ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 4, Fall 1992

# THEORY OF THE GENERAL $H$-FUNCTION OF TWO VARIABLES

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ABSTRACT. Rather simple criteria are provided for the determination of the convergence regions of the double Mellin-Barnes integrals which is used to define the $H$-functions of two variables. Several specific examples are included.

**1. Introduction.** In 1978 the general $H$-function of two variables was defined [**2**] by the double Mellin-Barnes integral of the form

$$(1)\quad H[x,y;(\alpha,a,A)_n^m;L_s,L_t] = \frac{1}{(2\pi i)^2}\int_{L_t}\int_{L_s}\frac{\prod_{j=1}^m\Gamma(\alpha_j+a_js+A_jt)}{\prod_{j=m+1}^n\Gamma(\alpha_j+a_js+A_jt)}x^sy^t\,ds\,dt,$$

where $\alpha_j\in\mathbf{C}$; $a_j,A_j\in\mathbf{R}$, $j=1,2,\dots,n$; $m,n$ are nonnegative integers and $m\le n$. Here $L_s$ and $L_t$ are infinite contours in the complex $s$- and $t$-planes, respectively, such that $\alpha_j+a_js+A_jt\ne 0,-1,-2,\dots$ for arbitrary $j=1,2,\dots,m$.

Two monographs [**6, 9**] and more than 250 papers are dedicated to the study of the $H$-functions (1) or somewhat specialized forms thereof. However, the works of many of these authors, for example [**1, 7, 8**], contain inaccuracies in finding the conditions on the parameters $\alpha_j,a_j,A_j$ that provide the convergence of the integral in (1). This fact was noticed and was made precise by Buschman [**2**].

In this paper rather simple criteria are provided for the determination of the convergence of integral (1) in terms of the parameters $\alpha_j,a_j,A_j$, $j=1,2,\dots,n$.

**2. Convergence regions results.**

**Theorem.** *Let the contours $L_s$ and $L_t$ have vertical form, i.e.,* $\operatorname{Re}(s)$ *and* $\operatorname{Im}(s)$ *are restricted for $s\in L_s$, $t\in L_t$. Then the integral*

Received by the editors on February 15, 1990, and in revised form on September 6, 1990.

*in* (1) *converges provided the following inequalities are satisfied for* $k = 1, 2, \dots, n$.
(2)
$$\rho_k = \sum_{j=1}^{n} \operatorname{sgn}(m + 1/2 - j) \left| \det \begin{bmatrix} a_j & A_j \\ a_k & A_k \end{bmatrix} \right| > \frac{2}{\pi} \left| \det \begin{bmatrix} \arg x & \arg y \\ a_k & A_k \end{bmatrix} \right|.$$

*If we replace the inequality symbol* ($>$) *in conditions* (2) *by the opposite one* ($<$) *for at least one* $k$, $1 \le k \le n$, *then the integral* (1) *will diverge.*

In order to prove this theorem we need the following additional lemma.

**Lemma.** *Let all* $\gamma_j$, $a_j$, $A_j \in \mathbf{R}$, $j = 1, 2, \dots, n$. *Then for*

$$f(u, v) = \sum_{j=1}^{n} \gamma_j |a_j u + A_j v| > 0, \qquad u, v \in \mathbf{R}, u^2 + v^2 \neq 0, \tag{3}$$

*it is necessary and sufficient that*
(4)
$$\hat{\rho}_k = f(A_k, -a_k) = \sum_{j=1}^{n} \gamma_j \left| \det \begin{bmatrix} a_j & A_j \\ a_k & A_k \end{bmatrix} \right| > 0, \qquad k = 1, 2, \dots, n.$$

*In this case the following inequalities are true*

$$f(u, v) \ge \frac{\hat{\rho}_k}{|a_k|} |v|, \qquad a_k \neq 0, k = 1, 2, \dots, n; \tag{5}$$

$$f(u, v) \ge \frac{\hat{\rho}_k}{|A_k|} |u|, \qquad A_k \neq 0, k = 1, 2, \dots, n. \tag{6}$$

