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# QUASI-DECOMPOSITION OF ABELIAN GROUPS AND BAER'S LEMMA

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**1. Introduction.** In this paper we consider an abelian group $A$. The *class of $A$-projective groups* (*of finite $A$-rank*) is obtained by closing $\{A\}$ under (finite) direct sums and direct summands. The *$A$-socle of an abelian group $G$*, which is denoted by $S_A(G)$, is the subgroup of $G$ which is generated by $\{\phi(A) | \phi \in \mathrm{Hom}\,(A,G)\}$. Finally, the group $A$ is *self-small* if, for all index-sets $I$ and all $\alpha \in \mathrm{Hom}\,(A, \oplus_I A)$, there is a finite subset $J$ of $I$ with $\alpha(A) \subseteq \oplus_j A$. Clearly, torsion-free abelian groups of finite rank are self-small. Other examples of self-small abelian groups can be found in [**7**].

The group $A$ has the (*finite*) *Baer-splitting property* if every exact sequence $0 \to B \overset{\alpha}{\to} G \overset{\beta}{\to} P \to 0$ such that $\alpha(B) + S_A(G) = G$ and $P$ is $A$-projective (of finite $A$-rank) splits. Baer verified in [**8**] that every subgroup of the rational numbers has the Baer-splitting property. In [**3**, Theorem 2.1 and Corollary 2.2], a complete characterization of the self-small abelian groups $A$ having the (finite) Baer-splitting property was obtained which extends Arnold's and Lady's results of [**6**].

Unfortunately, a splitting result like Baer's Lemma often has limited applications in the discussion of torsion-free groups of finite rank since the splitting of a short exact sequence of these groups occurs less frequently than its quasi-splitting. Because of this, we introduce the following weaker, but perhaps more useful version of the Baer-splitting property which is based on the idea of quasi-isomorphism introduced by Jonsson in the 1950s ([**10,11**]): A torsion-free abelian group $A$ has the (*finite*) *quasi-Baer-splitting property* if every exact sequence $0 \to B \overset{\alpha}{\to} C \overset{\beta}{\to} G \to 0$, in which $G$ is isomorphic to a torsion-free quasi-summand of an $A$-projective group (of finite $A$-rank), and $C$ and $\alpha(B) + S_A(C)$ are quasi-equal, quasi-splits. Theorem 2.3 and Corollary 2.4 give a complete characterization of the self-small abelian groups $A$ having the (finite) quasi-Baer-splitting property in terms of the $E(A)$-module structure of $A$ where the symbol $E(A)$ denotes the

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endomorphism ring of $A$. Moreover, Corollary 2.5 includes the partial results that were obtained in [**6**, Corollary 2.6]. However, we give an example of a torsion-free group $A$ of finite rank which has the quasi-Baer-splitting property, but is not one of the groups described in [**6**, Corollary 2.6] (Example 2.7).

On the other hand, one of the disadvantages usually associated with properties of an abelian group $A$, which can only be described in terms of the $E(A)$-module structure of $A$, is that this description provides virtually no information about the internal group-structure of $A$ itself. A typical example where this problem occurs is the Baer splitting property. We can however overcome this disadvantage for finite rank groups having the finite quasi-Baer-splitting property.

By [**9**, Theorem 92.5], every torsion-free abelian group $A$ of finite rank is quasi-isomorphic to a group of the form $A_1^{m_1} \oplus \cdots \oplus A_n^{m_n}$ where the $A_i$'s are strongly indecomposable, $m_1, \ldots, m_n > 0$, and $A_i$ is quasi-isomorphic to $A_j$ only if $i = j$. Theorem 3.1 gives a necessary and sufficient condition on the family $\{A_1, \ldots, A_n\}$ which ensures that $A$ has the finite quasi-Baer-splitting property. Moreover, this condition is independent of the chosen quasi-decomposition of $A$ into strongly indecomposable subgroups.

The notation of this paper mostly is the standard one introduced in [**9, 15**] while exceptions and additions are listed at the beginning of Section 2. Especially, all maps are written on the left.

**2. Endomorphism rings and quasi-splitting.** The purpose of the first part of this section is to introduce the notation used in this paper. We consider an abelian group, $A$, and define an adjoint pair of functors between $\mathcal{A}b$, the category of abelian groups, and $\mathcal{M}_{E(A)}$, the category of right $E(A)$-modules, in the following way: Since the group $A$ carries a natural left $E(A)$-module structure, the tensor-product $T_A(M) = M \otimes_{E(A)} A$ is well defined for all right $E(A)$-modules, and gives rise to a covariant functor $T_A : \mathcal{M}_{E(A)} \to \mathcal{A}b$. Conversely, composition of maps induces a right $E(A)$-module structure on $H_A(G) = \mathrm{Hom}\,(A, G)$ for all abelian groups $G$. The resulting functor $H_A : \mathcal{A}b \to \mathcal{M}_{E(A)}$ is a right adjoint to $T_A$ by [**13**, Theorem 2.11].

