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ON ABSOLUTE WEIGHTED MEAN SUMMABILITY

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1. Definitions and notation. Let Σa_n be an infinite series with a sequence of its partial sums (s_n), and let $A = (a_{nk})$ be an infinite matrix. Assume that

$$T_n = \sum_{v=0}^{\infty} a_{nv} s_v, \qquad n = 0, 1, \dots \tag{1}$$

exists (i.e., the series on the right-hand side converges for each n). If

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty, \tag{2}$$

then Σa_n is said to be $|A|_k$ summable, where $k \geq 1$. When $k = 1$, we say that Σa_n is absolutely summable by the matrix A or simply summable $|A|$.

Now let A be a Riesz matrix, i.e., weighted mean matrix defined by

$$a_{nv} = p_v / P_n \qquad \text{for } 0 \leq v \leq n, \quad \text{and} \quad a_{nv} = 0 \qquad \text{for } v > n$$

where (p_n) is a sequence of positive real numbers, and

$$P_n = p_0 + p_1 + \cdots + p_n, \qquad P_{-1} = 0, \qquad P_n \to \infty \text{ as } n \to \infty.$$

If no confusion is likely to arise, we say that Σa_n is summable $|R, p_n|_k$, $k \geq 1$, if (2) holds.

Using analytical techniques, it is shown in [**3**] that the summability methods $|R, p_n|_k$ and $|R, q_n|_k$, $k \geq 1$, are equivalent under certain conditions.

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In the present paper, using functional analytic techniques, we give the necessary and sufficient conditions in order that the series Σa_n should be summable $|R, q_n|_k$, $k \geq 1$, whenever it is summable $|R, p_n|$. Therefore, we extend the known results of [**1, 5**] to the case $k > 1$.

2. The main results. Let (p_n) and (q_n) be sequences of positive real numbers such that

$$P_n = p_0 + p_1 + \cdots + p_n \to \infty \qquad \text{as } n \to \infty$$

and

$$Q_n = q_0 + q_1 + \cdots + q_n \to \infty \qquad \text{as } n \to \infty.$$

We are now ready to prove the main theorem.

Theorem. *The $|R, p_n|$ summability implies the $|R, q_n|_k$, $k \geq 1$, summability if and only if the following conditions hold*:

$$\text{(i)} \quad \frac{q_v}{Q_v} \cdot \frac{P_v}{p_v} = O(v^{1/k-1}), \qquad \text{(ii)} \quad |\Delta Q_{v-1}| \cdot W_v = O\left(\frac{p_v}{P_v}\right)$$

$$\text{(iii)} \quad Q_v W_v = O(1) \quad \text{where } W_v = \left\{ \sum_{i=v+1}^{\infty} i^{k-1} \left(\frac{q_i}{Q_i Q_{i-1}} \right)^k \right\}^{1/k}$$

and we regard that the above series converges for each v and Δ is the forward difference operator.

Proof. Necessity. Let (t_n) and (T_n) be the sequences of Riesz means (R, p_n) and (R, q_n) of the series Σa_n, respectively, i.e.,

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v$$

and

$$T_n = \frac{1}{Q_n} \sum_{v=0}^{n} q_v s_v = \frac{1}{Q_n} \sum_{v=0}^{n} (Q_n - Q_{v-1}) a_v.$$

Hence, we have that, for $n \geq 1$,

$$c_n := t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \qquad c_0 = a_0, \tag{3}$$

and

$$C_n := T_n - T_{n-1} = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v, \qquad C_0 = a_0. \tag{4}$$

We are given that

$$\sum_{n=1}^{\infty} n^{k-1} |C_n|^k < \infty \tag{5}$$

whenever

$$\sum_{n=0}^{\infty} |c_n| < \infty. \tag{6}$$

Now the space of sequences (a_v) satisfying (6) is a Banach space if normed by

$$||c|| = \sum_{n=0}^{\infty} |c_n|. \tag{7}$$

We are also considering the space of those sequences (C_n) that satisfy (5). This is again BK-space (i.e., Banach space with continuous coordinates) with respect to the norm

$$||C|| = \left\{ |C_0|^k + \sum_{n=1}^{\infty} n^{k-1} |C_n|^k \right\}^{1/k}. \tag{8}$$

