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A RESTRICTION ON THE FIRST SUCCESSOR CARDINAL WHICH IS JONSSON

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ABSTRACT. This paper investigates Jonsson cardinals as successors of singular cardinals. In particular, Shelah's theory of possible cofinalities is used to show that the first successor cardinal which is Jonsson cannot be the successor of a singular cardinal whose cofinality is measurable.

Viewing an algebra as a nonempty set together with countably many finite argument functions on the set, an algebra is said to be Jonsson if any proper subalgebra must have smaller cardinality.

When the cardinal κ is taken as the nonempty set, then κ is said to bear a Jonsson algebra if countably many finite argument functions exist on κ, which cannot have their respective domains and ranges restricted to a proper subset of κ whose cardinality is κ. Otherwise, κ is said to be a Jonsson cardinal.

Jonsson's problem concerns which cardinals are Jonsson cardinals. For more background see [**2**].

Charting some of the classical and more recent results relating to Jonsson's problem in ZFC yields the following:

$\aleph_0, \aleph_1, ..., \aleph_n, ...$ are not Jonsson cardinals, see [**2**].

$\aleph_\omega$ open.

$\aleph_{\omega+1}$ is not a Jonsson cardinal (Shelah), see [**4**] or [**6**].

$\vdots$

$\aleph_{\alpha+1}$ is not a Jonsson cardinal if $\aleph_\alpha$ is regular,

(Tryba, Woodin) see [**1**] or [**7**].

$\vdots$

$\aleph_\beta$ is the first Jonsson cardinal implies either $cf(\aleph_\beta) = \omega$

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or $\aleph_\beta$ is a regular limit (Rowbottom), see [**2**].

$\vdots$

$\aleph_{\beta+1}$ is a Jonsson cardinal, then it cannot be the first

regular Jonsson cardinal (Shelah), see [**4**].

$\aleph_\eta$ is a Jonsson cardinal which is a regular limit,

implies $\aleph_\eta$ is $\aleph_\eta - Mahlo$ (Shelah), see [**3**].

The main result of this paper relies heavily on Shelah's theory of *possible cofinalities*, denoted, *pcf-theory*, see [**5**] and, in particular, how pcf-theory was applied to prove $\aleph_{\omega+1}$ is not Jonsson, see [**4**]. For an elegant and useful survey of Shelah's pcf-theory and among other things, its application to Jonsson's problem, see [**1**], which resulted from a series of lectures given by M. Magidor in the Fall of 1989 at the Mathematical Sciences Research Institute in Berkeley, California.

This paper will make use of exactly that from pcf-theory which is necessary for the proof of the main result.

Let μ be a singular cardinal of cofinality κ and D an ultrafilter on κ. Let

$$a \subset \mu \cap \{\text{Regular cardinals}\},$$

such that $|a| = \kappa$ and $min\ a > \kappa^+$. Let

$$a = \{a_\alpha : \alpha \in \kappa\},$$

$$\prod a = \{f : \kappa \to \cup a : f(\alpha) \in a_\alpha\},$$

$$\prod a/D = \left\{ \prod a, <_D \right\}, \quad \text{where}$$

$$f <_D g \quad \text{if and only if} \quad \{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in D.$$

Thus, $\prod a/D$ is linearly ordered and $cf(\prod a/D)$ denotes the cofinality of $\prod a/D$. Define

$$\lim_D a = \sup\{\gamma : \{\alpha \in \kappa : \gamma < a_\alpha\} \in D\}.$$

Finally, *the possible cofinalities of* a is defined as follows:

$$pcf a = \left\{ cf\left(\prod a/D \right) : D \text{ is an ultrafilter on } \kappa \right\}.$$

A result crucial to the main theorem of this paper states:

Lemma. (Shelah, see [1, Theorem 2.1]). *Let a, D and μ be as above with $\lim_D a = \mu$. If λ is a regular cardinal and*

$$\mu < \lambda < cf\left(\prod a/D\right),$$

then there is a set

$$a^1 \subset \mu \cap \{\textit{Regular cardinals}\},$$

$$|a^1| \leq |a|$$

and an ultrafilter D^1 on a^1 such that

$$\lim_{D^1} a^1 = \mu;$$

and

$$cf\left(\prod a^1/D^1\right) = \lambda.$$

The general idea for the proof of this lemma is to construct a sequence

$$\langle f_\alpha/D : \alpha < \lambda\rangle$$

in $\prod a/D$ which has a least upper bound g/D. Then it is demonstrated that $\prod cf(g(\alpha))/D$ has the desired properties. See [**1**].