There consequently exist natural homomorphisms $\theta_G : T_A H_A(G) \to G$ and $\phi_M : M \to H_A T_A(M)$ for all $G \in \mathcal{A}b$ and $M \in \mathcal{M}_{E(A)}$ which are defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\phi_M(m)](a) = m \otimes a$ where $\alpha \in H_A(G)$, $m \in M$ and $a \in A$. Arnold and Lady showed in [**6**] that $H_A$ and $T_A$ yield a category equivalence between the category of $A$-projective groups of finite $A$-rank and the category of finitely generated projective right $E(A)$-modules. If $A$ is self-small, then the finiteness conditions can be removed from Arnold's and Lady's result [**7**].

The concepts of quasi-isomorphism and quasi-splitting are strongly associated with the discussion of torsion-free abelian groups of finite rank, but are used in a more general setting in this paper. To avoid any possible confusion, in the next paragraph we give definitions of these concepts appropriate for modules over arbitrary rings $R$:

A submodule $U$ of a right $E(A)$-module $M$ is quasi-equal to $M$ if there exists a nonzero integer $n$ such that $nM \subseteq U \subseteq M$. We write $U \doteq M$ in this case. Two $R$-modules $M$ and $N$ are quasi-isomorphic if there exist a nonzero integer $n$ and maps $\alpha \in \mathrm{Hom}_R(M, N)$ and $\beta \in \mathrm{Hom}_R(N, M)$ with $\alpha\beta = n \cdot id_N$ and $\beta\alpha = n \cdot id_M$ where $id_N$ denotes the identity map on $N$. We write $M \dot{\sim} N$ and call $\alpha$ and $\beta$ quasi-isomorphisms. Finally, let $M$ and $N$ be $R$-modules and $\alpha \in \mathrm{Hom}_R(M, N)$ such that $\alpha(M) \doteq N$. We say that $\alpha$ quasi-splits if there are a map $\beta \in \mathrm{Hom}_R(N, M)$ and a nonzero integer $n$ with $\alpha\beta = n \cdot id_N$. Moreover, if the additive groups of $M$ and $N$ are torsion-free, then a map $\alpha \in \mathrm{Hom}_R(M, N)$ with $\alpha(M) \doteq N$ quasi-splits if and only if $\ker \alpha$ is a direct summand of a submodule of $M$ which is quasi-equal to $M$. We say that $N$ is isomorphic to a quasi-summand of $M$ in this case.

**Proposition 2.1.** *Let $A$ be a torsion-free abelian group.*

a) *The functors $H_A$ and $T_A$ preserve quasi-isomorphisms and quasi-splitting homomorphisms.*

b) *If $B$ and $G$ are abelian groups such that $\theta_G$ is a quasi-isomorphism, and if there exists a quasi-splitting map $\sigma : G \to B$, then $\theta_B$ is a quasi-isomorphism.*

*Proof.* Both a) and b) are an immediate consequence of the fact that $\theta$ is a natural transformation of $T_A H_A$ to the identity functor. $\square$

We continue with a technical lemma which is frequently used in the following:

**Lemma 2.2.** *Let $R$ be a ring.*

a) *If*

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\mu} \\ K & \xrightarrow[\beta]{} & L \end{array}$$

*is a commutative diagram of right $R$-modules and $R$-module homomorphisms such that $\mu$ and $\alpha$ quasi-split, then so does $\beta$.*

b) *If $P$ is a projective $R$-module, and the $R$-module $U$ admits a quasi-splitting map $\pi : P \to U$, then every map $\alpha \in \operatorname{Hom}_R(M, U)$ with $\alpha(M) \doteq U$ quasi-splits.*

*Proof.* a) Choose nonzero integers $m$ and $n$ as well as maps $\delta \in \operatorname{Hom}_R(N, M)$ and $\tau \in \operatorname{Hom}_R(L, N)$ such that $\alpha\delta = m \cdot id_N$ and $\mu\tau = n \cdot id_L$. Then $\beta(\nu\delta\tau) = \mu\alpha\delta\tau = \mu(m \cdot id_N)\tau = nm \cdot id_L$ which particularly yields $\beta(K) \doteq L$.

b) Choose nonzero integers $m$ and $n$ and a homomorphism $\delta : U \to P$ such that $\pi\delta = n \cdot id_U$ and $mU \subseteq \alpha(M)$. Since $P$ is projective, and $[(m \cdot id_U)\pi](P) \subseteq mU \subseteq \alpha(M)$, there is a map $\beta \in \operatorname{Hom}_R(P, M)$ such that $\alpha\beta = (m \cdot id_U)\pi$. Because of $\alpha(\beta\delta) = (m \cdot id_U)\pi\delta = (m \cdot n) \cdot id_U$, the map $\alpha$ quasi-splits. □

We are now able to characterize the self-small torsion-free abelian groups $A$ which have the (finite) quasi-Baer-splitting property in terms of their $E(A)$-module structure.

