\theta \rangle$. If $g_2 = 1$, we can easily find such a group $\overline{G}$ since G is residually finite by Theorem 2.1′ of Allenby and Tang [**2**]. So we may assume $g_2 \neq 1$. Furthermore, without loss of generality, we may assume that g_1 is reduced and g_2 is cyclically reduced in G. We divide the proof into several cases.

Case 1. Suppose $g_1 \notin \langle g_2 \rangle$ is implied by the lengths of g_1 and g_2, that is,

(1) if $||g_2|| = 0$, then $||g_1|| > 0$ or

(2) if $||g_2|| = 1$, then $||g_1|| > 1$ or

(3) if $||g_2|| = m > 1$, then $||g_1||$ is not divisible by m or $||g_1|| = 0$.

Suppose $g_1 = x_1x_2\cdots x_n$, $n \geq 1$, and $g_2 = y_1y_2\cdots y_m$, $m \geq 1$, where the x_i and y_i are not in the amalgamated subgroups. Let u_i denote those x_i and y_i in $A\backslash\langle a \rangle$ and v_i denote those x_i and y_i in $B\backslash\langle b \rangle$. Since A and B are π_c, we can find $M_1 \triangleleft_f A$ and $N_1 \triangleleft_f B$ such that $u_i\langle a \rangle \cap M_1 = v_i\langle b \rangle \cap N_1 = \varnothing$. Since M_1 and N_1 are of finite index in A and B, respectively, we have $M_1 \cap \langle a \rangle = \langle a^r \rangle$ and $N_1 \cap \langle b \rangle = \langle b^s \rangle$ for some integers r and s. Now let $t = \mathrm{lcm}\,(r, s)$. Since A is $\langle a \rangle$-Pot and B is $\langle b \rangle$-Pot , we can find $M_2 \triangleleft_f A$ and $N_2 \triangleleft_f B$ such that $M_2 \cap \langle a \rangle = \langle a^t \rangle$ and $N_2 \cap \langle b \rangle = \langle b^t \rangle$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Then, clearly $M \triangleleft_f A$, $N \triangleleft_f B$ and $M \cap \langle a \rangle = N \cap \langle b \rangle$. Thus we can form $\overline{G} = \langle \overline{A} * \overline{B}; \overline{H} \rangle$ where $\overline{A} = A/M$, $\overline{B} = B/N$ and $\overline{H} = \langle a \rangle/(M \cap \langle a \rangle)$. Let θ be the canonical map from G onto $\overline{G}$. Then the choice of M and N implies that $g_1\theta$ is reduced and $g_2\theta$ is cyclically reduced in $\overline{G}$. Furthermore, $||g_1\theta|| = ||g_1||$ and $||g_2\theta|| = ||g_2||$. Hence, $g_1\theta \notin \langle g_2\theta \rangle$. Since $\overline{G}$ is π_c by Lemma 2, our objective is achieved.

Case 2. Suppose $||g_1|| = 0$ and $||g_2|| \leq 1$ or $||g_1|| = ||g_2|| = 1$. The proof of this case can be derived easily from Lemma 3.1 of Allenby and Tang [**2**].

Case 3. Suppose $||g_1|| = km$ and $||g_2|| = m$ for $k \geq 1$ and $m > 1$. Let $g_1 = x_1x_2\cdots x_{km}$ and $g_2 = y_1y_2\cdots y_m$ where the x_i and y_i are not in the amalgamated subgroups. As in case 1, let u_i denote those x_i and y_i

in $A\backslash\langle a\rangle$ and v_i denote those x_i and y_i in $B\backslash\langle b\rangle$. Since A and B are π_c, we can find $M_1 \triangleleft_f A$ and $N_1 \triangleleft_f B$ such that $u_i\langle a\rangle \cap M_1 = v_i\langle b\rangle \cap N_1 = \varnothing$.

By Lemma 1(b), $g_1 = g_2^z$ if and only if $z = \pm k$. So we may consider $g_1 \neq g_2^{\pm k}$. Now $g_1 \neq g_2^{\pm k}$ implies that at least one equation in each of the sets of equations of Lemma 1(b) is not satisfied for both the cases $g_1 \neq g_2^k$ and $g_1 \neq g_2^{-k}$. Suppose $y_i^{-1}c_{i-1}x_i = c_i$ is the first such equation for the case $g_1 \neq g_2^k$, that is, c_i, $i \neq km$, is not in the amalgamated subgroup or $c_{km} \neq 1$. Without loss of generality, we may assume $c_i \in A$. Since A is π_c, we choose $M_2 \triangleleft_f A$ such that $c_i\langle a\rangle \cap M_2 = \varnothing$, $i \neq km$, or $c_{km} \notin M_2$.

Let M_3 be the similarly defined subgroup for the case $g_1 \neq g_2^{-k}$. Let $\overline{M} = M_1 \cap M_2 \cap M_3$. Then $\overline{M} \triangleleft_f A$ and $\overline{M} \cap \langle a\rangle = \langle a^r\rangle$ for some integer r. Also, N_1 is of finite index in B, we have $N_1 \cap \langle b\rangle = \langle b^s\rangle$ for some integer s. Let $t = \operatorname{lcm}(r, s)$. Since A is $\langle a\rangle$-Pot and B is $\langle b\rangle$-Pot, we can find $M_4 \triangleleft_f A$ and $N_2 \triangleleft_f B$ such that $M_4 \cap \langle a\rangle = \langle a^t\rangle$ and $N_2 \cap \langle b\rangle = \langle b^t\rangle$. Let $M = \overline{M} \cap M_4$ and $N = N_1 \cap N_2$. Then $M \triangleleft_f A$, $N \triangleleft_f B$ and $M \cap \langle a\rangle = N \cap \langle b\rangle$. Now we form $\overline{G} = \langle \overline{A} * \overline{B}; \overline{H}\rangle$ where $\overline{A} = A/M$, $\overline{B} = B/N$ and $\overline{H} = \langle a\rangle/(M \cap \langle a\rangle)$. Let θ be the canonical map θ from G onto $\overline{G}$. By the choice of M_1 and N_2, $g_1\theta$ is reduced and $g_2\theta$ is cyclically reduced in $\overline{G}$. Also, $||g_1\theta|| = ||g_1||$ and $||g_2\theta|| = ||g_2||$. Hence, we have $g_1\theta \neq g_2\theta^z$ for $z \neq \pm k$. Since $M \subseteq M_2 \cap M_3$, we also have $g_1\theta \neq g_2\theta^{\pm k}$. Thus, $g_1\theta \notin \langle g_2\theta\rangle$. By Lemma 2, G is π_c, and we are done. This completes the proof of the theorem. □

3. Remark. Our theorem can be used to prove the following generalization of Lemma 2.2 of Allenby and Tang [**3**].

Corollary. *Let* $G = \langle a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m; (u(a_i)v(b_i))^s = 1\rangle$ *where* $s \geq 1$ *and* $u(a_i)$ *and* $v(b_i)$ *are words on the disjoint sets of generators* $a_1, a_2, \dots, a_n$ *and* $b_1, b_2, \dots, b_m$. *Then* G *is* π_c.

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