In the spirit of Rowbottom's result that the first Jonsson cardinal must be of cofinality ω or weakly inaccessible; and Shelah's result that the first regular Jonsson cardinal cannot be a successor, the main result presented here shows that the first successor cardinal which is Jonsson cannot be the successor of a singular cardinal with measurable cofinality.

Theorem. *Let μ be a singular cardinal and μ^+ the first successor cardinal which is Jonsson; then $cf(\mu)$ is not measurable.*

Proof. Assume μ is a singular cardinal, μ^+ the first successor cardinal which is Jonsson and $cf(\mu)$ is measurable. Let D be a κ-complete, normal ultrafilter over κ.

Two facts must be established which are based on results of Shelah. The first fact follows from the above lemma combined with the assumption that $cf(\mu) = \kappa$ is measurable.

Fact 1. *If λ is a regular cardinal and $\mu < \lambda < cf(\prod a/D)$, where*

$$a \subset \mu \cap \{\textit{Regular cardinals}\},$$

such that $|a| = \kappa$ and $\min a > \kappa^+$ and $\lim_D a = \mu$, then there exists

$$a^1 \subset \mu \cap \{\textit{Regular cardinals}\},$$

such that

$$\sup a^1 = \mu$$

and

$$cf\left(\prod a^1/D\right) = \lambda.$$

To establish this fact, first, let

$$a = \{a_\alpha : \alpha < \kappa\}$$

enumerate a and

$$c_\alpha = \{\beta < \kappa : a_\alpha < a_\beta\}.$$

Since $\lim_D a = \mu$, $c_\alpha \in D$. The normality of D gives

$$\Delta c_\alpha = \{\beta < \kappa : \alpha < \beta \Rightarrow \beta \in c_\alpha\} \in D.$$

If $\alpha, \beta \in \Delta c_\alpha$ are such that $\alpha < \beta$, then $a_\alpha < a_\beta$. Hence a is increasing almost everywhere and $\sup a = \mu$.

Now, using the techniques from [**1**] in the proof of the lemma, an increasing sequence

$$\langle f_\alpha/D : \alpha < \lambda\rangle$$

is constructed in $\prod a/D$ using the fact that $cf(\prod a/D) > \lambda$. It is then argued (nontrivially) that this increasing sequence has a least upper bound in ON^κ/D. Let $g \in ON^\kappa$ be the least upper bound. By necessity, $g(\alpha)$ is almost everywhere a limit ordinal. Take it to be a limit ordinal everywhere. Let

$$a^1 = \{cf(g(\alpha)) : \alpha < \kappa\}.$$

Fix a cofinal sequence σ_α in $cf(g(\alpha))$ and set

$$\overline{f}_\beta(\alpha) = \inf\{\sigma_\alpha(i) : f_\beta(\alpha) \leq \sigma_\alpha(i)\}.$$

Now $\langle \overline{f}_\beta/D : \beta < \lambda\rangle$ witnesses $cf(\prod a^1/D) = \lambda$ and $\lim_D a^1 = \mu$. Hence $\sup a^1 = \mu$, since D is normal.

The second fact is implicit in the proof of Theorem 3.7 of [**1**] and explicit in Theorem 4.4(2) of [**4**].

Fact 2. *If μ is a singular cardinal of cofinality κ,*

$$a \subset \mu \cap \{\textit{Regular, nonJonsson cardinals}\}$$

with $\sup a = \mu$ *and $\mu^+ \in pcf(a)$, then μ^+ is not Jonsson.*

Back to the proof of the theorem. The strategy is to assume κ is measurable and repeatedly perform the construction from the discussion after Fact 1 in such a way so as to insure that the construction stops after a finite number of steps, producing a situation as in the hypothesis of Fact 2.