**Theorem 2.3.** *The following conditions are equivalent for a self-small torsion-free abelian group $A$:*

a) *$A$ has the quasi-Baer-splitting property.*

b) *A right $E(A)$-module $M$ such that $T_A(M)$ is bounded is itself bounded as abelian group.*

*Proof.* a) $\Rightarrow$ b). Let $M$ be a right $E(A)$-module such that $T_A(M)$ is a bounded abelian group, and choose an exact sequence $P_1 \overset{\phi_1}{\to} P_0 \overset{\phi_0}{\to} M \to 0$ of right $E(A)$-modules in which $P_0$ and $P_1$ are projective. Since $T_A$ is right exact, $T_A(P_0)/\operatorname{im} T_A(\phi_1) \cong T_A(M)$ is bounded; and consequently, $\operatorname{im} T_A(\phi_1)$ is quasi-equal to the $A$-projective group $T_A(P_0)$. Because $T_A(P_1)$ is also $A$-projective, a) implies that the canonical map $T_A(P_1) \to \operatorname{im} T_A(\phi_1)$ is quasi-splitting, and therefore, by an application of Lemma 2.2a, so is $T_A(\phi_1) : T_A(P_1) \to T_A(P_0)$. Now consider the commutative diagram

$$\begin{tikzcd}
H_AT_A(P_1) \arrow[r, "H_AT_A(\phi_1)"] \arrow[d, "\Phi_{P_0}"'] & H_AT_A(P_0) \arrow[d, "\Phi_{P_1}"] \\
P_1 \arrow[r, "\phi_1"'] & P_0
\end{tikzcd}$$

where the maps $\Phi_{P_0}$ and $\Phi_{P_1}$ are isomorphisms and $H_AT_A(\phi_1)$ quasi-splits by Proposition 2.1a. Another application of Lemma 2.2a yields the quasi-splitting of $\phi_1$, i.e., there is a map $\sigma \in \operatorname{Hom}(P_0, P_1)$ and a nonzero integer $k$ such that $\phi_1\sigma = k \cdot id_{P_0}$. Hence, $kP_0 \subseteq \phi_1(P_1)$ and $M \cong P_0/\phi_1(P_1)$ is bounded, as desired.

b) $\Rightarrow$ a). Consider the exact sequence $0 \to B \overset{\alpha}{\to} C \overset{\beta}{\to} G \to 0$ of torsion-free abelian groups where $G$ is isomorphic to a quasi-summand of an $A$-projective group $P$ and $C \doteq S_A(C) + \alpha(B)$. Since $\theta_P : T_AH_A(P) \to P$ is an isomorphism, and there is a quasi-splitting map $\delta : P \to G$, Proposition 2.1b implies that the map $\theta_G$ is a quasi-isomorphism. Consider the diagram

$$\begin{tikzcd}
T_AH_A(C) \arrow[r, "T_AH_A(\beta)"] \arrow[d, "\theta_C"'] & T_AH_A(G) \arrow[r] \arrow[d, "\theta_G"] & T_A(\operatorname{coker} H_A(\beta)) \arrow[r] & 0 \\
C \arrow[r, "\beta"'] & G \arrow[r] & 0 &
\end{tikzcd}$$

which is commutative and has exact rows. By our assumption on $C$, we obtain $G \doteq \beta(S_A(C)) = \beta\theta_C(T_AH_A(C)) = [\theta_GT_AH_A(\beta)](T_AH_A(C))$. Therefore, we can find a nonnegative integer $n$ such that $nG \subseteq$

$\operatorname{im}[\theta_G T_A H_A(\beta)]$. Since $\theta_G$ is a quasi-isomorphism there exist a map $\sigma : G \to T_A H_A(G)$ and a nonzero integer $k$ such that $\theta_G \sigma = k \cdot id_G$ and $\sigma\theta_G = k \cdot id_{T_A H_A(G)}$. It follows that the group

$$T_A(\operatorname{coker} H_A(\beta)) \cong \operatorname{coker} T_A H_A(\beta) = T_A H_A(G)/\operatorname{im} T_A H_A(\beta)$$

is bounded by $n \cdot k$: for if $x \in T_A H_A(G)$, then there is $y \in T_A H_A(C)$ such that $nkx = n\sigma\theta_G(x) = \sigma(n\theta_G(x)) = [\sigma\theta_G T_A H_A(\beta)](y) = [kT_A H_A(\beta)](y)$ is contained in $\operatorname{im} T_A H_A(\beta)$. By the hypothesis b), $\operatorname{coker} H_A(\beta)$ is also bounded; and consequently, $H_A(G) \doteq \operatorname{im} H_A(\beta)$. Therefore, since $H_A(\delta) : H_A(P) \to H_A(G)$ quasi-splits by Proposition 2.1a, and $H_A(P)$ is a projective right $E(A)$-module, Lemma 2.2b implies that the map $H_A(\beta) : H_A(C) \to H_A(G)$ quasi-splits. Recalling that $\theta_G$ is a quasi-isomorphism, we see that Lemma 2.2a can be applied to the diagram above to show that $\beta : C \to G$ quasi-splits. $\square$

The proof of the last result can be adopted easily to prove the equivalence of b) and c) in the following corollary.