*Proof of the Lemma.* Although the function $f(u, v)$ is not linear in the plane, it is linear in sectors of the plane. Note that all lines

$$L_j : a_j u + A_j v = 0, \qquad j = 1, 2, \dots, n$$

separate the $(u, v)$-plane into nonintersecting sectors. It is evident that in each sector the function $f(u, v)$ of (3) is linear in the variables $u$ and

$v$. Let the point $(u_0, v_0)$ be located in some sector which is bounded by two neighboring lines $L_{j_1}, L_{j_2}$. Then there are points $(u_1, v_1)$ and $(u_2, v_2)$ on the lines $L_{j_1}, L_{j_2}$, respectively, such that $u_0 = u_1 + u_2$, $v_0 = v_1 + v_2$. Hence, due to the linearity of $f(u, v)$ in this sector, we have $f(u_0, v_0) = f(u_1, v_1) + f(u_2, v_2)$. This last equality allows us to conclude that for $f(u, v) > 0$, $u^2 + v^2 \neq 0$, it is necessary and sufficient that $f(u, v) > 0$ for all points $(u, v)$ located on the lines $L_j$. This is equivalent to condition (4).

To obtain inequality (5) we transform $f(u, v)$, for $v \neq 0$, as follows.

$$f(u, v) = |v| f(u/v, 1) = |v| f_1(\gamma), \qquad \gamma = u/v.$$

The function

$$f_1(\gamma) = \sum_{j=1}^{n} \gamma_j |a_j \gamma + A_j|$$

is continuous and positive, hence

$$\min_{\mathbf{R}} f_1(\gamma) = \min_{a_k \neq 0} f_1(-A_k/a_k) = \min_{a_k \neq 0} \{\hat{f}_k / |a_k|\}.$$

Inequality (6) is proved by analogy. $\square$

*Proof of the theorem.* We use the following estimations from [**5**].

$$\Gamma(a + s) \sim H_1 \exp(-(\pi/2) \mathrm{Im}\,(s));$$
$$|x^s| \sim H_2 \exp(-\mathrm{Im}\,(s) \arg(x)),$$

where $\mathrm{Re}\,(s)$ is restricted, $\mathrm{Im}\,(s) \to \infty$ and $H_1$ and $H_2$ are of lower order than exponential. With the help of these estimations we have, for $\mathrm{Im}\,(s)$, $\mathrm{Im}\,(t) \to \infty$, and restricted $\mathrm{Re}\,(s), \mathrm{Re}\,(t)$ that

$$\left| \frac{\prod_{j=1}^{m} \Gamma(\alpha_j + a_j s + A_j t)}{\prod_{j=m+1}^{n} \Gamma(\alpha_j + a_j s + A_j t)} x^s y^t \right|$$
$$\sim H_3 \exp\left\{ -\left[ \sum_{j=1}^{n} \mathrm{sgn}\,(m + 1/2 - j) |a_j \mathrm{Im}\,(s) + A_j \mathrm{Im}\,(t)| \pi/2 \right.\right.$$
$$\left.\left. + \mathrm{Im}\,(s) \arg(x) + \mathrm{Im}\,(t) \arg(y) \right] \right\},$$

where $H_3$ is of lower order than exponential. Let $\mathrm{Im}\,(s) = u$, $\mathrm{Im}\,(t) = v$. Then for the convergence of the integral (1), it is sufficient that

$$g(u,v) = \sum_{j=1}^{n} (\pi/2)\,\mathrm{sgn}\,(m+1/2-j)|a_j u + A_j v| - |u \arg(x) + v \arg(y)| \to \infty$$

for $u, v \in \mathbf{R}$, $u^2 + v^2 \to \infty$. (Evidently, if there exist $u, v \in \mathbf{R}$ and $g(u,v) < 0$, then integral (1) diverges.)