Hence we are given that (4) transforms the space of sequences satisfying (6) into the space satisfying (5). Applying the Banach-Steinhaus theorem in the usual way, we find that there is a constant $M > 0$ such that

$$||C|| \leq M ||c|| \tag{9}$$

for all sequences satisfying (6). Taking any $v \geq 0$, we apply (9) with

$$a_{v+1} = 1, \qquad a_i = 0, \qquad i \neq v+1,$$

then (3) and (4) yield that

$$c_n = \begin{cases} 0, & n < v+1 \\ \dfrac{P_v p_n}{P_n P_{n-1}}, & n \geq v+1 \end{cases} \quad \text{and} \quad C_n = \begin{cases} 0, & n < v+1 \\ \dfrac{Q_v q_n}{Q_n Q_{n-1}}, & n \geq v+1. \end{cases}$$

By (7) and (8), it follows that $||c|| = 1$ and $||C|| = Q_v \cdot W_v$. Now (9) implies that $Q_v \cdot W_v \leq M$, hence (iii) is necessary. To get the necessity of the other conditions, we again apply (9) with

$$a_v = 1, \qquad a_{v+1} = -1, \qquad a_i = 0, \qquad i \neq v, \ i \neq v+1,$$

for any fixed $v \geq 0$. Then we have, by (7) and (8), that

$$c_n = \begin{cases} 0, & n < v \\ \dfrac{p_v}{P_v}, & n = v \\ \dfrac{-p_v p_n}{P_n P_{n-1}}, & n > v \end{cases} \quad \text{and} \quad C_n = \begin{cases} 0, & n < v \\ \dfrac{q_v}{Q_v}, & n = v \\ \dfrac{q_n \Delta Q_{v-1}}{Q_n Q_{n-1}}, & n > v. \end{cases}$$

So, by (7) and (8), we have

$$||c|| = \frac{2p_v}{P_v} \quad \text{and} \quad ||C|| = \left\{ v^{k-1} \left(\frac{q_v}{Q_v} \right)^k + |\Delta Q_{v-1}|^k \cdot W_v^k \right\}^{1/k}.$$

Now (9) implies that

$$v^{k-1}(q_v/Q_v)^k + |\Delta Q_{v-1}|^k \cdot W_v^k \leq (2M)^k (p_v/P_v)^k.$$

Since this holds for any $v \geq 0$, we get

$$v^{k-1}(q_v/Q_v)^k + |\Delta Q_{v-1}|^k \cdot W_v^k = O\{(p_v/P_v)^k\},$$

which in turn implies the necessity of (i) and (ii).

Sufficiency. By (3), we have

$$a_v = \frac{P_v}{p_v} c_v - \frac{P_{v-2}}{p_{v-1}} c_{v-1}, \qquad P_{-2} = P_{-1} = 0, \qquad c_{-1} = 0. \tag{10}$$

By (4) and (10), it follows that

$$\begin{aligned} C_n &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} \left(\frac{P_v}{p_v} c_v - \frac{P_{v-2}}{p_{v-1}} c_{v-1} \right) \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} (Q_{v-1} P_v - Q_v P_{v-1}) \frac{c_v}{p_v} + \frac{q_n P_n}{Q_n p_n} c_n. \end{aligned}$$

Now write $H_n := n^{(1-1/k)} \cdot C_n$, $n \geq 1$. Then we have $H_n = \sum_{v=1}^{n} a_{nv} c_v$ where

$$a_{nv} = \begin{cases} n^{(1-1/k)} \cdot \dfrac{q_n}{Q_n Q_{n-1}} \left(\dfrac{P_v}{p_v} \Delta Q_{v-1} + Q_v \right), & 1 \leq v \leq n-1; \\ n^{(1-1/k)} \cdot \dfrac{q_n P_n}{Q_n p_n}, & v = n; \\ 0, & v > n. \end{cases}$$