**Corollary 2.4.** *The following conditions are equivalent for a self-small, torsion-free abelian group $A$:*

a) *If $I$ is a right ideal of $E(A)$ such that $A/IA$ is bounded, then $E(A)/I$ is bounded as an abelian group.*

b) *A finitely generated right $E(A)$-module $M$ is bounded as abelian group if $T_A(M)$ is bounded.*

c) *$A$ has the finite quasi-Baer-splitting property.*

*Proof.* Since the implication b) $\Rightarrow$ a) is obvious, it remains to show its converse in view of the remarks preceding the corollary:

a) $\Rightarrow$ b). As a first step, we show that $\operatorname{Tor}^1_{E(A)}(N, A)$ is bounded for all cyclic right $E(A)$-modules $N$ whose additive group is bounded. Let $n$ be a nonzero integer, and suppose $\alpha : E(A) \to E(A)$ is multiplication by $n$. If $\pi : E(A) \to E(A)/nE(A)$ denotes the canonical projection, then the exact sequence $0 \to E(A) \xrightarrow{\alpha} E(A) \xrightarrow{\pi} E(A)/nE(A) \to 0$ induces the top-row of the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Tor}^1_{E(A)}(E(A)/nE(A), A) & \longrightarrow & T_A(E(A)) & \xrightarrow{T_A(\alpha)} & T_A(E(A)) \\
 & & & \downarrow \theta & & \downarrow \theta \\
 & 0 & \longrightarrow & A & \xrightarrow[n]{} & A
\end{array}$$

in which $\theta$ is defined by $\theta(r \otimes a) = r(a)$ for all $r \in E(A)$ and $a \in A$. The diagram commutes because $[\theta T_A(\alpha)](r \otimes a) = \theta(nr \otimes a) = nr(a)$ and $n\theta(r \otimes a) = nr(a)$. Furthermore, the map $\theta$ obviously is an isomorphism. Thus, $T_A(\alpha)$ is one-to-one and $\mathrm{Tor}^1_{E(A)}(E(A)/nE(A), A) = 0$.

Suppose that $I$ is a right ideal of $E(A)$ which contains $nE(A)$. The exact sequence $0 \to I/nE(A) \to E(A)/nE(A) \to E(A)/I \to 0$ induces the exact sequence

$$0 = \mathrm{Tor}^1_{E(A)}(E(A)/nE(A), A) \to \mathrm{Tor}^1_{E(A)}(E(A)/I, A) \to T_A(I/nE(A)).$$

Since $T_A(I/E(A))$ is bounded because $I/nE(A)$ is, the same is true for $\mathrm{Tor}^1_{E(A)}(E(A)/I, A)$. This shows that $\mathrm{Tor}^1_{E(A)}(N, A)$ is bounded for all cyclic right $E(A)$-modules $N$ which are bounded as abelian groups.

To complete the proof, suppose that $M = \langle m_1, \dots, m_r \rangle$ is a finitely generated right $E(A)$-module such that $T_A(M)$ is bounded. If $r = 1$, then $M$ is bounded as an abelian group by a). In the case $r > 1$, set $U = \langle m_1, \dots, m_{r-1} \rangle$; and consider the exact sequence $\mathrm{Tor}^1_{E(A)}(M/U, A) \to T_A(U) \to T_A(M) \to T_A(M/U) \to 0$ which is induced by the inclusion $U \subseteq M$. The module $M/U$ is a cyclic right $E(A)$-module such that $T_A(M/U)$ is bounded. By a), $M/U$ itself is bounded as an abelian group. The results of the first step of this proof guarantee that the same holds for $\mathrm{Tor}^1_{E(A)}(M/U, A)$. Consequently, $T_A(U)$ is bounded; and the same is true for the additive group of $U$ by induction. But this implies that the module $M$ is bounded as an abelian group. □

In the next corollary, we apply the preceding result to obtain a purely ring-theoretical condition on $\mathbf{Q}E(A)$ that is sufficient, but not necessary, for the group $A$ to have the finite quasi-Baer-splitting property.

**Corollary 2.5.** *A self-small, torsion-free abelian group $A$, such that every proper right ideal of $\mathbf{Q}E(A) = \mathbf{Q} \otimes_{\mathbf{Z}} E(A)$ has a nonzero left annihilator, has the finite quasi-Baer-splitting property.*

*Proof.* Suppose that $I$ is a right ideal of $E(A)$ such that $A/IA$ is bounded. If $E(A)/I$ is not bounded as an abelian group, then the $\mathbf{Z}$-purification $I_*$ of $I$ in $E(A)$ is a proper right ideal of $E(A)$. Consequently, $\mathbf{Q}I_*$ is a proper right ideal of $\mathbf{Q}E(A)$ which is annihilated from the left by a nonzero element $\alpha$ of $E(A)$. This yields $IA \subseteq \ker \alpha$. Hence, $\alpha(A) \cong A/\ker \alpha$ is bounded, a contradiction. □

Furthermore, it is obvious that the annihilator condition of Corollary 2.5 is satisfied in the following cases because of the results of [**6**]:

i) $\mathbf{Q}E(A)$ is semi-simple Artinian.

ii) $E(A)$ is commutative.

iii) $A$ is strongly indecomposable.