Inasmuch as $g(\lambda u, \lambda v) = |\lambda| g(u,v)$ this rescaling does not alter $g(u,v) > 0$ for $u^2 + v^2 \neq 0$. It follows from our Lemma that for the convergence of this integral (1) it is sufficient that for $k = 1, 2, \dots, n$ the following two sets of conditions are true

(7)
$$\rho_k = \sum_{j=1}^{n} \mathrm{sgn}\,(m+1/2-j)\left|\det \begin{bmatrix} a_j & A_j \\ a_k & A_k \end{bmatrix}\right| > \frac{2}{\pi}\left|\det \begin{bmatrix} \arg(x) & \arg(y) \\ a_k & A_k \end{bmatrix}\right|$$

$$\sum_{j=1}^{n} \mathrm{sgn}\,(m+1/2-j)\left|\det \begin{bmatrix} a_j & A_j \\ \arg(x) & \arg(y) \end{bmatrix}\right| > 0. \tag{8}$$

It remains to be proved that (8) follows from (7). Indeed, we see from (7) that $\rho_k > 0$, $k = 1, 2, \dots, n$. Then from the Lemma we get the inequality

$$g_1(u,v) = \sum_{j=1}^{n} \mathrm{sgn}\,(m+1/2-j)|a_j u + A_j v| > 0, \qquad u^2 + v^2 \neq 0.$$

In particular, then

$$g_1(\arg(y), -\arg(x)) = \sum_{j=1}^{n} \mathrm{sgn}\,(m+1/2-j)\left|\det \begin{bmatrix} a_j & A_j \\ \arg(x) & \arg(y) \end{bmatrix}\right| > 0.$$

This completes the proof of the Theorem. $\square$

*Consequence* 1. Let $\rho_k > 0$ for $k = 1, 2, \dots, n$ and

$$H_x = \min_{A_k \neq 0} \left\{ \frac{\rho_k}{|A_k|} \right\}, \qquad H_y = \min_{a_k \neq 0} \left\{ \frac{\rho_k}{|a_k|} \right\}.$$

Then the integral in (1) converges if the sufficient condition holds

$$\frac{|\arg(x)|}{H_x} + \frac{|\arg(y)|}{H_y} < \frac{\pi}{2}. \tag{9}$$

*Proof.* From (5) and (6) of the Lemma we have the following inequalities

$$g_1(u,v) = \sum_{j=1}^{n} \operatorname{sgn}(m+1/2-j)|a_j u + A_j v| \geq H_x|u|;$$
$$g_2(u,v) = \sum_{j=1}^{n} \operatorname{sgn}(m+1/2-j)|a_j u + A_j v| \geq H_y|v|.$$

Hence, it follows that

$$g_1(u,v) \geq \varepsilon H_x|u| + (1-\varepsilon)H_y|v|, \qquad 0 < \varepsilon < 1.$$

Consequently, we get the inequality

$$\begin{aligned} g(u,v) &= (\pi/2)g_1(u,v) - |u\arg(x) + v\arg(y)| \\ &\geq |u|(\varepsilon H_x\pi/2 - |\arg(x)| + |v|)((1-\varepsilon)H_y\pi/2 - |\arg(y)|). \end{aligned}$$