Let $a \subset \mu$ such that min $a > \kappa^+$, $|a| = \kappa$, $\sup a = \mu$ and if $a_\alpha \in a$, then almost every such member, a_α, is a successor cardinal which is not the successor of a limit cardinal (call such a cardinal a *successor of order* 2 *or higher*). If $\mu^+ \in pcf(a)$, then μ^+ is not Jonsson by the Tryba-Woodin result in the chart on page 1 combined with Fact 2. Otherwise, since D extends the dual of the ideal of bounded subsets of κ,

$$cf\left(\prod a/D\right) = \lambda > \mu^+.$$

Let $\langle f_\beta : \beta < \lambda \rangle$ witness this. By the discussion after Fact 1, there exists $g^1 \in \prod a$ and $\langle \overline{f}^1_\beta : \beta < \mu^+ \rangle$ witnessing

$$cf\left(\prod_{\alpha \in \kappa} cf(g^1(\alpha))/D \right) = \mu^+.$$

Let

$$a^1 = \{cf(g^1(\alpha)) : \alpha < \kappa\}.$$

So $\sup a^1 = \mu$ by Fact 1. Let $a^1_\alpha = cf(g^1(\alpha))$. Hence $a^1_\alpha < a_\alpha$. If

$$\{\alpha < \kappa : a^1_\alpha \text{ is a successor cardinal of order 2 or higher}\} \in D,$$

then again, by the Tryba-Woodin result combined with Fact 2, μ^+ would not be a Jonsson cardinal, contrary to the hypothesis. Let

$$b = \{\alpha < \kappa : a^1_\alpha \text{ is a Jonsson cardinal }\};$$

hence, $b \in D$. By the hypothesis of the theorem, this means a^1_α is a regular limit cardinal for every $\alpha \in b$. As before, for $\alpha < \kappa$, let

$$c_\alpha = \{\beta < \kappa : a^1_\alpha < a^1_\beta\}.$$

Since $\sup a^1 = \mu$, $c_\alpha \in D$. The normality of D gives

$$\Delta c_\alpha = \{\beta < \kappa : \alpha < \beta \Rightarrow \beta \in c_\alpha\} \in D.$$

If $\alpha, \beta \in \Delta c_\alpha$ are such that $\alpha < \beta$, then $a^1_\alpha < a^1_\beta$. Set

$$b^*_2 = b \cap \Delta c_\alpha.$$

Hence b^*_2 is strictly increasing and $\sup b^*_2 = \mu$.

Set

$$\bar{b}_2 = \{\alpha \in b^*_2 : a^1_\alpha > |a|^+ = \kappa^+\} \in D.$$

Let α_ω be the ω^{th} member of $\bar{b}_2$ and let

$$\gamma_\omega = \cup\{a^1_\gamma : \gamma \in \bar{b}_2 \text{ and } \gamma < \alpha_\omega\}.$$

Also, let

$$b_2 = \{\alpha \in \overline{b}_2 : \gamma_\omega < a^1_\alpha\}.$$

So $b_2 \in D$. For $\alpha \in b_2$, let

$$a^2_\alpha = (\cup\{a^1_\delta : \delta < \alpha \text{ and } a^1_\delta < a^1_\alpha\})^{++}.$$

Assume, without loss of generality, that $a^2_\alpha < a^1_\alpha$. This is possible since a^1_α is a regular limit cardinal and $a^1_\alpha > \kappa^+$ for all $\alpha \in b_2$.

Next, let

$$a^2 = \{a^2_\alpha : \alpha \in b_2\}.$$

Hence, a^2 is strictly increasing, $\sup a^2 = \mu$ and

$$a^2 \subset \{\text{Successors of order two or higher}\}.$$

If $\mu^+ \in pcf(a^2)$, then again μ^+ is not Jonsson, resulting in a contradiction. Otherwise, construct a^3 from a^2 in the same way that a^1 was constructed from a. Again, by the discussion after Fact 1, it is possible to construct a $g^2 \in \prod a^2$ and $\{\overline{f}^2_\beta : \beta < \mu^+\}$, witnessing

$$cf\left(\prod_{\alpha\in b_2} cf(g^2(\alpha))/D\right) = \mu^+.$$

Setting

$$a^3_\alpha = cf(g^2(\alpha))$$

and

$$a^3 = \{a^3_\alpha : \alpha \in b_2\};$$

if

$$\{\alpha \in b_2 : a^3_\alpha \text{ is a successor of order 2 or higher}\} \in D,$$

then, as above, μ^+ is not a Jonsson cardinal. So let

$$b^*_4 = \{\alpha \in b_2 : a^3_\alpha \text{ is a Jonsson cardinal}\};$$

and assume that $b^*_4 \in D$. Construct, $b_4 \in D$ and a_4 in the same way that b_2 and a^2 were constructed above. This will result in

$$a^4_\alpha < a^3_\alpha < a^2_\alpha, \qquad \text{for } \alpha \in b_4;$$

and $\sup a^4 = \mu$. Now, either this process can be continued through every finite stage, or it must stop at some point. In the case of the former, since D is at least countably complete, what would result is

$$b_2 \supset b_4 \supset b_6 \supset \cdots, \qquad \text{with } b_{2n} \in D.$$