While these three cases represent the classes of groups considered in [**6**, Corollary 2.6], there nevertheless exists a torsion-free group $A$ of finite rank which has the quasi-Baer-splitting property, but does not satisfy the annihilator condition of Corollary 2.5. The existence of such an $A$ is a consequence of

**Corollary 2.6.** *A self-small torsion-free abelian group $A$, which is faithfully flat as an $E(A)$-module, has the quasi-Baer-splitting property.*

*Proof.* Let $M$ be a right $E(A)$-module such that $mT_A(M) = 0$ for some nonzero integer $m$. Suppose that $\phi : M \to mM$ is the epimorphism which is induced by multiplication with $m$. Let $i : mM \to M$ be the inclusion map; and consider the commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] & \\ T_A(M) \arrow[r, "T_A(\phi)"] \arrow[d, Rightarrow, no head] & T_A(mM) \arrow[r] \arrow[d, "T_A(i)"] & 0 \\ T_A(M) \arrow[r, "m"'] & T_A(M) & \end{tikzcd}$$

whose rows and columns are exact. We obtain $0 = mT_A(M) = \operatorname{im} T_A(i)T_A(\phi)$. Hence $0 = \operatorname{im} T_A(\phi) = T_A(mM)$. Since $A$ is a faithfully flat left $E(A)$-module, $mM = 0$. $\square$

*Example* 2.7. In [**4**, Theorem 2.8], we showed that every cotorsion-free ring $R$ can be realized as the endomorphism ring of an abelian group $A$ which is self-small and faithfully flat as an $E(A)$-module. Moreover, if the additive group of $R$ has finite rank, then $A$ can be chosen to have finite rank [**9**, Theorem 110.2 and **4**, Theorem 2.8]. Hence, Corollary 2.6 and [**4**] yield the existence of a torsion-free abelian group $A$ of finite rank which has the quasi-Baer splitting property and satisfies $\mathbf{Q}E(A) \cong \left\{ \left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) | a, b, c \in \mathbf{Q} \right\}$. On the other hand, the right ideal $I = \left\{ \left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right) | a, b, \in \mathbf{Q} \right\}$ of $\mathbf{Q}E(A)$ is proper. If $r = \left(\begin{smallmatrix} x & 0 \\ y & z \end{smallmatrix}\right)$ with $x, y, z \in \mathbf{Q}$ an element of $\mathbf{Q}E(A)$ with $rI = 0$, then we obtain $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & 0 \\ y & z \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax & 0 \\ ay+bz & 0 \end{smallmatrix}\right)$ for all $a, b \in \mathbf{Q}$. Choosing $a = 1$ and $b = 0$ gives $x = y = 0$. On the other hand, the choice $a = 0$ and $b = 1$ yields $z = 0$. Thus, $I$ has a zero left annihilator in $\mathbf{Q}E(A)$.

Therefore, in contrast to Corollary 2.4, neither [**6**, Corollary 2.6] nor Corollary 2.5 completely characterize the torsion-free abelian groups of finite rank with the finite quasi-Baer-splitting property.

**3. The finite rank case.** In this section we describe up to quasi-isomorphism the structure of the torsion-free abelian groups of finite rank which have the finite quasi-Baer-splitting property. To simplify our notation, we say that torsion-free abelian groups $A_1, \dots, A_n$ have *incomparable socles* if, for all $i, j \in \{1, \ldots, n\}$, the following two conditions are satisfied:

i) If $S_{A_j}(A_i) \doteq A_i$ for some $j \in \{1, \ldots, n\}$, then $i = j$.

ii) If $U_1, \ldots, U_n$ are subgroups of $A_i$ such that $S_{A_j}(U_j) = U_j$ for all $j$, then $A_i \doteq U_1 + \cdots + U_n$ only if $A_i \doteq U_j$ for some $j$.

**Theorem 3.1.** *The following conditions are equivalent for a torsion-free abelian group $A$ of finite rank*:

a) *$A$ has the finite quasi-Baer-splitting property.*

b) *Whenever $A \dot{\sim} A_1^{m_1} \oplus \cdots \oplus A_n^{m_n}$ is a quasi-decomposition of $A$ into strongly indecomposable groups $A_1, \ldots, A_n$ such that $A_i \dot{\sim} A_j$ only if $i = j$, and $0 < m_1, \ldots, m_n < \omega$, then $A_1, \ldots, A_n$ have incomparable socles.*

c) *There exists a quasi-decomposition $A \dot{\sim} A_1^{m_1} \oplus \cdots \oplus A_n^{m_n}$ where the $A_i'$s are strongly indecomposable groups such that $A_1, \ldots, A_n$ have incomparable socles.*

*Proof.* a) $\Rightarrow$ b). Suppose $A \dot{\sim} A_1^{m_1} \oplus \cdots \oplus A_n^{m_n}$ where each $A_i$ is strongly indecomposable, $A_i \dot{\sim} A_j$ only if $i = j$ and $0 < m_1, \ldots, m_n < \omega$. If $S_{A_j}(A_i) \doteq A_i$ for some $i, j \in \{1, \ldots, n\}$, then $S_{A_j}(A_i)$ itself is a quasi-summand of $A$, and there exists an exact sequence $\oplus_I A_j \xrightarrow{\beta} S_{A_j}(A_i) \to 0$ for some index-set $I$. Since $\oplus_I A_j$ is a quasi-summand of $\oplus_I A$, we have $S_A(\oplus_I A_j) \doteq \oplus_I A_j$ by Proposition 2.1b. Hence, $\beta$ quasi-splits by a); and $\oplus_I A_j \dot{\sim} S_{A_j}(A_i) \oplus \ker \beta$. Since the groups $A_j$ and $A_i \doteq S_{A_j}(A_i)$ are strongly indecomposable, we obtain $A_j \dot{\sim} S_{A_j}(A_i) \doteq A_i$ by [**9**, Theorem 92.5]. The choice of the $A_i$'s yields $i = j$.