Therefore, the integral in (1) converges if

$$|\arg(x)| < \varepsilon H_x\pi/2, \qquad |\arg(y)| < (1-\varepsilon)H_y\pi/2,$$

which is equivalent to (9). □

*Consequence* 2. After combining all pairs such that $(a_{j_1}, A_{j_1}) = \lambda(a_{j_2}, A_{j_2})$, we have

$$\sum_{j=1}^{n} \operatorname{sgn}(m+1/2-j)|a_j u + A_j v| = \sum_{j=1}^{p} c_j|b_j u + B_j v|,$$

where $p \leq n$ and $b_j B_k - b_k B_j \neq 0$ for $j \neq k$. Then the integral in (1) converges if for all $k = 1, 2, \ldots, p$ the following inequalities are satisfied

$$\rho_k^* = \sum_{j=1}^{p} c_j \left| \det \begin{bmatrix} b_j & B_j \\ b_k & B_k \end{bmatrix} \right| > \frac{2}{\pi} \left| \det \begin{bmatrix} \arg(x) & \arg(y) \\ b_k & B_k \end{bmatrix} \right|. \tag{10}$$

If in (10) the inequality symbol ($>$) is reversed ($<$) for at least one $k$, $1 \le k \le p$, then the integral in (1) will diverge.

*Consequence* 3. Let $\rho_k^* > 0$ for $k = 1, 2, \dots, n$ and

$$H_x^* = \min_{B_k \neq 0} \left\{ \frac{\rho_k^*}{|B_k|} \right\}, \qquad H_y^* = \min_{b_k \neq 0} \left\{ \frac{\rho_k^*}{|b_k|} \right\}.$$

Then, analogous to Consequence 1, the sufficient condition is

$$\frac{|\arg(x)|}{H_x^*} + \frac{|\arg(y)|}{H_y^*} < \frac{\pi}{2}. \tag{11}$$

*Note* 1. In Buschman [**2**], the sufficient condition

$$\sum_{j=1}^{n} \operatorname{sgn}(m + 1/2 - j)|a_j u + A_j v| \geq K_x|u| + K_y|v|,$$

where $|\arg(x)| < K_x\pi/2$, $|\arg(y)| < K_y\pi/2$ with $K_x, K_y > 0$. Meanwhile, it follows from Consequence 1 that it is possible to set $K_x = \varepsilon H_x^*$, $K_y = (1-\varepsilon)H_y^*$, for $0 < \varepsilon < 1$. But, see also Example 1 of Section 3.

*Note* 2. The following conditions are encountered in the works [**1, 6, 7, 8**], among others

$$|\arg(x)| < \frac{\pi}{2} \sum_{j=1}^{n} \operatorname{sgn}(m + 1/2 - j)|a_j|; \tag{12}$$

$$|\arg(y)| < \frac{\pi}{2} \sum_{j=1}^{n} \operatorname{sgn}(m + 1/2 - j)|A_j|. \tag{13}$$

Although these conditions are necessary, they are not sufficient. They do not provide the convergence of the integral in (1). This may be illustrated by the example

$$\int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \Gamma(1 + s + t) x^s y^t \, ds \, dt. \tag{14}$$

Indeed, the integral (14) satisfies the conditions (12), (13), but at the same time this integral diverges for all $(x, y)$, $x^2 + y^2 \neq 0$.

**3. Examples of regions.** We provide several examples in order to illustrate some of the various types of regions of convergence which occur. For the study of these examples, we introduce the notation

$$\theta(s,t) = \frac{\prod_{j=1}^{m} \Gamma(\alpha_j + a_j s + A_j t)}{\Gamma_{j=m+1}^{n} \Gamma(\alpha_j + a_j s + A_j t)}.$$

*Example* 1. $\theta(s,t) = \Gamma(1+s)\Gamma(1+t)$. The domain of convergence is given by $\{|\arg(x)| < \pi/2, |\arg(y)| < \pi/2\}$. Note that it is impossible to extend this domain to $\{|\arg(x)| + |\arg(y)| < \pi\}$. In fact, it is not possible to extend the domain at all which illustrates the error in Buschman [**3**].

The best that can be done by the method in [**3**], after removing the error, is a mild extension if $K_x \neq K_y$. For example, if we assume that $K_x > K_y > 0$, $|u| > |v|$, then we have

$$K_x|u| + K_y|v| \geq \gamma|u| + (K_x + K_y - \gamma)|v|, \qquad \text{for } \gamma \leq K_x.$$

This provides extensions to points for $K_x \leq \gamma \leq (K_x + K_y)/2$. (A similar situation occurs if $K_y > K_x > 0$.)