Hence, Σa_n is summable $|R, q_n|_k$, $k \geq 1$, whenever Σa_n is summable $|R, p_n|$ if and only if $\sum_n |H_n|^k < \infty$ whenever $\Sigma |c_n| < \infty$ or, equivalently, by [**2**, Theorem 5],

$$\sup_v \sum_n |a_{nv}|^k < \infty. \tag{11}$$

By the definition of $A = (a_{nv})$, we have

$$\sum_{n=v}^{\infty} |a_{nv}|^k = v^{k-1} \left(\frac{q_v P_v}{Q_v p_v} \right)^k + \left| \frac{P_v}{p_v} \Delta Q_{v-1} + Q_v \right|^k W_v^k.$$

Now the conditions (i), (ii) and (iii) imply that $\sum_{n=v}^{\infty} |a_{nv}|^k = O(1)$ as $v \to \infty$, which completes the proof. $\square$

We now provide some examples to illustrate situations in which the sequences (p_n) and (q_n) satisfy (i)–(iii) and also in which they don't.

Consider the sequences (p_n) and (q_n) defined by $p_v = x^v$ and $q_v = (v+1)^\alpha$ where $x > 1$ and $\alpha > -1$. A few calculations reveal that

$$\frac{P_v}{p_v} \sim \frac{x}{x-1}, \qquad Q_v \sim \frac{vq_v}{\alpha} \quad \text{and} \quad W_v = O(1/v+1).$$

One can now see that conditions (i)–(iii) hold.

As a second example, consider the sequence (p_v) defined by $p_v = v^\alpha$ where $\alpha \neq -1$. Then $P_v/p_v \sim v/\alpha$ (i.e. P_v/p_v is asymptotic to v/α). In this case, there is no sequence (q_n) of positive real numbers for which $Q_n \to \infty$ and conditions (i)–(iii) hold when $k > 1$. Actually, by (i), we must have

$$q_v/Q_v = O(1/v^{2-1/k}), \qquad k > 1. \tag{12}$$

Since the series $\sum 1/v^{2-1/k}$ is convergent for $k > 1$ and the series $\sum q_v/Q_v$ fails to converge by the familiar Abel-Dini theorem, (12) does not hold.

We now turn our attention to a result of [**4**] which claims that, if $k > 0$ and $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$, then there exist two positive constants M and N, depending only on k, for which

$$\frac{M}{P_{v-1}^k} \leq \sum_{n=v}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \leq \frac{N}{P_{v-1}^k}$$

for all $v \geq 1$, where M and N are independent of (p_n).

Using this result and taking $q_n = 1$, for all n in the Theorem, we get the following Corollary immediately.

Corollary. *The $|R, p_n|$ summability implies the $|C, 1|_k$, $k \geq 1$, summability if and only if*

$$P_v/p_v = O(v^{1/k}).$$

We conclude the paper with the following observation. If we take $q_n = p_n$, for all n, in the Theorem, and if the $|R, p_n|$ summability implies the $|R, p_n|_k$, $k \geq 1$, summability, then, by (i), we must have $v^{1-1/k} = O(1)$, which is impossible when $k > 1$. This means that there is a series Σa_n which is $|R, p_n|$ summable but not $|R, p_n|_k$, $k > 1$, summable. Actually we can construct such a series as follows.

Let us define

$$c_n = t_n - t_{n-1} = \begin{cases} 1/(v+1)^2, & n = 2^v,\ v = 0, 1, \dots, \\ 0, & n \neq 2^v. \end{cases}$$

Then $\Sigma |c_n| < \infty$. But, for $k > 1$, we have

$$\sum_{n=1}^{\infty} n^{k-1} |c_n|^k = \sum_{v=0}^{\infty} (2^v)^{k-1} (1/(v+1)^{2k}) = \infty.$$

On the other hand, by (3), we get $c_0 = t_0 = a_0$ and

$$a_n = \frac{P_n}{p_n} c_n - \frac{P_{n-2}}{p_{n-1}} c_{n-1}.$$

Hence, we have

$$a_n = \begin{cases} \dfrac{P_{2^v}}{(v+1)^2 P_{2^v}}, & n = 2^v,\ v = 0, 1, 2, \dots, \\ \dfrac{-P_{2^v}}{(v+1)^2 P_{2^v}}, & n = 2^v + 1, \\ 0 & \text{otherwise}. \end{cases}$$

So, Σa_n is the series we are seeking.

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