Now assume that $U_1, \ldots, U_n$ are subgroups of $A_i$ for some $i \in \{1, \ldots, n\}$ such that $A_i \doteq U_1 + \cdots + U_n$ and $S_{A_j}(U_j) = U_j$ for all $j = 1, \ldots, n$. The codiagonal map $\sigma : \oplus_{j=1}^n U_j \to A_i$ satisfies $\operatorname{im} \sigma = U_1 + \cdots + U_n \doteq A_i$. Once we have shown that $S_A(U_j) \doteq U_j$ for all $j = 1, \ldots, n$, we obtain $S_A(U_1 \oplus \cdots \oplus U_n) = S_A(U_1) \oplus \cdots \oplus S_A(U_n) \doteq U_1 \oplus \cdots \oplus U_n$. Thus, the sequence $U_1 \oplus \cdots \oplus U_n \to \operatorname{im} \sigma \to 0$ quasi-splits by a). Since $\operatorname{im} \sigma \doteq A_i$, we obtain that $A_i$ is isomorphic to a quasi-summand of $U_1 \oplus \cdots \oplus U_n$. Considering quasi-decompositions of the groups $U_j$ into strongly indecomposable subgroups, we obtain that $A_i$ has to be quasi-isomorphic to an indecomposable quasi-summand $V$ of

$U_j$ for some $j \in \{1, \dots, n\}$. Because $V$ is strongly indecomposable, [**1**, Theorem 7.3 and Proposition 7.6] implies that the embedding $V \subseteq U_j \subseteq A_i$ quasi-splits. Since $A_i$ is strongly indecomposable, we conclude $V \doteq U_j \doteq A_i$, as required.

To show $S_A(U_j) \doteq U_j$, we observe that there exist a nonzero integer $k$ and homomorphisms $\lambda : A_j \to A$ and $\pi : A \to A_j$ with $\pi\lambda = k \cdot id_{A_j}$. If $x \in U_j$, then there are $\alpha_1, \dots, \alpha_s \in \operatorname{Hom}(A_j, U_j)$ and $a_1, \dots, a_s \in A$ with $x = \alpha_1(a_1) + \cdots + \alpha_s(a_s)$. Hence, $kx = \alpha_1(ka_1) + \cdots + \alpha_s(ka_s) = (\alpha_1\pi)\lambda(a_1) + \cdots + (\alpha_s\pi)\lambda(a_s) \in S_A(U_j)$. Thus, $kU_j \subseteq S_A(U_j)$.

b) $\Rightarrow$ c). This is obvious because of [**9**, Theorem 92.5].

c) $\Rightarrow$ a). As a first step, we prove the following. Given a map $\alpha : F \to P$ where $P = A_1^{s_1} \oplus \cdots \oplus A_n^{s_n}$, $F \doteq \oplus_{j=1}^n F_j$ with $F_j \cong \oplus_{I_j} A_j$, and $\alpha(F) \doteq P$, then there exists a quasi-splitting map $\sigma : P \to F$ with $\alpha\sigma = k \cdot id_P$ for some nonzero integer $k$. With $F$ fixed, we establish the existence of $\sigma$ by induction on $s = s_1 + \cdots + s_n$. In the case $s = 1$, we may assume without loss of generality that $P = A_1$. Then, taking $U_j = \alpha(F_j)$, we obviously have $U_j = S_{A_j}(U_j)$ and $A_1 = P \doteq \alpha(F) \doteq U_1 + \cdots + U_n$. By hypothesis c), there is a $j$ with $A_1 \doteq U_j = S_{A_j}(U_j) \subseteq S_{A_j}(A_1)$; and, consequently, $j = 1$ by the definition of incomparable socles. Hence, $\alpha(F_1) \doteq A_1$. If $i : F_1 \to F$ is the inclusion map, then $\alpha i = \alpha|_{F_1}$ quasi-splits by Corollary 2.5. Thus, there is a mapping $\sigma' : P \to F_1$ such that $(\alpha i)\sigma' = k \cdot id_P$ for some positive integer $k$, and therefore $\sigma = i\sigma' : P \to F$ is the desired map.

Now suppose $s > 1$. To simplify our notation, let $H = A_1 \oplus \cdots \oplus A_n$. No generality is lost if we assume $s_1 > 0$. Write $P = A_1 \oplus D$ where $D = A_1^{s_1 - 1} \oplus A_2^{s_2} \oplus \cdots \oplus A_n^{s_n}$, and denote the projection of $P$ onto $D$, whose kernel is $A_1$, by $\pi$. Because of $\pi\alpha(F) \doteq \pi(P) = D$, the map $\pi\alpha : F \to D$ quasi-splits by induction. Choose a subgroup $U$ of $F$ with $F \doteq \ker \pi\alpha \oplus U$. We obtain $\ker \pi\alpha = \alpha^{-1}(A_1)$ and $\alpha(\ker \pi\alpha) \doteq A_1$. Moreover, since $\ker \alpha$ is a quasi-summand of $F$, we obtain $S_A(\ker \pi\alpha) \doteq \ker \pi\alpha$ by Proposition 2.1b. Thus, we can find index-sets $J_1, \dots, J_n$ and a map $\beta : \oplus_{j=1}^n [\oplus_{J_j} A_j] \to \ker \pi\alpha$ such that $\ker \pi\alpha \doteq \operatorname{im} \beta$. Consider the map $\alpha\beta : \oplus_{j=1}^n [\oplus_{J_j} A_j] \to A_1$. It satisfies $\operatorname{im} \alpha\beta \doteq \alpha(\ker \pi\alpha) \doteq A_1$. By the result of the case $s = 1$, the map $\alpha\beta$ quasi-splits, and the same holds for $\alpha|_{\ker \pi\alpha}$. We write $\ker \pi\alpha \doteq V \oplus \ker \alpha$. Hence, $F \doteq \ker \pi\alpha \oplus U \doteq \ker \alpha \oplus V \oplus U$. Thus, the map $\alpha : F \to P$ quasi-splits.