*Example* 2. $\theta(s,t) = (\Gamma(1+2s+2t)\Gamma(1-3s)\Gamma(1-2t))/(\Gamma(1-s-t)\Gamma(1/2-s))$. We have

$$\theta(s,t) \sim H_4 \exp\{(-\pi/2)(|u+v| + 2|u| + 2|v|)\};$$

hence

$$h(u,v) = |u+v| + 2|u| + 2|v| > \frac{2}{\pi}|u \arg(x) + v \arg(y)|.$$

After some computations we obtain the inequalities

$$|\arg(x) - \arg(y)| < 2\pi, \qquad |\arg(y)| < 3\pi/2, \qquad |\arg(x)| < 3\pi/2$$

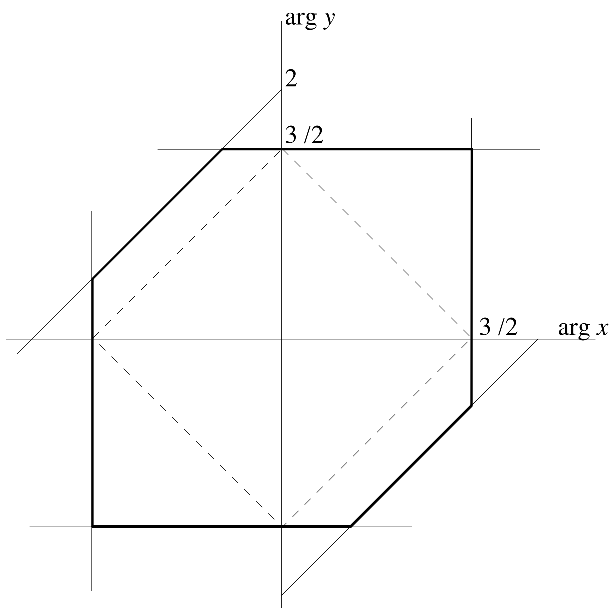


Diagram for Example 2.

which describe the region of convergence. The sufficient condition (14) leads to the somewhat smaller region

$$|\arg(x)|/3 + |\arg(y)|/3 < \pi/2.$$

The boundaries of these domains are shown in the diagram by heavy solid and by dashed line segments, respectively.

*Example* 3. $\theta(s,t) = (\Gamma(1+s+t)\Gamma(1-3s)\Gamma(1-4t))/\Gamma(1+2s+3t)$. Here the results of Section 2 lead to the inequalities

$$|\arg(x)+\arg(y)| < \pi, \qquad |\arg(y)| < \pi, \qquad |\arg(x)| < \pi,$$
$$|3\arg(x) - 2\arg(y)| < 11\pi.$$

The first three inequalities determine the region. Since $H_x^* = H_y^* = 1$, we again see that condition (14) is more restrictive.

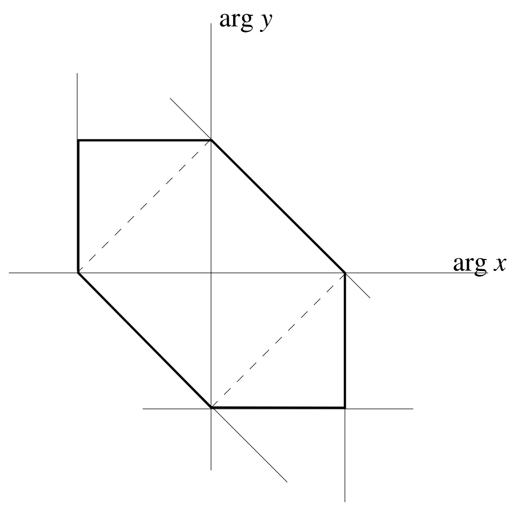


Diagram for Example 3.