Hence, $\cap b_{2n} \in D$. Taking $\alpha \in \cap b_{2n}$ yields,

$$\cdots < a_\alpha^{2n} < \cdots < a_\alpha^4 < a_\alpha^2,$$

producing an impossibility. Therefore, at some finite step n the construction must stop, with

$$\{\alpha \in b_{2n} : a_\alpha^{2n+1} \text{ is a successor of order 2 or higher}\} \in D,$$

and

$$cf\left(\prod a^{2n+1}/D\right) = \mu^+.$$

But by the Tryba-Woodin result and Fact 2, μ^+ is not Jonsson; a contradiction. □

In conclusion: What limits this proof from extending to the general case of a Jonsson cardinal which is the successor of a singular, is the nature of

$$\{a_\alpha^{2n-1} : \alpha < \kappa\}.$$

With D a normal measure on κ, it can be assumed, without loss of generality, that

$$\{a_\alpha^{2n-1} : \alpha < \kappa\}$$

is strictly increasing, with $\sup \mu$. For the general case of a successor of a singular cardinal, the possibility of a measure one set of Jonsson cardinals which are successors of singular cardinals must be considered. Say, for almost all α,

$$a_\alpha^{2n-1} = \eta_\alpha^+.$$

There are three possibilities for almost all α:

(i) $cf(\eta_\alpha) < \alpha$,

(ii) $cf(\eta_\alpha) > \alpha$,

(iii) $cf(\eta_\alpha) = \alpha$.

It is a possibility that (iii) occurs almost everywhere which prevents a uniform selection of a^{2n}_α less than a^{2n-1}_α from being made to hold almost everywhere, since this allows for the possibility that

$$cf(\eta_\alpha) = \cup\{a^{2n-1}_\beta : \beta < \alpha\}.$$

Since Fact 1 is an existence theorem for

$$a^{2n-1} = \{a^{2n-1}_\alpha : \alpha \in \kappa\},$$

$$\sup a^{2n-1} = \mu;$$

and

$$cf\left(\prod a^{2n-1}/D\right) = \mu^+,$$

there is little control over what types of regular cardinals the a^{2n-1}_α can be. However, a result of Shelah suggests that (iii), above, may be a "natural" possibility for the a^{2n-1}_α.

Fact 3 ([**4**, Claim 2.1]). *Let $\{\beta_\gamma : \gamma < \kappa\}$ be a strictly increasing, continuous sequence with limit μ. Then there exists a closed and unbounded subset C of κ such that the true cofinality*

$$tcf\left(\prod_{\gamma \in C} \beta^+_\gamma/(J^{bd}_C)^*\right) = \mu^+,$$

where

$$J^{bd}_C = \{b \subset C : b \text{ is bounded}\}$$

and $(J^{bd}_C)^*$ is its dual filter. Hence $\mu^+ = \max pcf\{\beta^+_\gamma : \gamma < \kappa\}$.

Item (iii) would give rise to a situation in the hypothesis of Shelah's result, since in this case if $b \in D$, $b \subset$ (weak inaccessibles) and $\{a^{2n-1}_\alpha : \alpha \in b\}$ then both can be extended respectively to a closed unbounded subset of κ and a strictly increasing, continuous sequence. Hence, Shelah's result would coincide with the construction from Fact 1 used in the proof of the Theorem at this stage.

Replacing a normal measure on κ with a nonnormal measure ν presents similar difficulties. Let f_ν be a minimal function which is regressive mod ν, but never bounded mod ν. Again, if

$$a_\alpha^{2n-1} = \eta_\alpha^+,$$

for almost all α, then a difficulty similar to case (iii) above would result when the case of

$$cf(\eta_\alpha) = f_\nu(\alpha),$$

with

$$\eta_\alpha = \cup\{a_\beta^{2n-1} : \beta < \alpha \quad \text{and} \quad a_\beta^{2n-1} < a_\alpha^{2n-1}\}$$

is considered.

Nevertheless, it follows immediately from Facts 2 and 3 that:

Corollary. *If μ^+ is a Jonsson cardinal and $cf(\mu) = \kappa$, where κ is measurable, then there is already a measure one set of successor cardinals which are Jonsson below μ, with respect to any normal measure on κ.*

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