To complete the proof, suppose $0 \to B \overset{\alpha}{\to} C \overset{\beta}{\to} G \to 0$ is an exact sequence of torsion-free groups such that $C \doteq \alpha(B) + S_A(C)$ and $G$ is a quasi-summand of an $A$-projective group of finite $A$-rank. In particular, $G$ itself has finite rank and $G \dot{\sim} P = A_1^{s_1} \oplus \cdots \oplus A_n^{s_n}$ for some choice of nonnegative integers $s_1, \ldots, s_n$. Fix a quasi-isomorphism $\gamma : G \to P$ and select a homomorphism $\delta : F \to C$ which satisfies $\delta(F) = S_A(C)$, where $F = \oplus_I A$ for some index-set $I$. Then, $\alpha = \gamma\beta\delta$ is a map from $F$ to $P$ with $\alpha(F) = \gamma\beta(S_A(C)) \doteq \gamma(G) \doteq P$. Since $A \dot{\sim} A_1^{m_1} \oplus \cdots \oplus A_n^{m_n}$, there are index sets $I_1, \ldots, I_n$ such that $F \doteq \oplus_{j=1}^n [\oplus_{I_j} A_j]$. By our first step, there is a map $\sigma : P \to F$ and a nonzero integer $k$ such that $\alpha\sigma = k \cdot id_P$. Since $\gamma$ is monic, and $\gamma\beta\delta\sigma\gamma = \alpha\sigma\gamma = (k \cdot id_P)\gamma = \gamma(k \cdot id_G)$, we obtain that $\sigma' = \delta\sigma\gamma : G \to C$ is the desired map with $\beta\sigma' = k \cdot id_G$. □

**Corollary 3.2.** *An almost completely decomposable group $A$ of finite rank has the finite quasi-Baer splitting property if and only if $A \dot{\sim} A_1^{m_1} \oplus \cdots \oplus A_n^{m_n}$ where the $A_i$'s are the subgroups of* $\mathbf{Q}$ *of pairwise incomparable type.*

*Proof.* Suppose that $A$ has the finite quasi-splitting property, and write $A \dot{\sim} A_1^{m_1} \oplus \cdots \oplus A_n^{m_n}$ where the $A_i$'s are pairwise nonisomorphic subgroups of $\mathbf{Q}$. If type $(A_i) \leq$ type $(A_j)$ for some $i \neq j$ in $\{1, \ldots, n\}$, then $S_{A_i}(A_j) = A_j$ in contradiction to Theorem 3.1.

On the other hand, the condition in Theorem 3.1b is satisfied in particular if there are non nonzero homomorphisms between nonquasi-isomorphic, strongly indecomposable quasi-summands of $A$. Thus, the converse is an immediate consequence of Theorem 3.1. □

Furthermore, there is a special case in which it is easy to check that torsion-free abelian groups $A_1$ and $A_2$ have incomparable socles:

**Corollary 3.3.** *Let $A_1$ and $A_2$ be strongly indecomposable torsion-free abelian groups of finite rank such that $E(A_1)$ and $E(A_2)$ do not have zero divisors. A torsion-free group $A \dot{\sim} A_1^n \oplus A_2^m$ such that $0 < n$, $m < \omega$ has the finite quasi-Baer-splitting property if and only if $S_{A_1}(A_2) \neq A_2$ and $S_{A_2}(A_1) \neq A_1$.*

*Proof.* It suffices to show that two such groups $A_1$ and $A_2$ always satisfy the second condition for incomparability of socles. Every nonzero endomorphisms of a strongly indecomposable abelian group of finite rank is either nilpotent or a monomorphism. Since $A_1$ has no divisors of zero, every nonzero endomorphism of $A_1$ is one-to-one. Hence, every nonzero subgroup $U$ of $A_1$ with $S_{A_1}(U) = U$ contains a subgroup $W$ which is isomorphic to $A_1$. By [**1**, Theorem 7.3], $W$ is quasi-summand of $A_1$. This yields $U \doteq A_1$.

Let $V_1$ and $V_2$ be subgroups of $A_1$ which satisfy $S_{A_i}(V_i) = V_i$ for $i = 1, 2$ and $A_1 \doteq V_1 + V_2$. If $V_1 \neq 0$, then we have $V_1 \doteq A_1$. However, $V_1 = 0$ implies $V_2 \doteq A_1$. The corollary immediately follows from Theorem 3.1. $\square$

However, if $A$ has more than two pairwise nonquasi-isomorphic strongly indecomposable quasi-summands, then the second condition for incomparability of socles is no longer automatically satisfied as the next result shows.