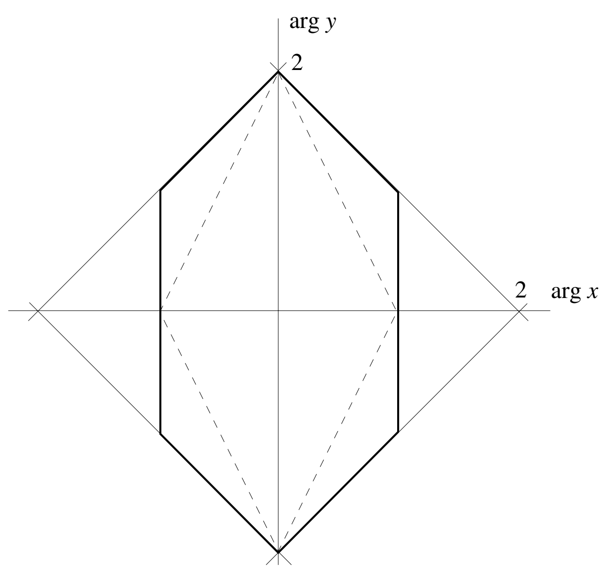


Diagram for Example 5.

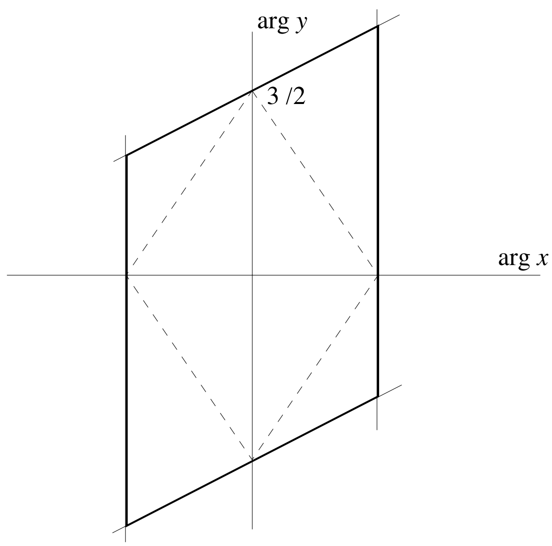


Diagram for Example 6.

*Example* 4. $\theta(s,t) = (\Gamma(1+2s+3t)\Gamma(1+s-t)\Gamma(s))/\Gamma(1+2s-3t)$. Since $\rho_1 = -4 < 0$, the integral diverges, although the necessary conditions (12), (13) are satisfied.

*Example* 5. (Tandon [**10**], see also the sequel, Buschman and Srivastava [**4**]). $\theta(s,t) = (\Gamma(a+s-t)\Gamma(b+s+t)\Gamma(c+t)\Gamma(-s)\Gamma(-t))/\Gamma(d+s)$. The inequalities which describe the convergence region are

$$|\arg(x)+\arg(y)| < 2\pi, \qquad |\arg(x)-\arg(y)| < 2\pi, \qquad |\arg x| < \pi.$$

Since $H_x^* = 2$, $H_y^* = 4$, the smaller region is described by

$$\frac{|\arg(x)|}{4} + \frac{|\arg(y)|}{2} < \frac{\pi}{2}.$$

*Example* 6. $\theta(s,t) = (\Gamma(a+2s-t)\Gamma(b_1+t)\Gamma(b_2+t)\Gamma(-s)\Gamma(-t))/\Gamma(c+s)$. In this case we have only two inequalities

$$|-\arg(x)+2\arg(y)| < 3\pi, \qquad |\arg(x)| < \pi.$$

Further, we have, from $H_x^* = 3$, $H_y^* = 2$, the restricted region

$$\frac{|\arg(x)|}{3} + \frac{|\arg(y)|}{2} < \frac{\pi}{2}.$$

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