**Theorem 3.4.** a) *There exists a torsion-free abelian group $A$ of finite rank which does not have the finite quasi-Baer splitting property, but admits a quasi-decomposition $A \dot{\sim} A_1^{m_1} \oplus \cdots \oplus A_n^{m_n}$ such that $A_1, \dots, A_n$ satisfy the first condition for incomparability of socles.*

b) *There exists a torsion-free group $A$ of finite rank which does not have the finite quasi-Baer splitting property, but admits a quasi-decomposition $A \dot{\sim} A_1^{m_1} \oplus \cdots \oplus A_n^{m_n}$ such that $A_1, \dots, A_n$ satisfy the second condition for incomparability of socles.*

*Proof.* If $p_1, \dots, p_n$ are primes of $\mathbf{Z}$, then $\mathbf{Z}_{p_1,\dots,p_n}$ denotes the subgroup of $\mathbf{Q}$ whose elements can be written in the form $r/s$ where $r$ and $s$ are relatively prime integers and the $p_i$'s do not divide $s$ for $i = 1, \dots, n$. We fix three distinct primes $p_1, p_2$, and $p_3$ of $\mathbf{Z}$.

a) Let $V = \mathbf{Q}e_1 \oplus \mathbf{Q}e_2$ be a two-dimensional rational vector space, and define $G_1 = \mathbf{Z}_{p_1,p_2}e_1$, $G_2 = \mathbf{Z}_{p_1,p_3}e_2$, and $G = \langle G_1, G_2, p_1^{-n}(e_1 + e_2) | n < \omega \rangle$. By [**5**, Example 2.4], $G$ is a strongly indecomposable abelian group of rank 2 whose endomorphism ring is a subring of $\mathbf{Q}$. Moreover, $G_1$ and $G_2$ are pure fully invariant subgroups of $G$; and the type of $e_1 + e_2$ in $G$ is represented by the characteristic $(m_p)_p$

where $m_p = \infty$ for $p \neq p_2, p_3$ and $m_{p_2} = m_{p_3} = 0$. Set $B_1 = \mathbf{Z}_{p_1,p_2}$, $B_2 = \mathbf{Z}_{p_1,p_3}$, and $B_3 = \mathbf{Z}_{p_2,p_3}$.

We show that the group $A = G \oplus B_1 \oplus B_2 \oplus B_3$ does not have the finite quasi-Baer splitting property and that $G, B_1, B_2$ and $B_3$ satisfy the first but not the second condition for incomparability of socles.

Suppose $S_{B_1}(G) \doteq G$. Since $B_1 = p_3 B_1$, we obtain $p_3 G = G$. Hence, $p_3 G_2 = p_3 G \cap G_2 = G_2$, which is not possible since $G_2 \cong \mathbf{Z}_{p_1,p_3}$. In the same way, we show that $S_{B_2}(G)$ and $S_{B_3}(G)$ are not quasi-equal to $G$.

On the other hand, suppose that $\phi$ is an element of $\operatorname{Hom}(G, B_1)$. Because the types of $G_2$ and of $e_1 + e_2$ are incomparable with the type of $B_1$, we obtain $\phi(e_2) = 0$ and $\phi(e_1 + e_2) = 0$. Thus, $\operatorname{Hom}(G, B_1) = 0$. Since $\operatorname{Hom}(B_i, B_j) = 0$ for $i \neq j \in \{1, 2, 3\}$, we have shown that $G, B_1, B_2$, and $B_3$ satisfy the first condition for incomparability of socles.

Furthermore, if we let $G_3$ be the $\mathbf{Z}$-purification of $\langle e_1 + e_2 \rangle$ in $G$, then we obtain $G \doteq G_1 + G_2 + G_3 + H$ where $S_G(H) = H = 0$ and $S_{B_i}(G_i) = G_i$ for $i = 1, 2, 3$, although none of the groups $G_1, G_2, G_3$ or $H$ is quasi-equal to $G$. This shows that $G, B_1, B_2$ and $B_3$ do not satisfy the second condition for incomparability of socles. Moreover, the codiagonal map $\beta : G_1 \oplus G_2 \oplus G_3 \to G$ does not quasi-split by [**9**, Theorem 92.5]; and $A$ does not have the finite quasi-Baer splitting property.

b) We continue to use the notation of part a) of this proof. Let $K \cong \mathbf{Z}_{p_2}$, and consider the group $B = G \oplus K$. Since $B_1$ and $B_3$ are isomorphic to subgroups of $G/G_2$, we see that $G/G_2$ has type at least the type of $K$. If $G/G_2$ is divisible, then $p_2 G_2 = G_2$ and $p_2(G/G_2) = G/G_2$ yield $p_2 G = G$. Hence, $p_2 G_1 = p_2 G \cap G_1 = G_1$, which is not possible since $G_1 \cong \mathbf{Z}_{p_1,p_2}$. Hence, $G/G_2 \cong K$. Therefore, $G$ and $K$ do not satisfy the first condition for incomparability of socles. By Corollary 3.3, $B$ does not have the finite quasi-Baer splitting property. On the other hand, the arguments used in the proof of Corollary 3.3 show that $G$ and $K$ satisfy the second condition for incomparability of socles. $\square$

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